Vocabulary:
- linear scale (the length of an interval represents the difference between the values);
- logarithmic scale (the length of an interval represents the ratio between the values).

The logarithmic scale is often useful, especially if the range of possible values is large. Common uses of the logarithmic scale include:
- earthquake strength (the Richter scale);
- sound loudness (measured in decibels);
- pH of solutions;
- etc. etc. etc.

The logarithmic scale can also be misleading, sometimes deliberately, if it is confused with the linear scale.

The logarithm to base $a$ of $b$, denoted $\log_a(b)$, is the power to which one needs to raise $a$ to get $b$. On a logarithmic scale, if multiplying by $a$ is represented by an interval of unit length, then multiplying by $b$ will be represented by an interval of length $\log_a(b)$.

**Examples.**

(a) We have:

\[
\begin{align*}
\log_2(8) &= 3, \text{ since } 8 = 2^3; \\
\log_3(9) &= 2, \text{ since } 9 = 3^2; \\
\log_5(5) &= 1, \text{ since } 5 = 5^1.
\end{align*}
\]

(b) Let us determine the value of $\log_4(8)$. Note that $4^1 = 4$, which is less than 8, while $4^2 = 16$, which is greater than 8. Therefore, $\log_4(8)$ is somewhere between 1 and 2. To locate it precisely, notice that

\[
4 = 2^2 = 2 \cdot 2 \quad \text{and} \quad 8 = 2^3 = 2 \cdot 2 \cdot 2.
\]

Imagine that multiplying by 4 is represented on a logarithmic scale by an interval of unit length. Since multiplying by 4 is the same as multiplying by 2 twice, the interval corresponding to 2 must be of length 1/2. But then multiplying by 8 is the same as multiplying by 2 three times, so the interval corresponding to 8 has length $3 \cdot 1/2 = 3/2$. We conclude that $\log_4(8) = 3/2$.

(c) Let us compute the value of $\log_{27}(9)$. First, we see that $27^0 = 1$ and $27^1 = 27$, so $\log_{27}(9)$ is somewhere between 0 and 1. Notice that

\[
27 = 3^3 = 3 \cdot 3 \cdot 3 \quad \text{and} \quad 9 = 3^2 = 3 \cdot 3.
\]

As in part (b), imagine that multiplying by 27 is represented on a logarithmic scale by a unit interval. Since multiplying by 27 is the same as multiplying by 3 three times, the interval corresponding to 3 must be of length 1/3. Multiplying by 9 is the same as multiplying by 3 twice, so the interval corresponding to 9 has length $2 \cdot 1/3 = 2/3$. We conclude that $\log_{27}(9) = 2/3$. 