Question 1

Find a particular solution for the differential equation

\[ y'' + y' - 2y = e^x. \]

Solution to homogeneous equation:

\[ r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1)=0 \]

so \( y_c(x) = c_1 e^x + c_2 e^{-2x} \) i.e. \( e^x \) is a solution.

Take \[ y_p(x) = Ae^x \]

Then \[ y_p'(x) = Ae^x(x+1) \]

\[ y_p''(x) = A e^x (x+1+1) = A e^x (x+2). \]

so \[ y_p'' + y_p' - 2y_p = A e^x (x+x-2x+2+1) = 3A e^x \]

Hence we must take \( A = \frac{1}{3} \), i.e.

\[ y_p(x) = \frac{1}{3} x e^x \]
Question 2

Consider the eigenvalue problem

\[ y'' + \lambda y = 0 \]
\[ y(0) = 0 \]
\[ y(1) = y'(1) \]

(a) Show that \( \lambda = 0 \) is an eigenvalue and find the associated eigenfunction.

When \( \lambda = 0 \):

\[ y'' = 0 \implies y(x) = A + Bx \]

\[ y(0) = 0 \implies 0 = A \implies y(x) = Bx \text{ (and } y'(x) = B) \]
Thus
\[ y(1) = y'(1) \implies B = B \text{, which is true for any } B \]

Hence \( \boxed{y(x) = x} \) will satisfy all the conditions and is thus the required eigenfunction.
(b) Show that \( \lambda = \alpha^2 \) is an eigenvalue if \( \tan(\alpha) = \alpha \), i.e. if \( \alpha \) is a positive root of the equation \( \tan(\theta) = \theta \).

If \( \lambda = \alpha^2 \):
\[
y'' + \alpha^2 y = 0
\]
\[
\Rightarrow y = A \cos \alpha x + B \sin \alpha x
\]
\[
y(0) = 0 \Rightarrow 0 = A
\]
So
\[
y(x) = B \sin \alpha x
\]
and
\[
y'(x) = \alpha B \cos \alpha x
\]

Hence
\[
y(1) = y'(1) \Rightarrow B \sin \alpha = \alpha B \cos \alpha.
\]

i.e.
\[
B(\sin \alpha - \alpha \cos \alpha) = 0
\]

For a non-trivial solution we thus need
\[
\sin \alpha = \alpha \cos \alpha
\]

i.e.
\[
\alpha = \tan(\alpha)
\]
Question 3

(a) Without explanation or calculation, fill in the blanks:

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\pi x) \sin(m\pi x) = \begin{cases} \frac{0}{1} & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}
\]

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\pi x) \cos(m\pi x) = \begin{cases} \frac{0}{1} & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}
\]

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\pi x) \cos(m\pi x) = \frac{0}{1}
\]

(b) If \( f_1(x) \) and \( f_2(x) \) are even functions of \( x \), and \( g_1(x) \) and \( g_2(x) \) are odd, state (without proof) whether the following are even, odd, or neither:

\[ f_1(x)f_2(x) \quad \text{even} \]
\[ f_1(x)g_1(x) \quad \text{odd} \]
\[ g_1(x)g_2(x) \quad \text{even} \]
\[ g_1(x) + g_2(x) \quad \text{odd} \]
\[ g_1(x) + f_1(x) \quad \text{neither} \]

(c) Prove that if \( f(x) \) is an even function, then its derivative \( f'(x) \) is odd.

\[ f \text{ even } \Rightarrow f(-x) = f(x) \]
\[ \Rightarrow f'(x) = -f'(-x) \quad \text{(Chain rule!)} \]
\[ \Rightarrow f' \text{ is odd} \]
Question 4

