Section 16.3 Solutions

1. \( C \) appears to be a smooth curve, and since \( \nabla f \) is continuous, we know \( f \) is differentiable. Then Theorem 2 says that the value of \( \int_C \nabla f \cdot dr \) is simply the difference of the values of \( f \) at the terminal and initial points of \( C \). From the graph, this is \( 50 - 10 = 40 \).

2. \( C \) is represented by the vector function \( \mathbf{r}(t) = (t^3 + 1) \mathbf{i} + (t^3 + t) \mathbf{j}, 0 \leq t \leq 1 \), so \( \mathbf{r}'(t) = 3t^2 \mathbf{i} + (3t^2 + 1) \mathbf{j} \). Since \( 3t^2 + 1 \neq 0 \), we have \( \mathbf{r}'(t) \neq 0 \), thus \( C \) is a smooth curve. \( \nabla f \) is continuous, and hence \( f \) is differentiable, so by Theorem 2 we have \( \int_C \nabla f \cdot dr = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6 \).

3. \( \partial(2x - 3y)/\partial y = -3 = \partial(-3x + 4y - 8)/\partial x \) and the domain of \( \mathbf{F} \) is \( \mathbb{R}^2 \) which is open and simply connected, so by Theorem 6 \( \mathbf{F} \) is conservative. Thus, there exists a function \( f \) such that \( \nabla f = \mathbf{F} \), that is, \( f_x(x, y) = 2x - 3y \) and \( f_y(x, y) = -3x + 4y - 8 \). But \( f_x(x, y) = 2x - 3y \) implies \( f(x, y) = x^2 - 3xy + g(y) \) and differentiating both sides of this equation with respect to \( y \) gives \( f_y(x, y) = -3x + g'(y) \). Thus \( -3x + 4y - 8 = -3x + g'(y) \) so \( g'(y) = 4y - 8 \) and \( g(y) = 2y^2 - 8y + K \) where \( K \) is a constant. Hence \( f(x, y) = x^2 - 3xy + 2y^2 - 8y + K \) is a potential function for \( \mathbf{F} \).

4. \( \partial(e^x \cos y)/\partial y = -e^x \sin y \), \( \partial(e^x \sin y)/\partial x = e^x \sin y \). Since these are not equal, \( \mathbf{F} \) is not conservative.

5. \( \partial(xy \cos xy + \sin xy)/\partial y = -x^2 y \sin xy + 2x \cos xy = \partial(xy \cos xy)/\partial x \) and the domain of \( \mathbf{F} \) is \( \mathbb{R}^2 \). Hence \( \mathbf{F} \) is conservative, so there exists a function \( f \) such that \( \nabla f = \mathbf{F} \). Then \( f_x(x, y) = x^2 \cos xy \) implies \( f(x, y) = x \sin xy + g(x) \) and \( f_y(x, y) = xy \cos xy + \sin xy + g'(x) \). But \( f_x(x, y) = xy \cos xy + \sin xy \) so \( g(x) = K \) and \( f(x, y) = x \sin xy + K \) is a potential function for \( \mathbf{F} \).

6. (a) \( f_x(x, y) = x^2 \) implies \( f(x, y) = \frac{1}{2}x^3 + g(y) \) and \( f_y(x, y) = 0 + g'(y) \). But \( f_y(x, y) = y^2 \) so \( g'(y) = y^2 \) \( \Rightarrow \) \( g(y) = \frac{1}{2}y^3 + K \). We can take \( K = 0 \), so \( f(x, y) = \frac{1}{2}x^3 + \frac{1}{2}y^3 \).

(b) \( \int_C \mathbf{F} \cdot dr = f(2, 8) - f(-1, 2) = \left( \frac{3}{2} + \frac{32}{2} \right) - \left( -\frac{1}{2} + \frac{8}{2} \right) = 171 \).

7. (a) \( f_x(x, y, z) = yz \) implies \( f(x, y, z) = xyz + g(y, z) \) and so \( f_y(x, y, z) = xz + g_y(y, z) \). But \( f_y(x, y, z) = xz \) so \( g_y(y, z) = 0 \) \( \Rightarrow \) \( g(y, z) = h(z) \). Thus \( f(x, y, z) = xyz + h(z) \) and \( f_z(x, y, z) = xy + h'(z) \). But \( \int_C \mathbf{F} \cdot dr = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77 \).

