Math 415 - Midterm 1

September 30, 2004

- Answer all questions.
- Be as clear as possible. Unless otherwise stated, give reasons for your answers.
- Use both sides of the paper, if necessary.
- Good luck!

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<thead>
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<th>Question</th>
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<td><strong>TOTAL</strong></td>
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Question 1

Indicate (without reasons) whether the following are True (T) or False (F):

(1) Using the usual rule for matrix addition, the sum $A + B$ is defined for any two matrices $A$ and $B$.

(2) If $A$ is a $10 \times 10$ matrix and $|A| = 5$, then $A$ has an inverse.

(3) Any homogeneous system of 22 linear equations in 35 unknowns has an infinite number of solutions.

(4) Elementary row operations do not change the determinant of a matrix.

(5) If $A$ and $B$ are matrices which can be multiplied in either order, i.e. such that both $AB$ and $BA$ are defined, then $A$ and $B$ must both be square matrices of the same size.

(6) The half-plane $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0 \}$ is a vector subspace of $\mathbb{R}^2$.

(7) If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of linearly dependent vectors in a vector space $V$, then at least one of the vectors in $S$ can be expressed as a linear combination of the others.

(8) If $A$ is a square matrix in echelon form then $A$ is upper triangular.

(9) If $A$ and $B$ are both $n \times n$ matrices, then $\text{det}(A + B) = \text{det}(A) + \text{det}(B)$.

(10) If $A^T$ is the transpose of an invertible matrix $A$, then $(A^T)^{-1} = (A^{-1})^T$.

(10 \times 2 = 20 points)
Question 2

State whether the following sets of vectors are linearly dependent or linearly independent. Give reasons.

\[
\begin{bmatrix}
1 \\
-2 \\
0 \\
3 \\
4
\end{bmatrix}, \begin{bmatrix}
3 \\
-6 \\
0 \\
9 \\
12
\end{bmatrix}
\]

(a)

\[
\begin{bmatrix}
3 \\
-6 \\
9 \\
12
\end{bmatrix} = 3 \begin{bmatrix}
1 \\
-2 \\
0 \\
4
\end{bmatrix}
\]

Hence the vectors are linearly dependent
\[
(b) \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 12 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 12 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\det(A) = \begin{vmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 12 \\ 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 12 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 \\ 12 & 1 \end{vmatrix}
\]

\[
= -12 + 2(3-12) \\
= -12 - 18 \\
= -30 
eq 0
\]

Hence the columns of \( A \) are linearly independent.
Question 3

If \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \) find \( A^{-1} \) or prove that it does not exist.

\[ |A| = 1 \cdot 2 \cdot 2 \neq 0 \] Hence \( A^{-1} \) exists

Compute \( A^{-1} \) from \( A^{-1} = \frac{\text{adj}(A)}{|A|} \) as follows:

\[
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[ \therefore \quad A^{-1} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \]
Question 4

(a) Find all the solutions for the following system of linear equations

\[ x_1 + 2x_2 - x_4 = 2 \]
\[ 2x_1 + 5x_2 + x_3 = 6 \]

\[ \text{Augmented matrix of coefficients:} \]
\[
\begin{bmatrix}
1 & 2 & 0 & -1 & 2 \\
2 & 5 & 1 & 0 & 6 \\
1 & 2 & 0 & -1 & 2 \\
0 & 1 & 1 & 2 & 2
\end{bmatrix}
\]

\[ \Rightarrow \quad \begin{align*}
X_2 &= 2 - x_3 - 2x_4 \\
\text{and} \quad X_1 + 2(2 - x_3 - 2x_4) - x_4 &= 2 \\
\Rightarrow \quad X_1 &= -2 + 2x_3 + 5x_4
\end{align*} \]

\[ \text{Solutions:} \quad X = \begin{bmatrix}
-2 + 2x_3 + 5x_4 \\
2 - x_3 - 2x_4 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \]
(b) If $A$ is the matrix of coefficients for the system of equations in part (a), find a spanning set of vectors for $\text{Nul}(A)$, i.e. find vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^4$ such that $\text{Nul}(A) = \text{Span}\{v_1, v_2, \ldots, v_n\}$.

From the row reduction in part (a), we get:

$$AX = 0 \Rightarrow x = x_3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 5 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ span $\text{Nul}(A)$.