Finding the continued fraction expansion

Let us compute the continued fraction expansion of $\frac{43}{30}$. We start by separating the integer part:

$$\frac{43}{30} = 1 + \frac{13}{30}.$$ 

Notice that the remainder, $\frac{13}{30}$, is less than 1. Our goal is to arrive at an expression where the numerator of every fraction is 1. Since $13 \neq 1$, we invert (or “flip”) the remainder:

$$\frac{43}{30} = 1 + \frac{13}{30} = 1 + \frac{1}{\frac{30}{13}}. \quad (*)$$

Now we apply the same operations to $\frac{30}{13}$:

$$\frac{30}{13} = 2 + \frac{4}{13} = 2 + \frac{1}{\frac{13}{4}}. \quad (†)$$

And again to $\frac{13}{4}$:

$$\frac{13}{4} = 3 + \frac{1}{4} \quad (‡)$$

The last remainder, $\frac{1}{4}$, is a fraction with numerator 1, so we stop here. To obtain the desired continued fraction, we trace the steps back: First, we plug the result of (‡) into (†):

$$\frac{30}{13} = 2 + \frac{1}{\frac{13}{4}} = 2 + \frac{1}{3 + \frac{1}{4}}.$$ 

Then we plug that into (*):

$$\frac{43}{30} = 1 + \frac{1}{\frac{30}{13}} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}. \quad (†)$$

The final answer is

$$\frac{43}{30} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}.$$ 

The terms of this continued fraction are 1, 2, 3, 4:

$$\frac{1}{1} + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}$$
Computing the convergents

To quickly find the convergents of a continued fraction, the following method can be used.

**Step 1.** Draw a table with three rows and several columns (you will need as many columns as many convergents you wish to find plus two).

**Step 2.** The first two columns are filled as shown below (the first two cells of the first row are empty):

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 3.** The first row is filled, starting with the third cell, with the terms of the continued fraction.

**Step 4.** The cells of the second and the third rows are filled one by one according to the following rule: *The value in each cell is the sum of*:

- *the value two cells to the left; and*
- *the value in the cell immediately to the left times the value in the top row.*

**Step 5.** The convergents are precisely the fractions formed by the elements of the second and the third rows taken from the same column.

**Example:** Let us calculate the convergents of

\[ 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}} \]

We start by setting up the table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To fill in the second row, we start with the leftmost empty cell (the relevant values in the table are circled):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Its value is

\[ 0 + (1 \cdot 1) = 1. \]

Then we look at the next cell:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ 1 + (1 \cdot 2) = 3. \]
And so on:

\[
\begin{array}{cccc}
0 & 1 & 1 & 3 \\
1 & 0 & & \\
\end{array}
\]

1 + (3 \cdot 3) = 10.

Thus, we have filled in the second row:

\[
\begin{array}{cccc}
0 & 1 & 1 & 3 & 10 & 43 \\
1 & 0 & & & & \\
\end{array}
\]

Now we do the same with the third row:

\[
\begin{array}{cccc}
0 & 1 & 1 & 3 & 10 & 43 \\
1 & 0 & & & & \\
\end{array}
\]

1 + (0 \cdot 1) = 1.

0 + (1 \cdot 2) = 2.

Finally, the table is complete:

\[
\begin{array}{cccc}
0 & 1 & 1 & 3 & 10 & 43 \\
1 & 0 & 1 & 2 & 7 & 30 \\
\end{array}
\]

We conclude that the convergents are

\[
\frac{1}{1} = 1, \quad \frac{3}{2}, \quad \frac{10}{7}, \quad \text{and} \quad \frac{43}{30}.
\]