Tuesday, October 25  ** Multiple integrals & Polar coordinates

1. The function $P(x) = e^{-x^2}$ is fundamental in probability.

   (a) Sketch the graph of $P(x)$. Explain why it is called a “bell curve.”

   (b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2} \, dx$ using the following brilliant strategy of Gauss.

      i. Instead of computing $I$, compute $I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right)$.

      ii. Rewrite $I^2$ as an integral of the form $\int \int_{R} f(x, y) \, dA$ where $R$ is the entire Cartesian plane.

      iii. Convert that integral to polar coordinates.

      iv. Evaluate to find $I^2$. Deduce the value of $I$.

   Amazingly, it can be mathematically proven that there is NO elementary function $Q(x)$ (that is, function built up from sines, cosines, exponentials, and roots using “usual” operations) for which $Q'(x) = P(x)$.

2. Let $E$ be the polar triangle $E = \{ (r, \theta) \mid 0 \leq r \leq \pi/2, 0 \leq \theta \leq r \}$.

   (a) Sketch $E$ and compute its area.

   (b) Let $D$ be the region in the cartesian plane corresponding to $E$. Sketch $D$ and find its area.

3. We have discussed the fact that the area of a disc of radius $r$ is $\pi r^2$ and that the volume of a sphere of radius $r$ is $\frac{4}{3} \pi r^3$.

   (a) Use a quadruple integral to find the volume of the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$

      of radius $r$ in $\mathbb{R}^4$. You may wish to use either of the following integration formulas:

      $\int \cos^4 \theta \, d\theta = \frac{1}{16} \left[ 4 \cos^3 \theta \sin \theta + 6 \theta + 3 \sin 2\theta \right]$,  

      or $\int \sin^4 \theta \, d\theta = \frac{1}{16} \left[ -4 \sin^3 \theta \cos \theta + 6 \theta - 3 \sin 2\theta \right]$.

   (b) Use an iterated integral to find the volume of the hypersphere of radius $r$ in $\mathbb{R}^n$ to be

      $V_n = \begin{cases} 
      \frac{2^{(n+1)/2}}{3 \cdot 5 \cdots \cdot n} \pi^{(n-1)/2} r^n, & n \text{ odd} \\
      \frac{2^{n/2}}{2 \cdot 4 \cdots \cdot n} \pi^{n/2} r^n, & n \text{ even}. 
      \end{cases}$

      You may wish to use the reduction formula

      $\int \cos^n \theta \, d\theta = \frac{1}{n} \cos^{n-1} \theta \sin \theta + \frac{n-1}{n} \int \cos^{n-2} \theta \, d\theta$

      or $\int \sin^n \theta \, d\theta = -\frac{1}{n} \sin^{n-1} \theta \cos \theta + \frac{n-1}{n} \int \sin^{n-2} \theta \, d\theta$. 