MATH 181: SUMMARY 4

Euler’s Theorem

Read Euler, read Euler, he is the master of us all.

Pierre-Simon Laplace

Theorem. A connected graph has an Euler circuit if and only if the degree of every vertex is even.

Hierholzer’s algorithm is a way to find an Euler circuit (if it exists) without trial and error.

Step 1. Calculate the degrees to make sure the graph is Eulerian!

Step 2. Choose any starting vertex. Call it \( v \).

Step 3. Follow an arbitrary trail of edges from \( v \) until you return back to \( v \). This gives a tour that starts and ends at \( v \) (but it might not use all the edges of the graph).

Step 4. As long as there is a vertex \( u \) on the current tour that is incident to some unused edges, start another trail from \( u \), following unused edges until returning back to \( u \), and insert the tour formed in this way into the previous tour.

Remark: You have a lot of freedom in the above algorithm!

Another way of finding Euler circuits is using Fleury’s algorithm (see page 12 in the textbook*).

Historical note: Even though Euler stated the above Theorem in 1741, he did not give a complete proof. The first proof was published by Carl Hierholzer in 1873.

Further reading:

L. Euler (1741). Solutio problematis ad geometriam situs pertinentis, Commentarii academiae scientiarum Petropolitanae, 8, 128–140.

This is the original paper by Euler in which he solves the Seven Bridges of Königsberg problem. You can access it through the Euler Archive†, an extensive on-line collection of Euler’s works.


This book is a compilation of a large number of important mathematical papers, essays, etc., and I encourage you to take a look at it (you can find it at the library). Part IV, Chapter 4 presents an English translation of Euler’s paper.


This is an article about an application of Euler circuits in bioinformatics. If you are interested, you can read the introduction, which is written in an accessible manner. (You can access it through the library or at http://www.pnas.org/content/98/17/9748.long.)


In this article you can see some artwork that was made by traversing a single closed “loop,” akin to an Euler circuit in a graph. (The easiest way to access this article is through W.T. Ross’s website, at https://facultystaff.richmond.edu/~wross/PDF/Jordan-revised.pdf.)

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*Or the last paragraph on page 11 of the 9th edition.
†http://eulerarchive.maa.org/pages/E053.html
Example of using Hierholzer’s algorithm to find an Euler circuit:

This graph is connected and has all vertices of even degree, so it has an Euler circuit.

Start with any vertex, say, 1.

Follow a trail of edges starting at 1 until returning back. In this example, the resulting trail is 1–2–3–7–6–1.

2 is still incident to some unused edges (namely 2–7 and 2–6).

Follow a trail of edges starting at 2 until returning back. In this example, the resulting trail is 2–7–5–6–2.

Insert the new trail into the previous one, getting 1–2–7–5–6–2–3–7–6–1 (the inserted part is underlined).

3 is still incident to some unused edges (namely 3–4 and 3–5).

Follow a trail of edges starting at 3 until returning back. The resulting trail is 3–4–5–3.

Insert the new trail into the previous one, getting 1–2–7–5–6–2–3–4–5–3–7–6–1 (the inserted part is underlined).

Since there are no more unused edges, we have found an Euler circuit, namely 1–2–7–5–6–2–3–4–5–3–7–6–1.