Last Time Limits for $f: \mathbb{R}^n \to \mathbb{R}$.

To understand $\lim_{x \to \bar{c}} f(x)$, approach $\bar{c}$ along lines or curves.

1) Some of these are different, then

$$\lim_{x \to \bar{c}} f(x) \text{ DNE}$$

OR 2) All of these limits agree, then

probably $\lim_{x \to \bar{c}} f(x)$ is same.

Example: Find $\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$.

Along any line, find limit of 0

Along line $y = mx$, get

$$\lim_{x \to 0} \frac{mx^2}{1x^2(1+m^2)} = \lim_{x \to 0} \frac{m}{1+m^2} \quad |x| = 0$$

Also along parabola. How to show it is 0?

Use the Squeeze Theorem:

If $f \leq g \leq h$ and $\lim_{(x,y) \to (a,b)} f(x,y) = L$,

then $\lim_{(x,y) \to (a,b)} g(x,y) = L \text{ too.}$
Enough (by Squeeze Thm) to show

\[
\lim_{(x,y) \to (0,0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = 0.
\]

But \(1/|y| = \frac{1}{\sqrt{y^2}} \leq \frac{1}{\sqrt{x^2 + y^2}}\), so

\[
0 \leq 1/x \cdot \frac{1/|y|}{\sqrt{x^2 + y^2}} \leq 1/x.
\]

Since \(\lim_{(x,y) \to (0,0)} 1/x = 0\), done by Squeeze Theorem.

Now understand limits, so can discuss continuity.

**Definition** \(f : \mathbb{R}^n \to \mathbb{R}\) is **continuous** at \(x_0\) if

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

**Examples**
- polynomial functions \(f(x) = p(x)\) (continuous everywhere)
- rational functions
  \[
  g(x) = \frac{p(x)}{q(x)}
  \]
  continuous wherever \(q(x) \neq 0\).
- \(f(x,y) = \frac{xy}{\sqrt{x^2 + y^2}}\)
  
  Easy to see continuous away from origin.

At \((x,y) = (0,0)\), \(f\) is not defined.

Define \(g(x,y) = \left\{ \begin{array}{ll} \frac{xy}{\sqrt{x^2 + y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{array} \right.\)

Then \(g\) is continuous everywhere since

\[
\lim_{(x,y) \to (0,0)} g(x,y) = \lim_{(x,y) \to (0,0)} f = 0 \quad \text{(above example)}
\]
6.14.3 (Partial) Derivatives

For \( f: \mathbb{R} \rightarrow \mathbb{R} \), \( f'(c) = \frac{df}{dx}(c) \) measures rate of change of \( f(x) \) at \( x = c \). (Change as \( x \rightarrow c \)).

For \( f: \mathbb{R}^n \rightarrow \mathbb{R} \), \( f \) may change differently depending on direction of approach.

Example: \( f(x,y) = \frac{x}{y} \)

At \((1,1)\), \( f \) increasing as \( x \) increases, decreasing as \( y \) increases.

Rate of change in \( x \) direction is partial derivative with respect to \( x \), written \( \frac{df}{dx}(x,y) \) or \( f'_x(x,y) \).

Definition: Let \( g(x) = f(x,d) \) (for fixed \( d \)).

Then \( \frac{df}{dx}(c,d) = g'(c) \).

Similarly, let \( h(y) = f(c,y) \) (fixed \( c \)).

Then \( \frac{df}{dy}(c,d) = h'(d) \).

Example: \( f(x,y) = \frac{x}{y} \), at \((1,1)\).

\( \frac{df}{dx}(1,1) = \frac{d}{dx}\left(\frac{x}{y}\right) \bigg|_{x=1} = 1 \).

\( \frac{df}{dy}(1,1) = \frac{d}{dy}\left(\frac{x}{y}\right) \bigg|_{y=1} = \left(\frac{-1}{y^2}\right) \bigg|_{y=1} = -1 \).

Expanded definition:

Recall that \( g'(c) \) means \( \lim_{h \to 0} \frac{g(c+h) - g(c)}{h} \)

or \( \lim_{x \to c} \frac{g(x) - g(c)}{x-c} \)
So really \[ \frac{\partial f}{\partial x}(c,d) = \lim_{h \to 0} \frac{f(c+h,d) - f(c,d)}{h} \]

or \[ = \lim_{x \to c} \frac{f(x,d) - f(c,d)}{x - c} \]

and \[ \frac{\partial f}{\partial y}(c,d) = \lim_{h \to 0} \frac{f(c,d+h) - f(c,d)}{h} \]

\[= \lim_{y \to d} \frac{f(c,y) - f(c,d)}{y - d} \]

For \[ f(x,y) = \frac{x}{y}, \]

\[ f_x(c,d) = \lim_{h \to 0} \frac{\frac{c+h}{d} - \frac{c}{d}}{h} = \lim_{h \to 0} \frac{h}{d \cdot h} = \frac{1}{d}. \]

and \[ f_y(c,d) = \lim_{h \to 0} \frac{\frac{c}{dh} - \frac{c}{d}}{h} = \lim_{h \to 0} \frac{c \cdot d - c \cdot (d+hn)}{(d+hn) \cdot d} \]

\[= \lim_{h \to 0} \frac{-c \cdot h}{(d+hn) \cdot d} = \lim_{h \to 0} \frac{-c}{(d+hn) d} = \frac{-c}{d^2}. \]

Even more notation:

\[ \frac{\partial f}{\partial x}(c,d) = f_x(c,d) = D_x f(c,d) \]

and \[ \frac{\partial f}{\partial y}(c,d) = f_y(c,d) = D_y f(c,d). \]

For more than 2 variables, all works the same way.

**Ex.** \[ f(x_1, y_1, z_1, w) = 2x_1^3y_1w - 6y_1^2z_1w + z_1^2w^2 \]

\[ f_1 = \frac{\partial f}{\partial x} = 6x_1^2y_1w \]

\[ f_2 = \frac{\partial f}{\partial y} = 2x_1^3w - 12y_1^2zw \]

\[ f_3 = \frac{\partial f}{\partial z} = 6y_1^2w + 2z_1zw^2 \]
\[ f_y = \frac{df}{dy} = 2x^3 y - 6y^2z + 2z^2w \]

Higher derivatives \( \frac{df}{dx}, \frac{df}{dy} \) etc. are still functions \( \mathbb{R}^n \rightarrow \mathbb{R} \), so can consider their partial derivatives.

**Ex.** Same as above

\[ f_{yx} = D_z D_x f = 6x^2w \quad \text{same} \]
\[ f_{xx} = D_x D_x f = 12xyw \]
\[ f_{xy} = D_x D_y f = 6x^2w \quad \text{same} \]
\[ f_{yy} = D_y D_y f = 6x^2y \]
\[ f_{xx} = D_x D_x f = 6x^2y \]

In fact, for any \( f \), if \( f_{xy} \) and \( f_{yx} \) are both continuous, then \[ f_{xy} = f_{yx}. \] (This is Clairaut's Theorem)

Alternative notation for higher partial derivatives:

\[ f_{x,y} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \]

\[ f_{x,x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} \quad \text{Careful: This is just notation. Do not try to interpret it literally.} \]