Let
\[ L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0, \]
where \( n \) is any positive integer and \( a_0, a_1, \ldots, a_n \) are constants with \( a_n \neq 0 \). The characteristic polynomial and characteristic equation are
\[ p(r) = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_0 = 0. \]

Suppose that \( r_1, r_2, \ldots, r_n \) are the roots of \( p(r) = 0 \). The order of a characteristic root is the number of times \( r_k \) appears in the list \( r_1, r_2, \ldots, r_n \). Suppose that a certain root \( r_k \) appears \( m \) times, i.e., has order \( m \).

Recall that \( p(r_k) = p'(r_k) = \ldots = p^{(m-1)}(r_k) = 0 \), but \( p^{(m)}(r_k) \neq 0 \), since \( m \) is the order of the root \( r_k \).

We want to show that
\[ e^{r_k t}, te^{r_k t}, \ldots, t^{m-1}e^{r_k t} \]
are solutions of \( L[y] = 0 \). Let us begin with the fact
\[ L[e^{r_k t}] = p(r_k)e^{r_k t} = 0, \]

since \( p(r_k) = 0 \), i.e., it is a characteristic root.

Recall that \( L[y] \) is a sum of derivatives of \( y \) with respect to \( t \). We will replace \( r_k \) by a general variable \( r \). Thus,
\[ L[e^{rt}] = p(r)e^{rt}. \] (1)

The key idea is to differentiate both sides of (1) with respect to \( r \). Since we are differentiating exponentials, it doesn't make any difference if we differentiate first with respect to \( t \) or with respect to \( r \). Thus, differentiating (1),
\[ \frac{\partial}{\partial r} L[e^{rt}] = L[\frac{\partial}{\partial r} e^{rt}] = L[te^{rt}] \]
\[= p'(n) e^{nt} + p(n) e^{nt}. \quad (2)\]

Note that in the first equality at the bottom of the previous page we used the fact that the order of differentiation does not matter. Let \( r = n_k \), and suppose that the order of \( n_k \) is at least 2. Then, recall that \( p'(n_k) = 0 \). Thus,

\[
L \left[ te^{nt} \right] = p'(n_k) + p(n_k) te^{nt} = 0 + 0 = 0.
\]

Thus \( te^{nt} \) is a solution of our original differential equation. Differentiate (2) again. Then

\[
\frac{d^2}{dt^2} L \left[ e^{nt} \right] = L \left[ \frac{d}{dt} te^{nt} \right] = L \left[ te^{nt} \right]
\]

\[= p''(n) e^{nt} + 2tp'(n) e^{nt} + p(n) t^2 e^{nt}. \quad (3)\]

Let \( n = n_k \) and assume that the order of \( n_k \) is at least 3. Then \( p(n_k) = p'(n_k) = p''(n_k) = 0 \). Thus, from (3),

\[
L \left[ t^2 e^{nt} \right] = 0 + 0 + 0 = 0,
\]

i.e. \( t^2 e^{nt} \) is a solution of \( L[y] = 0 \).

We now prove by induction a general formula for which we already proved for \( m = 1, 2 \). Suppose that

\[
\frac{d^{m-1}}{dt^{m-1}} L \left[ e^{nt} \right] = L \left[ t^{m-1} e^{nt} \right]
\]

\[= p^{(m-1)}(n) e^{nt} + \text{terms involving derivatives of } (4)\]

\( p(n) \) of order < \( m-1 \). Differentiate once again to get

\[
\frac{d^m}{dt^m} L \left[ e^{nt} \right] = L \left[ t^m e^{nt} \right] = p^{(m)}(n) e^{nt} + \text{terms}
\]

involving derivatives of \( p(n) \) of order < \( m \). (5)

Now let \( n = n_k \) and assume \( n_k \) has order \( N \). Then \( p^{(j)}(n_k) = 0 \) for \( j = 0, 1, \ldots, N-1 \). Thus, from (5), we see that

\[
L \left[ t^j e^{nt} \right] = 0
\]

for \( j = 0, 1, \ldots, N-1 \), because \( p(n_k) = p'(n_k) = \cdots = p^{(N-1)}(n_k) = 0 \).
But we know that \( p^{(N)}(v_v) \neq 0 \). Thus, if we set \( m = N \) in (5), we get, if \( v = v_v \),

\[
L[t^N e^{v_v t}] = p^{(N)}(v_v) e^{v_v t} + \text{terms involving derivatives of order } < N. \text{ So},
\]

\[
L[t^N e^{v_v t}] = p^{(N)}(v_v) e^{v_v t} + 0 \neq 0.
\]

Let us now consider the nonhomogeneous differential equation with constant coefficients, namely,

\[
L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = f(t),
\]

where \( n \) is any positive integer. We consider the cases when \( f(t) \) contains an exponential on the right side. We separate cases as follows:

\[
y(t) = y_p(t) + t^n e^{\lambda t} p_n(t).
\]

Here \( p_n(t) \), \( q_n(t) \) are polynomials of degree \( n \). We first need to solve the homogenous equation \( L[y] = 0 \).

If \( \lambda \) is a characteristic root (we assume \( \lambda \) is real), \( \lambda \) is the order of the root, i.e., the number of times \( \lambda \) appears in the list of characteristic roots. Of course, if \( \lambda \) is not a characteristic root \( n = 0 \). Now suppose that \( f(t) = e^{\alpha t} \cos(\beta t) P_n(t) \) or \( e^{\alpha t} \sin(\beta t) P_n(t) \).

Again \( P_n(t) \) is a polynomial in \( t \) of degree \( n \). We now ask if \( \alpha \pm i\beta \) are characteristic roots. Let \( n \) be the order of the (possible) characteristic roots \( \alpha \pm i\beta \), i.e., the number of times \( \alpha \pm i\beta \) appears in our list of characteristic roots. Then

\[
y_p(t) = t^n \left[ e^{\alpha t} \cos(\beta t) Q_n(t) + e^{\alpha t} \sin(\beta t) Q_{n+1}(t) \right].
\]
show that if $\alpha$ is a polynomial of degree $n$, the product $\alpha \cdot x^\alpha$ is a polynomial of degree $n + 1$. 

Provided that $x = 0$, then $\alpha \cdot x^\alpha = 0$. 

(4)