Problem Set 2

Question 1

In class we showed the correspondence between

- a meromorphic function \( f(z) \) with a single pole at \( z = 0 \), and
- a holomorphic map \( F : \mathbb{C} \rightarrow \mathbb{CP}^1 \) with \( F^{-1}([0, 1]) = \{0\} \).

Prove the general version of this result, i.e. prove that a meromorphic function with poles at \( \{p_1, p_2, \ldots, p_l\} \) is equivalent to a holomorphic map (from \( \mathbb{C} \) to \( \mathbb{CP}^1 \)) with \( F^{-1}([0, 1]) = \{p_1, p_2, \ldots, p_l\} \).

Question 2

(1) Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a holomorphic map such that \( f(0) = 0 \). Show that there is always a biholomorphism, say \( \phi \), defined on a neighborhood of 0 such that

\[
f(\phi(z)) = z^\nu
\]

for some positive integer \( \nu \in \mathbb{Z} \).

(2) Let \( \Sigma \) and \( \Sigma' \) be closed Riemann surfaces, and let \( F : \Sigma \rightarrow \Sigma \) be a holomorphic map between them. Suppose that \( F(p) = q \), where \( p \) is a point in \( \Sigma \). Prove that there are local co-ordinates (denoted by \( z \) and \( z' \) respectively) on neighborhoods of \( p \in \Sigma \) and of \( q \in \Sigma' \) such that \( z(p) = 0 \), \( z'(q) = 0 \), and with respect to these coordinates \( F \) is given by a map of the form \( z \mapsto z^\nu \) for some positive integer \( \nu \).
**Question 3**

Let $\Sigma$ and $\Sigma'$ be closed Riemann surfaces. Let $f \in K(\Sigma)$ be a meromorphic function, and let $F : \Sigma \rightarrow \Sigma'$ be a holomorphic map. For each point $p \in \Sigma$ we can pick local coordinates (denoted by $z$) in a neighborhood of $p$ such that $z(p) = 0$ and such that with respect to these coordinates $f$ is given by the map $z \mapsto z^\nu h(z)$, where $h(z)$ is holomorphic and non-vanishing at $z = 0$. Define the order of $f$ at $p$ to be

$$ord_p(f) = \nu .$$

Suppose that $F(p) = q$ and that with respect to local coordinates chosen as in Question 1(b) $F$ is given by the map $z \mapsto z^m$. Define the multiplicity (or ramification number) of $F$ at $p$ to be

$$mult_p(F) = m .$$

(we called this $\nu_F(p)$ in class).

Now suppose that $\Sigma' = \mathbb{CP}^1$ and that $F$ is the holomorphic map corresponding to $f \in K(\Sigma)$, i.e.

$$F(p) = \begin{cases} [0, 1] & \text{if } f \text{ has a pole at } p \\ [1, f(p)] & \text{otherwise} \end{cases}$$

Prove that

$$mult_p(F) = \begin{cases} ord_p(f) & \text{if } f(p) = 0 \\ -ord_p(f) & \text{if } f \text{ has a pole at } p \\ ord_p(f - f(p)) & \text{otherwise} \end{cases}$$

**Question 4**

Prove that every non-constant meromorphic function on a closed Riemann surface has at least one zero and one pole.
Question 5

(a) Let \( f(z, w) \) and \( g(z, w) \) be homogeneous polynomials in \((z, w)\). Prove that the map

\[
[z, w] \mapsto [f(z, w), g(z, w)]
\]
is a well defined holomorphic map from \( \mathbb{CP}^1 \) to \( \mathbb{CP}^1 \) if and only if \( f \) and \( g \) have no common factor.

(b) Suppose that \( f(z, w) \) and \( g(z, w) \) are both degree 1, say

\[
\begin{align*}
f(z, w) &= az + bw \\
g(z, w) &= cz + dw,
\end{align*}
\]

and let \( F_A : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be the map defined by

\[
[z, w] \mapsto [az + bw, cz + dw]
\]
Show that \( F_A \) is a well defined biholomorphic map if and only if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is an invertible \( 2 \times 2 \) matrix over \( \mathbb{C} \). For which \( A \) is \( F_A \) the identity?

(c) Let \( F_A : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be as in (b), and let \( f \in K(\mathbb{CP}^1) \) be the corresponding meromorphic function on \( \mathbb{CP}^1 \). Find the rational function corresponding to \( f \) (under the correspondence between rational functions on \( \mathbb{C} \) and meromorphic functions on \( \mathbb{CP}^1 \)).

Question 6

Define the degree of a holomorphic map \( F : \Sigma \to \Sigma' \) to be the integer

\[
deg(F) = \sum_{F(p) = q} \text{mult}_p(F),
\]
where \( q \in \Sigma' \) is any point. Let \( f \) be the meromorphic function on \( \mathbb{CP}^1 \) defined by the rational function

\[
z \mapsto \frac{z^3}{(1 - z^2)},
\]
and let \( F : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be the corresponding holomorphic map. Compute the degree of \( F \).