MATH 181: SUMMARY 14

Musical Intervals

From the point of view of physics, a sound is just a propagating vibration of air, or a wave. The area of mathematics that studies waves and wave-like phenomena is called **harmonic analysis**. The applications of harmonic analysis are numerous: from the ones in physics (the study of sound, electro-magnetic waves such as light, quantum mechanics, etc.) to the ones in computer science (for example, the JPEG format stores images in the computer memory by representing them as combinations of waves).

The pitch of a sound is determined by its **frequency**, i.e., how many times per second the wave oscillates. The higher the frequency, the higher the perceived pitch. The **interval** between two sounds is the ratio of their frequencies. For instance, the musical note A is standardly set at 440 Hz (440 hertz, i.e., oscillations per second). If we double the frequency, we get 880 Hz, which corresponds to A an octave higher. If we double the frequency once again, we get 1760 Hz, an even higher A.

The intervals that are perceived as especially “harmonious” (or **consonant**) can be expressed as ratios of small whole numbers. (This observation dates back to **Pythagoras**.) For instance,

- the ratio $2 : 1$ corresponds to the interval called the **octave**;
- the ratio $3 : 2$ corresponds to the interval called the (perfect) **fifth**;
- the ratio $4 : 3$ corresponds to the interval called the (perfect) **fourth**.

So, if we start with A at 440 Hz, then a fifth higher is the frequency $440 \cdot (3/2) = 660$ Hz (corresponding to the note E), a fourth higher the frequency is $440 \cdot (4/3) = 586.66 \ldots$ Hz (corresponding to D), and an octave higher is $440 \cdot 2 = 880$ Hz (again called A). Similarly, if we start at 440 Hz and go down by a fifth, we will get $440 \cdot (2/3) = 293.33 \ldots$ Hz (a lower D), going down by a fourth we get $440 \cdot (3/4) = 330$ Hz (a lower E), and going down by an octave gives $440 \cdot (1/2) = 220$ Hz (a lower A).

Notice that if you start at any frequency—call it $f$—then go up by a fifth, and then go even higher up by a fourth, then the resulting frequency will be

$$f \cdot (3/2) \cdot (4/3) = f \cdot 2,$$

i.e., precisely an octave above $f$. 