We would like to devise a musical system with the following properties:

- the “steps” (i.e., the intervals) between consecutive notes are equal to each other;
- starting from any note, after some number of steps we reach the note that is a fifth (i.e., \(3/2\) times) higher and, at some later point, the note that is an octave (i.e., \(2\) times) higher.

We do not know yet what the interval between two consecutive notes in such a system would be, so let us denote it simply by \(x\). In other words, with each step going up, the frequency of a note is multiplied by \(x\). Then after two steps, the frequency is multiplied by \(x \cdot x = x^2\); after three steps—by \(x \cdot x \cdot x = x^3\); after four steps—by \(x \cdot x \cdot x \cdot x = x^4\); and so on. After any number \(n\) of steps, the frequency of the note becomes multiplied by \(x \cdot x \cdot \ldots \cdot x = x^n\).

If, starting from any note, the fifth is reached after \(n\) steps, then we must have
\[
x^n = \frac{3}{2}.
\]
Similarly, if the octave is reached in \(m\) steps, then
\[
x^m = 2.
\]
We have therefore reduced the original musical problem to a purely mathematical one; namely, we want to find a number \(x\) and two whole numbers \(n\) and \(m\) such that
\[
x^n = \frac{3}{2} \quad \text{and} \quad x^m = 2.
\]

**Theorem.** These conditions cannot be satisfied.

The conclusion is that it is actually impossible to build a musical system with equally spaced intervals between consecutive notes that includes perfect octaves and perfect fifths. Even more surprisingly, the reason boils down to the fact that an even and an odd number cannot be equal to each other!

**Proof.** Suppose that \(x\), \(n\), and \(m\) are as desired. What happens if we go up by \(n + m\) steps? The outcome is the same as if we first went up by \(n\) steps, and then by \(m\) more steps. Thus,
\[
x^{n+m} = x^n \cdot x^m = \frac{3}{2} \cdot 2 = 3. \quad (\ast)
\]
Now, what if we take \(m(n + m)\) steps? On the one hand, that is the same as taking \(m\) steps \((n + m)\) times; \((\sharp)\)
on the other hand, it is the same as taking \((n + m)\) steps \(m\) times. \((\natural)\)
(For instance, taking \(6 = 2 \cdot 3\) steps is the same as either taking 2 steps 3 times, or 3 steps 2 times.) Using \((\natural)\), we conclude that
\[
x^{m(n+m)} = \underbrace{x^m \cdot x^m \cdots x^m}_{n+m \text{ times}} = \underbrace{2 \cdot 2 \cdots 2}_{n+m \text{ times}} = 2^{n+m}.
\]
But from \((\natural)\) and \((\ast)\), we obtain
\[
x^{m(n+m)} = \underbrace{x^{n+m} \cdot x^{n+m} \cdots x^{n+m}}_{m \text{ times}} = \underbrace{3 \cdot 3 \cdots 3}_{m \text{ times}} = 3^m.
\]
Combining the results of the above computations, we see that
\[
2^{n+m} = 3^m.
\]
But \(2^{n+m}\) is an even number, while \(3^m\) is odd! So this equality is impossible.