Last time: The Chain Rule.

For \( F: \mathbb{R}^3 \to \mathbb{R}, \quad g, h, k: \mathbb{R}^2 \to \mathbb{R}, \)

\[ F(s, t) = F(g(s, t), h(s, t), k(s, t)) \]

Then

\[ \frac{\partial F}{\partial s} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial s}. \]

Book notation: \( x(s, t), y(s, t), z(s, t) \).

Chain Rule:

\[ \frac{\partial F}{\partial s} (= \frac{\partial z}{\partial s}) = \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} + \frac{\partial F}{\partial z} \frac{dz}{ds} \]

Example

\[ f(x, y, z) = x^2 + yz \]

\[ x(s, t) = s, \quad y(s, t) = t, \quad z(s, t) = \sqrt{1-s^2-t^2} \]

What are \( \frac{\partial F}{\partial s} (0, 0), \frac{\partial F}{\partial t} (0, 0) ? \)

By Chain Rule,

\[ \frac{\partial F}{\partial s} = \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} + \frac{\partial F}{\partial z} \frac{dz}{ds} \]

\[ = z \cdot 1 + \sqrt{1-s^2-t^2} \cdot 0 + t \cdot \frac{-3}{\sqrt{1-s^2-t^2}} \]

\[ \frac{\partial F}{\partial s} (0, 0) = 0 \]

\[ \frac{\partial F}{\partial t} = z \cdot 0 + \sqrt{1-s^2-t^2} \cdot 1 + t \cdot \frac{-t}{\sqrt{1-s^2-t^2}} \]

\[ \frac{\partial F}{\partial t} (0, 0) = -1 \]
6.4.6 **Directional Derivatives**

Same idea as $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Measure rate of change along a given direction $\vec{u}$.

If $\vec{u} = (u_1, u_2)$ then

$$D_{\vec{u}} f(\vec{c}) = \lim_{h \to 0} \frac{f(\vec{c} + h \vec{u}) - f(\vec{c})}{h} = \lim_{h \to 0} \frac{f(c_1 + hu_1, c_2 + hu_2) - f(c_1, c_2)}{h}$$

**Note:** $D_{\vec{u}} f = f_x \cdot u_1 + f_y \cdot u_2$.

**Example:** $f(x, y) = x - y$.

Then $f_x(x, y) = 1$, $f_y(x, y) = -1$,

$$D_{(1, 0)} f(x, y) = \lim_{h \to 0} [ (x + h) - (y + h) ] - (x - y) = 0$$

$$D_{(1, -1)} f(x, y) = \lim_{h \to 0} [ (x + h) - (y - h) ] - (x - y) = \lim_{h \to 0} \frac{2h}{h} = 2$$

How to calculate in general?

Related to $f_x$ & $f_y$ by

**Theorem:** If $f$ is differentiable at $\vec{c}$, then

$$D_{\vec{u}} f(\vec{c}) = f_x(\vec{c}) u_1 + f_y(\vec{c}) u_2$$

**Why?** Intersect tangent plane

$$z = f(\vec{c}) + f_x(\vec{c})(x - c_1) + f_y(\vec{c})(y - c_2)$$

with parameterized line $(c_1, c_2, f(\vec{c})) + t (a_1, a_2, D_{\vec{u}} f(\vec{c}))$
\( x = c_1 + t u_1, \quad y = c_2 + t u_2, \quad z = f(c) + t \left[ D_u f (c) \right] \)  

get

\[
\frac{\partial f (c) + t D_u f (c)}{\partial t} = \frac{\partial f (c)}{\partial x} + \frac{\partial f (c)}{\partial y} (c y_1) + \frac{\partial f (c)}{\partial z} (c y_2) = \left[ \frac{\partial f (c)}{\partial x} u_1 + \frac{\partial f (c)}{\partial y} u_2 \right] + \epsilon
\]

So

\[
D_u f (c) = \frac{\partial f (c)}{\partial x} u_1 + \frac{\partial f (c)}{\partial y} u_2.
\]

Formula can also be written

\[
D_u f (c) = (\frac{\partial f (c)}{\partial x}, \frac{\partial f (c)}{\partial y}) \cdot \mathbf{u}
\]

**Definition** For \( f : \mathbb{R}^2 \to \mathbb{R} \) differentiable, define

\[
\nabla f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{(gradient of \( f \))}
\]

\[
\nabla f (x, y) = \left( \frac{\partial f}{\partial x} (x, y), \frac{\partial f}{\partial y} (x, y) \right) = \left( \frac{\partial f}{\partial x} \right)_x + \left( \frac{\partial f}{\partial y} \right)_y.
\]

Similarly, for \( f : \mathbb{R}^n \to \mathbb{R}, \ \nabla f : \mathbb{R}^n \to \mathbb{R}^n, \)

\[
\nabla f (x) = \left( \frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right).
\]

**Example** \( f (x, y) = x - y \).

\[
\nabla f (x, y) = (1, -1)
\]

**Example** \( f (x, y) = xy \).

\[
\nabla f (x, y) = (y, x)
\]
Geometric meaning

1. \( |D_{\hat{u}} f| = \| \nabla f \| \cdot \| \hat{u} \| \cos \theta \) (usually take \( \hat{u} \) a unit vector)

largest when \( \nabla f \) parallel to \( \hat{u} \).

Take \( \hat{u} \) unit vector in direction \( \nabla f \) (of \( f \)).

Then \( D_{\hat{u}} f = \| \nabla f \| \)

Interpretation: \( \nabla f(\hat{u}) \) is direction of maximal increase of \( f \) at \( \hat{u} \).

Example: \( f(x, y) = xy \) graph is

\( \nabla f = (y, x) \) hyperbolic paraboloid

At \((1, 1)\), \( \nabla f(1, 1) = (1, 1) \).

Parabola opening up along line \( y = x \).

At \((1, -1)\), \( \nabla f(1, -1) = (-1, 1) \)

Parabola opening down along \( y = -x \).

Steepest ascent back to vertex.