Problem Set #1, Math 532

1. a. Prove that \( f(a, b) = (a, ab)_{\infty} (b, ab)_{\infty} \), where we assume the Jacobi triple product in the form

\[
\sum_{n=0}^{\infty} z^n q^n = (zq; q^2)_\infty (-zq; q^2)_\infty (q^2; q^2)_\infty
\]

b. Show two proofs of

\[
f(a, b) = a_{n(n+1)/2} b_{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}), \quad n \in \mathbb{Z}.
\]

c. Show two proofs of

\[
\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} = 0.
\]

Proof.

Let \( q^2 = ab \) and \( z = \frac{a}{b} \). Then

\[
q^n z^n = (ab)^{n/2} (a/b)^{n/2} = a_{n(n+1)/2} b_{n(n-1)/2}
\]

Thus,

\[
\sum_{n=0}^{\infty} z^n q^{n^2+n} = f(a, b) = (a, ab)_{\infty} (b, ab)_{\infty} (ab, ab)_{\infty}
\]

b. Replace the summation index \( k \) by \( k+n \). Thus,

\[
f(a, b) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{k+n+1} (b(k+n+1)/2, b(k+n-1)/2)
\]

Then

\[
= a_{n(n+1)/2} b_{n(n-1)/2} \sum_{k=0}^{\infty} a_{k(n+2)+1} (b(k+2n+1)/2, b(k+2n-1)/2)
\]

Then

\[
= a_{n(n+1)/2} b_{n(n-1)/2} \sum_{k=0}^{\infty} \frac{(a, ab)^n}{(b, ab)^{-n} f(a(ab)^n, b(ab)^{-n})}
\]

C. By the Jacobi triple product identity

\[
f(q^2-1) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} = (1, q^2)_{\infty} (q^2, q^2)_{\infty} = 0 \cdot (q^2, q^2)_{\infty}
\]

Replacing \( n \) by \( -n-1 \) in the second sum below, we obtain

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} = (\sum_{n=0}^{\infty} + \sum_{n=-1}^{0}) (-1)^n q^{n^2+n} = \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} + \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} = 0.
\]
By the triple product identity,
\[
f((ab)_n, (blab)_n, (blab)_n) = (-a(ab)_n(ab;ab)_\infty)
\]
\[\times (-b(ab)_n(ab;ab)_\infty)
\]
\[= a^{-n(n+1)/2} b^{-n(n-1)/2} (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty
\]

3. \textbf{Reduce the Jacobi triple product identity from Ramanujan's } \(\psi_1\)-\textit{summation.}
Let \(b = 0\) in the \(\psi_1\)-\textit{summation} theorem:
\[
\sum_{n=0}^{\infty} (a)_n z^n = \frac{(a z; a z)_\infty (b/a z; b/a z)_\infty}{(z; z)_\infty (b/a z; b/a z)_\infty}.
\]
Replace: \(a \to c, z \to ab, z = -b/c\) to get
\[
\sum_{n=0}^{\infty} (c)_n (-b/c)^n = \frac{(-b;ab)_\infty (-a;ab)_\infty (ab;ab)_\infty}{(-b/c; ab)_\infty (ab/c; ab)_\infty}.
\]
Let \(c \to \infty\). Note that
\[
(c)_n (-b/c)^n = (-b)^n \frac{1 - c^n}{1 - c} \frac{1 - c^n}{c} \cdots \frac{1 - c^{n-1}}{c}
\]
\[\to (-b)^n (-1)^n (ab)_n (n(n-1)/2 = a n(n-1)/2 b n(n+1)/2
\]
Thus,
\[
\sum_{n=0}^{\infty} a n(n-1)/2 b n(n+1)/2 = (-b;ab)_\infty (-a;ab)_\infty (ab;ab)_\infty
\]

8. \textbf{Prove that } \(\tau(n) \leq d(n) n^{1/2}\) assuming the theorem of Deligne: \(\tau(p) \leq 2 p^{1/2}\).
Write \(\tau(p) p^{-1/2} = 2 \cos \theta\)
Let \(a_n = \tau(p^n) p^{-n/2}\)
We know that
\[
\tau(p^{n+1}) = \tau(p^n) \tau(p) - p^{n/2} \tau(p^{n-4})
\]
or
\[
\tau(p^{n+1}) p^{-n(n+1)/2} = \tau(p^n) p^{-n/2} - \tau(p^{n-1}) p^{-n/2} (n-1)
\]
i.e.
\[a_{n+1} = a_n a_1 - a_{n-1}.
\]
We will prove by induction that
\[ a_n = \frac{\sin(n+1)\theta}{\sin \theta}, \quad n \geq 0 \]  
\[ (*) \]

Now \( a_0 = 2(1)p^0 = 2 = \frac{\sin \theta}{\sin \theta} \).

Also, \( a_1 = 2 \cos \theta = \frac{\sin 2\theta}{\sin \theta} \).