Suppose that the $f(t)$ is the function

$$f(t) = t^4 \quad -\pi < t < \pi$$
$$f(t + 2\pi) = f(t)$$

Given that the Fourier series for $f$ is

$$f(t) = \sum_{n=1}^{\infty} (-1)^n \left( 4\pi^2 \frac{1}{n^2} - \frac{48}{n^4} \right) \cos(nt) - \sum_{n=1}^{\infty} (-1)^n \left( \frac{4\pi^2}{n^2} - \frac{48}{n^4} \right) \cos(nt),$$

find the Fourier series for $g(t)$ where

$$g(t) = t^3 \quad -\pi < t < \pi$$
$$g(t + 2\pi) = g(t)$$

Since $\boxed{g(t) = \frac{1}{4} f'(t)}$, we see that

$$g(t) = \frac{1}{4} \left\{ \sum_{n=1}^{\infty} (-1)^n \left( 4\pi^2 \frac{1}{n^2} - \frac{48}{n^4} \right) (-n) \sin(nt) \right\} - 0$$

$$= \sum_{n^2=1}^{8} (-1)^{n+1} \left( \frac{\pi^2}{n} - \frac{12}{n^3} \right) \sin(nt)$$
Question 5

Consider the boundary value problem for the function $u(x,t)$:

$$
5u_t = u_{xx} \quad 0 < x < 10 \ ; \ t > 0
$$
$$
u(0,t) = u(10,t) = 0 \quad \text{(endpoint conditions)}
$$
$$
u(x,0) = \sin\left(\frac{\pi x}{10}\right) - 16 \sin(\pi x) \quad \text{(initial condition)}
$$

(a) Suppose that $u(x,t) = X(x)T(t)$, i.e. suppose that the variables can be separated. Re-write the differential equation as a pair of equations, one for the function $X(x)$ and one for the function $T(t)$.

$$
\begin{align*}
U_t(x,t) &= X \dot{T} \\
U_{xx}(x,t) &= X'' \ T
\end{align*}
$$

Hence the equation becomes

$$
5 \cdot \dot{X} = X'' \ T
$$

i.e.

$$
\frac{X''}{X} = \frac{\dot{X}}{T}
$$

Hence we must have

$$
\frac{X''}{X} = \frac{\dot{X}}{T} = \lambda, \ \text{i.e.}
$$

$$
\begin{cases}
X'' - \lambda X = 0 \\
\dot{T} - \frac{\lambda}{5} T = 0
\end{cases}
$$

8.
(b) Find all solutions to the equations which satisfy the endpoint conditions.
(You may assume that the eigenvalues are positive)

\[ g \phi - \lambda \phi = 0 \text{ then } \]
\[ \phi(x) = A \cosh \alpha x + B \sin \alpha x \]

Also \( u(0,t) = x(0)T(t) = 0 \Rightarrow x(0) = 0 \)

i.e. \( 0 = A \), so \( x(x) = B \sin \alpha x \).

Then \( u(0,t) = x(0)T(t) = 0 \Rightarrow x(0) = 0 \)

so \( B \sin 10x = 0 \)

Hence \( \alpha = \frac{n \pi}{10} \), i.e. \( \lambda_n = -\frac{n^2 \pi^2}{100} \)

When \( \lambda_n = -\frac{n^2 \pi^2}{100} \):

\[ X_n(x) = \sin \frac{n \pi}{10} x \]

Also \( \hat{T} - \frac{n^2 \pi^2}{500} T = 0 \Rightarrow \hat{T} + \frac{n^2 \pi^2}{500} T = 0 \)

\[ \Rightarrow T(t) = T_0 e^{-\frac{n^2 \pi^2}{500} t} \]

Take \( T_n(t) = e^{-\frac{n^2 \pi^2}{500} t} \)

Hence the solutions are

\[ u_n(x,t) = \sin \frac{n \pi}{10} x e^{-\frac{n^2 \pi^2}{500} t} \]
(c) Find the solution which satisfies both the endpoint conditions and the initial condition.

\[
T(x,t) = \sum_{n=1}^{\infty} a_n U_n(x,t), \quad \text{with}
\]

\[
a_n \text{ chosen so that}
\]

\[
U(x,0) = \sin \frac{\pi x}{10} - 16 \sin \pi x, \quad \text{i.e. so that}
\]

\[
\sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{10} = \sin \frac{\pi x}{10} - 16 \sin \frac{10 \pi x}{10} \quad \text{already a Fourier \sin series!}
\]

Hence we need
\[
a_1 = 1
\]
\[
a_{10} = -16
\]
\[
a_n = 0 \quad \forall \ n \neq 1 \text{ or } 10
\]

\[
U(x,t) = \sin \frac{\pi x}{10} e^{-\frac{\pi^2}{500} t} - 16 \sin \pi x e^{-\frac{\pi^2}{5} t}
\]