(6) By Exercise 8 (p. 68) — or by Exercise 7 (p. 128) — a unit speed plane curve with curvature \( \kappa(s) \) is given by

\[
\beta(s) = (\int \cos \psi(t) \, dt, \int \sin \psi(t) \, dt)
\]

where \( \psi(t) = \int \kappa(x) \, dx \).

The ambiguity in these formulae is due to the constants of integration. Thus if \( \alpha(s) \) and \( \beta(s) \) are unit speed curves with the same curvature, then

\[
\psi_{\beta}(t) = \psi_{\alpha}(t) + \delta
\]

for some fixed \( \delta \).

Now use

\[
\int \cos (\psi_{\beta}(t)) \, dt = \int \cos (\psi_{\alpha}(t) + \delta) \, dt
\]

\[
= \cos \delta \int \cos (\psi_{\alpha}(t)) \, dt - \sin \delta \int \sin (\psi_{\alpha}(t)) \, dt
\]

and

\[
\int \sin (\psi_{\beta}(t)) \, dt = \sin \delta \left( \int \cos (\psi_{\alpha}(t)) \, dt + \cos \delta \int \sin (\psi_{\alpha}(t)) \, dt \right)
\]

Hence, if \( \alpha(s) = (x_{a}(s), y_{a}(s)) \), then

\[
\beta(s) = \begin{pmatrix}
\cos \delta \left( x_{a}(s) + a \right) - \sin \delta \left( y_{a}(s) + b \right), \\
\sin \delta \left( x_{a}(s) + a \right) + \cos \delta \left( y_{a}(s) + b \right)
\end{pmatrix}
\]
where \(a, b\) are constants of integration from \(\int \cos y(t) \, dt\) and \(\int \sin y(t) \, dt\) respectively.

In "column notation":

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
\cos \delta & -\sin \delta \\
\sin \delta & \cos \delta
\end{bmatrix}
\begin{bmatrix}
x + a \\
y + b
\end{bmatrix}
\]

\(\uparrow\text{Rotation}\) \quad \uparrow\text{Translation}

\text{thru } \delta \quad \text{by } (a, b).

Thus, curves \(x\) and \(y\) are related by an isometry, i.e., they are congruent!
For a unit speed plane curve, say
\[ \beta(s) = (x(s), y(s)) \quad \text{SEI} \]
we can write
\[ T(s) = (x'(s), y'(s)) = (\cos \gamma(s), \sin \gamma(s)) \]
Then
\[ \kappa(s) = y'(s) \]
Hence to have \( \kappa(s) = f(s) \) we must have
\[ y(s) = \int_0^s f(t) \, dt \]
Then
\[ x'(s) = \cos \gamma(s) \Rightarrow x(s) = \int \cos \gamma(t) \, dt \]
\[ y'(s) = \sin \gamma(s) \Rightarrow y(s) = \int \sin \gamma(t) \, dt \]
Thus the required curve is
\[ \beta(s) = \left( \int \cos \gamma(t) \, dt, \int \sin \gamma(t) \, dt \right) \]
where \( \gamma(s) = \int f(t) \, dt \).