(b) \( \int_C \mathbf{F} \cdot dr = f(\pi, 0, \pi) - f(0, 0, 0) = 0 - 0 = 0 \).
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18. (a) \( f_x(x, y, z) = e^y \) implies \( f(x, y, z) = xe^y + g(y, z) \) and so \( f_y(x, y, z) = xe^y + g_y(y, z) \). But \( f_x(x, y, z) = xe^y \) so \( g_y(y, z) = 0 \) \( \Rightarrow \) \( g(y, z) = h(z) \). Thus \( f(x, y, z) = xe^y + h(z) \) and \( f_x(x, y, z) = 0 + h'(z) \). But 
\[ f_z(x, y, z) = (z + 1)e^z, \]
so \( h'(z) = (z + 1)e^z \) \( \Rightarrow \) \( h(z) = ze^z + K \) (using integration by parts). Hence 
\[ f(x, y, z) = xe^y + xe^z \] (taking \( K = 0 \)).

(b) \( r(0) = (0,0,0), r(1) = (1,1,1) \) so \( \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 2e - 0 = 2e. \)

21. \( \mathbf{F}(x, y) = 2y^{3/2} \mathbf{i} + 3x \sqrt{y} \mathbf{j} \), \( W = \int_C \mathbf{F} \cdot d\mathbf{r} \). Since \( \partial (2y^{3/2})/\partial y = 3 \sqrt{y} = \partial (3x \sqrt{y})/\partial x \), there exists a function \( f \) such that \( \nabla f = \mathbf{F} \). In fact, \( f_x(x, y) = 2y^{3/2} \Rightarrow f(x, y) = 2xy^{3/2} + g(y) \Rightarrow f_y(x, y) = 3xy^{1/2} + g'(y) \). But 
\[ f_y(x, y) = 3x \sqrt{y} \] so \( g'(y) = 0 \) or \( g(y) = K \). We can take \( K = 0 \) \( \Rightarrow \) \( f(x, y) = 2xy^{3/2} \). Thus 
\[ W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,4) - f(1,1) = 2(2)(8) - 2(1) = 30. \]

22. \( \mathbf{F}(x, y) = e^{-y} \mathbf{i} - xe^{-y} \mathbf{j}, W = \int_C \mathbf{F} \cdot d\mathbf{r} \). Since \( \frac{\partial}{\partial y} (e^{-y}) = -e^{-y} = \frac{\partial}{\partial x} (-xe^{-y}) \), there exists a function \( f \) such that 
\[ \nabla f = \mathbf{F}. \] In fact, \( f_x = e^{-y} \Rightarrow f(x, y) = xe^{-y} + g(y) \Rightarrow f_y = -xe^{-y} + g'(y) \Rightarrow g'(y) = 0 \), so we can take 
\[ f(x, y) = xe^{-y} \] as a potential function for \( \mathbf{F} \). Thus 
\[ W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,0) - f(0,1) = 2 - 0 = 2. \]

29. \( D = \{ (x, y) \mid x > 0, y > 0 \} \) is the first quadrant (excluding the axes).

(a) \( D \) is open because around every point in \( D \) we can put a disk that lies in \( D \).

(b) \( D \) is connected because the straight line segment joining any two points in \( D \) lies in \( D \).

(c) \( D \) is simply-connected because it's connected and has no holes.

30. \( D = \{ (x, y) \mid x \neq 0 \} \) consists of all points in the \( xy \)-plane except for those on the \( y \)-axis.

(a) \( D \) is open.

(b) Points on opposite sides of the \( y \)-axis cannot be joined by a path that lies in \( D \), so \( D \) is not connected.

(c) \( D \) is not simply-connected because it is not connected.

31. \( D = \{ (x, y) \mid 1 < x^2 + y^2 < 4 \} \) is the annular region between the circles with center \((0, 0)\) and radii 1 and 2.

(a) \( D \) is open.

(b) \( D \) is connected.

(c) \( D \) is not simply-connected. For example, \( x^2 + y^2 = (1.5)^2 \) is simple and closed and lies within \( D \) but encloses points that are not in \( D \). (Or we can say, \( D \) has a hole, so it is not simply-connected.)
32. \( D = \{ (x, y) \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9 \} = \) the points on or inside the circle \( x^2 + y^2 = 1 \), together with the points on or between the circles \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \).

(a) \( D \) is not open because, for instance, no disk with center \((0, 2)\) lies entirely within \( D \).

(b) \( D \) is not connected because, for example, \((0, 0)\) and \((0, 2.5)\) lie in \( D \) but cannot be joined by a path that lies entirely in \( D \).

(c) \( D \) is not simply-connected because, for example, \( x^2 + y^2 = 9 \) is a simple closed curve in \( D \) but encloses points that are not in \( D \).