SOLUTION TO WORKSHEET FOR TUESDAY, SEPTEMBER 13, 2001

1. Let \( f(x, y) = y^2 \ln(x^3 + 1) + \sqrt{y} \). Find the partial derivatives

\[ f_x, \quad D_2f, \quad \frac{\partial^2 f}{\partial x^2}, \quad D_1D_2f, \quad f_{yy}, \quad \frac{\partial^2 f}{\partial y \partial x}. \]

**SOLUTION:**

\[
f_x = \frac{3x^2y^2}{x^3+1} \quad D_2f = 2y \log(x^3 + 1) + \frac{1}{2\sqrt{y}} \quad \frac{\partial^2 f}{\partial x^2} = \frac{6xy^2}{x^3+1} - \frac{9x^4y^2}{(x^3 + 1)^2} \]

\[
D_1D_2f = \frac{6x^2y}{x^3+1} \quad f_{yy} = 2 \log(x^3 + 1) - \frac{1}{4y^{3/2}} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{6x^2y}{x^3+1}
\]

2. (Harmonic functions) The partial differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0
\]

is called the **Laplace equation**. Any function \( u(x, y, z) \) satisfying the Laplace equation is called a **harmonic function**.

(a) Let \( u(x, y) = e^{ax} \cos(by) \). Find \( u_{xx} \) and \( u_{yy} \). What must be true of \( a \) and \( b \) in order for \( u \) to be harmonic of two variables?

**SOLUTION:**

\[
u_{xx} = a^2e^{ax} \cos(by) \quad \text{and} \quad u_{yy} = -b^2e^{ax} \cos(by).
\]

\( u_{xx} + u_{yy} = 0 \) if and only if \( a^2 = b^2 \) or \(|a| = |b|\).

(b) Show that \( u(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \) is a harmonic function of three variables.

**SOLUTION:**

\[
u_{xx} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{3/2}} \quad u_{yy} = \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{3/2}} \quad u_{zz} = \frac{-x^2+y^2+2z^2}{(x^2+y^2+z^2)^{3/2}}
\]

From this we can see that \( u_{xx} + u_{yy} + u_{zz} = 0 \) and \( u \) is harmonic.

3. (Counterexample to Clairaut) Let

\[
f(x) = \begin{cases} x y \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}
\]

(a) Find \( f_x(0, y) \).

(b) Find \( f_y(x, 0) \).

(c) Show that \( f_{xy}(0, 0) \neq f_{yx}(0, 0) \).
SOLUTION:

First, \( f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \). Similarly we see that \( f_y(0, 0) = 0 \).

When \((x, y) \neq (0, 0)\),

\[
f_x = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}
\]

and

\[
f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}
\]

Substituting, we see that \( f_x(0, y) = -y \) and \( f_y(x, 0) = x \). Since we computed that \( f_x(0, 0) = f_y(0, 0) = 0 \), these equations are also valid when \( x = y = 0 \). Now

\[
f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_x(0, h) - f_x(0, 0)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1
\]

and

\[
f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1
\]

So \( f_{xy}(0, 0) \neq f_{yx}(0, 0) \).

4. The wind-chill index \( W = f(T, v) \) is the perceived temperature when the actual temperature is \( T \) and the wind speed is \( v \). Here is a table of values for \( W \).

<table>
<thead>
<tr>
<th>Actual temperature (°C)</th>
<th>Wind speed (km/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>−10</td>
<td>−18</td>
</tr>
<tr>
<td>−15</td>
<td>−24</td>
</tr>
<tr>
<td>−20</td>
<td>−30</td>
</tr>
<tr>
<td>−25</td>
<td>−37</td>
</tr>
</tbody>
</table>

(a) Use the table to estimate \( \frac{\partial f}{\partial T} \) and \( \frac{\partial f}{\partial v} \) at \((T, v) = (-20, 40)\).

SOLUTION:

\[
\frac{\partial f}{\partial T} \big|_{(-20, 40)} \approx \frac{f(-15, 40) - f(-20, 40)}{-15 + 20} = 7/5 \text{ and } \frac{\partial f}{\partial v} \big|_{(-20, 40)} \approx \frac{f(-20, 50) - f(-20, 40)}{50 - 40} = -1/10.
\]

(b) Use your answer in (a) to write down the linear approximation to \( f \) at \((-20, 40)\).

SOLUTION:

\[
L = f(-20, 40) + \frac{\partial f}{\partial T} \big|_{(-20, 40)}(T + 20) + \frac{\partial f}{\partial v} \big|_{(-20, 40)}(v - 40)
\]

\[
\approx -34 + 7/5(T + 20) - 1/10(v - 40)
\]

(c) Use your answer in (b) to approximate \( f(-22, 45) \).

SOLUTION:

\[
f(-22, 45) \approx -34 + 7/5(-2) - 1/10(5) = -373/10 = -37.3
\]