Thus, \( (*) \) is true for \( n = 0, 1 \). Assume that \( (*) \) holds up to \( n \). Then, by induction,
\[ a_{n+1} = a_n a_1 - a_{n-1} \]
\[ = \frac{\sin (n+1)\theta}{\sin \theta} \cdot \frac{\sin 2\theta}{\sin \theta} - \frac{\sin(n)\theta}{\sin \theta} \frac{\sin(0)\theta}{\sin \theta} \]
\[ = \frac{2 \sin(n+1)\theta \cos \theta - \sin(n)\theta}{\sin \theta} \]
\[ = \frac{\sin(n+1)\theta \cos \theta - \sin(n)\theta + \sin(n+2)\theta - \cos(n+1)\theta \sin \theta}{\sin \theta} \]
\[ = \frac{\sin(n+1)\theta - \theta^2 - \sin(n)\theta + \sin(n+2)\theta}{\sin \theta} \]
\[ = \frac{\sin(n+2)\theta}{\sin \theta} \]

Thus, we have established \( (*) \) by induction.

We now quote the known identities,
\[ \frac{\sin(n+1)\theta}{\sin \theta} = \begin{cases} 2(\cos \theta + \cos 3\theta + \ldots + \cos(2k+1)\theta), & \text{if } n+1 = 2k+1, \\ 2(\cos 2\theta + \cos 4\theta + \ldots + \cos(2k)\theta), & \text{if } n+1 = 2k. \end{cases} \]

Hence,
\[ |a_n| = \left| \frac{\sin(n+1)\theta}{\sin \theta} \right| \leq 2 \left| \frac{n+1}{2} \right| \leq n+1 \]
i.e.,
\[ |2(p^n)| \leq (n+1)p^{\frac{n+1}{2}}. \]

By multiplicativity of \( \tau(m) \),
\[ |\tau(m)| \leq d(m)^{\frac{n}{2}}. \]
10. Prove that \( n \) can be represented as a sum of two squares if and only if
\[
\eta = 2^{a} \prod_{j=1}^{n} p_{j}^{d_{j}} \prod_{j=1}^{s} q_{j}^{2 \beta_{j}},
\]
where \( p_{j} \) and \( q_{j} \) are primes, \( p_{j} \equiv 1 \pmod{4} \) and \( q_{j} \equiv 3 \pmod{4} \), and \( a, d_{j}, \) and \( \beta_{j} \) are positive integers, and where \( a \) is a non-negative integer. We are obliged to use the formula
\[
\mu_{2}(\eta) = 4 \sum_{d \mid \eta} (-1)^{(d-1)/2} = 4 \sum_{d \mid \eta} \chi(d) = 4f(\eta),
\]
where \( \chi \) is the unique, non-trivial primitive character modulo 4. Since \( \chi \) is multiplicative, \( f \) is multiplicative. From the formula above, it is clear that \( f(2^{a}) = 1 \). If \( p \equiv 1 \pmod{4} \), then \( \chi(p) = 1 \), and since \( \chi \) is completely multiplicative, or from the definition of \( \chi \) directly,
\[
f(p^{d}) = \sum_{d \mid p^{d}} \chi(d) = \chi(1) + \chi(p) + \ldots + \chi(p^{d}) = a + 1.
\]
If \( g \equiv 3 \pmod{4} \), then \( \chi(g) = -1 \), and so by complete multiplicativity, or from the definition of \( \chi \),
\[
f(g^{\beta}) = \sum_{d \mid g^{\beta}} \chi(d) = \chi(1) + \chi(g) + \ldots + \chi(g^{\beta})
\]
\[
= 1 - 1 + \ldots + (-1)^{\beta}
\]
\[
= \begin{cases} 
1, & \text{if } \beta \text{ is even} \\
0, & \text{if } \beta \text{ is odd}.
\end{cases}
\]
Assume that \( n \) can be represented as a sum of two squares, i.e. \( \mu_{2}(\eta) > 0 \). Write
\[
n = 2^{a} \prod_{j=1}^{n} p_{j}^{d_{j}} \prod_{j=1}^{s} q_{j}^{2 \beta_{j}},
\]
where \( p_{j} \equiv 1 \pmod{4} \), \( q_{j} \equiv 3 \pmod{4} \). Now \( \mu_{2}(\eta) > 0 \iff f(\eta) > 0 \). As \( f(\eta) \) is multiplicative,
\[
0 < f(\eta) = f(2^{a}) \prod_{j=1}^{n} f(p_{j}^{d_{j}}) \prod_{j=1}^{s} f(q_{j}^{2 \beta_{j}})
\]
\[= \prod_{d=1}^{n} (x_d + 1) \prod_{j=1}^{s} f(q_j, b_j),\]

Thus, \(f(q_j, b_j) > 0, \ 1 \leq j \leq s. \) Hence \(b_j \) is even, i.e. \(b_j = 2\beta_j.\)

Thus, we have established (*).

Conversely, if we assume (*), then by the multiplicativity of \(f(n), \) we find that \(f(n) > 0, \) i.e., \(\gamma_2(n) > 0.\)