Problem Set 1

Question 1
(a) Show, by explicitly describing a diffeomorphism, that as a smooth manifold $\mathbb{CP}^1$ is diffeomorphic to $S^2$.

(b) Let
\[ \Lambda = \{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z} \} \]
be the lattice defined by $\omega_1, \omega_2 \in \mathbb{C}$, and let $T_\Lambda = \mathbb{C}/\Lambda$ be the corresponding torus. Construct an atlas of coordinate charts on $T_\Lambda$ and hence prove that it is a (compact) one-dimensional complex manifold, i.e. prove that it is a Riemann surface.

Question 2
Let $f(x, y)$ and $g(x, y)$ be polynomials with complex coefficients. Show that the following are equivalent:

- $f$ and $g$ have the same irreducible factors (possibly with different multiplicities)
- There are positive integers $m$ and $n$ such that $f$ divides $g^m$ and $g$ divides $f^n$. 

1
Question 3

(a) Show that if \( f(x, y) \) is a (non-trivial) homogeneous polynomial of degree \( d \) then it factors as a product of linear polynomials

\[
f(x, y) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y)
\]

for some \( \alpha_i, \beta_i \in \mathbb{C} \).

(b) Let \( F(x, y, z) \) be a non-trivial homogeneous polynomial of degree \( d \), and let \( C_F \) be the curve in \( \mathbb{CP}^2 \) defined by \( F(x, y, z) = 0 \). If \( \mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 \), where \( \mathbb{CP}^1 \subset \mathbb{CP}^2 \) is the ‘line at infinity’ defined by \( z = 0 \), show that \( C_F \cap \mathbb{CP}^1 \) is either a finite set of points or the entire \( \mathbb{CP}^1 \).

(c) Let \( f(x, y) = y^2 - 2x^2 - xy + 5x \), and let \( C_f = f^{-1}(0) \) be the algebraic curve in \( \mathbb{C}^2 \) defined by \( f \). Let \( C_F \) be the corresponding curve in \( \mathbb{CP}^2 \), i.e. let \( C_F \) be the compactification of \( C_f \) obtained by the usual compactification \( \mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 \). Find all the points in \( C_F - C_f \) and hence all the lines in \( \mathbb{C}^2 \) on which the intersection with \( C_f \) has less than two points.