Thursday, September 8  **  Functions of several variables.

1. Consider the function

\[ f(x, y) = \frac{2xy}{x^2 + y^2}. \]

(a) What does this function look like along a line \( y = mx \)?
(b) Sketch the graph of \( f(x, y) \).

2. Consider the function

\[ f(x, y) = xy. \]

(a) Sketch the level sets of \( f \).
(b) Sketch the graph of \( f(x, y) \). What is the name of this surface?

3. Let \( f(x, y) = 3x + 5y - 1 \). This problems deals with

\[ \lim_{(x,y) \to (1,1)} 3x + 5y - 1. \]

(a) Let \( \varepsilon = 1 \). Find a \( \delta > 0 \) such that if \( \|(x, y) - (1, 1)\| < \delta \), then \( |f(x, y) - 7| < \varepsilon \).
(b) Now find a \( \delta > 0 \) for arbitrary \( \varepsilon \) (your answer should be in terms of \( \varepsilon \)).

4. In class, we showed that

\[ \lim_{(x,y) \to (1,0)} \frac{x}{y} \]

does not exist, by approaching the point \((1,0)\) along different lines. This can also be shown directly from the \( \varepsilon, \delta \) definition. To do this, for each possible real number \( L \), you must show that the limit cannot be \( L \).

(a) Let \( L \) be any real number. For the value \( \varepsilon = 1 \), show that no matter which \( \delta > 0 \) is chosen, there is always a point \((x, y)\) such that \( \|(x, y) - (1, 0)\| < \delta \) but \( \left| \frac{x}{y} - L \right| \geq 1 \). This shows that the limit is not \( L \).
    **(Hint: Take any value for \( x \) in the interval \((1 - \delta, 1 + \delta)\). Show that there is a value for \( y \) that makes the above inequalities true.)**
(b) More generally, show that for any \( \varepsilon > 0 \), no good \( \delta \) can be found.