**Cross products and quadrics in \( \mathbb{R}^3 \).**

1. (a) Show that if \( \mathbf{u} \) is any vector then \( \mathbf{u} \times \mathbf{u} = 0 \).

(b) If \( \mathbf{u} \) and \( \mathbf{v} \) are any two nonzero vectors such that \( \mathbf{u} \times \mathbf{v} = 0 \), what can you say about the vectors \( \mathbf{u} \) and \( \mathbf{v} \)?

(c) Let \( \mathbf{u} \) and \( \mathbf{v} \) be nonzero vectors that are not parallel to each other. Show that the vector

\[
\mathbf{u} \times (\mathbf{u} \times \mathbf{v})
\]

can never be zero.

2. Suppose that \( \mathbf{v} \cdot \mathbf{w} = 0 \). Find an expression for \( \| \mathbf{v} \times \mathbf{w} \| \) (in terms of \( \mathbf{v} \) and \( \mathbf{w} \)).

3. (Volume of Prisms) Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors. Consider the prism (parallelepiped) with vertex at the origin and with sides given by the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \). The formula for the volume of a prism, like that of a cylinder, is

\[
\text{volume} = \text{base} \cdot \text{height}
\]

(a) Consider the face containing \( \mathbf{u} \) and \( \mathbf{v} \) as the “base”, give the formula for the area of the base.

(b) Find a formula for the height. This formula should be expressed in terms of \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) and the angle \( \theta \) between \( \mathbf{w} \) and the plane containing \( \mathbf{u} \) and \( \mathbf{v} \).

(c) Put your answers together to arrive at a formula for the volume of the prism.

4. (A quadric surface in nonstandard form) Consider the surface described by the equation

\[4x^2 - 4xy + 4y^2 - 10x + 2y - 2z + 9 = 0.\]

(a) Introduce new variables \( u = x + y \) and \( v = x - y \). Solve for \( x \) and \( y \) in terms of \( u \) and \( v \).

(b) Substitute your answer above into the original equation to get a new equation in terms of \( u, v, \) and \( z \).

(c) Complete the square in both \( u \) and \( v \) to in order to arrive at a quadric equation in “standard form”.

(d) What type of surface is this? If you don’t remember the classification of quadric surfaces, try drawing the cross-sections with the standard coordinate planes.