Problems on Nonhomogeneous Differential Equations



#1 $y^{(iv)} - y''' - 3y'' + 5y' - 2y = e^t$

Characteristic roots $1, 1, 1, -2$.

Thus, $y_h(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 e^{-2t}$

1 is of order 3. Hence

$$y_p(t) = At^3 e^t$$

$$y_p' = 3At^2 e^t + At^3 e^t = (3t^2 + t^3) A e^t$$

$$y_p'' = [(6t + 3t^2) + (3t^2 + t^3)] A e^t$$

$$= (6t + 6t^2 + t^3) A e^t$$

$$y_p''' = [(6 + 12t + 3t^2) + (6t + 6t^2 + t^3)] A e^t$$

$$= (6 + 18t + 9t^2 + t^3) A e^t$$

$$y_p^{(iv)} = [(18 + 18t + 3t^2) + (6 + 18t + 9t^2 + t^3)] A e^t$$

$$= [24 + 36t + 12t^2 + t^3] A e^t$$

$$L[y_p] = \Big( [24 + 36t + 12t^2 + t^3]$$
$$+ [-6 - 18t - 9t^2 - t^3]$$
$$+ [-18t - 18t^2 - 3t^3]$$
$$+ [15t^2 + 5t^3] - 2At^3 \Big) A e^t$$

$$= 18 A e^t = e^t \Rightarrow A = \frac{1}{18}$$

$$\therefore y_p(t) = \frac{1}{18} t^3 e^t$$

#2 $y^{(iv)} - 6y''' + 10y'' - 6y' + 9y = t^2 \cos t - 3t^2 e^t + t e^{3t}$

$p(r) = r^4 - 6r^3 + 10r^2 - 6r + 9$

$p(3) = 81 - 162 + 90 - 18 + 9 = 0$ $\therefore$ 3 is a root

$p'(r) = 4r^3 - 18r^2 + 20r - 6$

$p'(3) = 108 - 162 + 60 - 6 = 0$ $\therefore$ 3 is a double root

$\dfrac{p(r)}{(r-3)^2} = r^2 + 1$ $\therefore$ $\pm i$ are roots

$\therefore y_h(t) = c_1 e^{3t} + c_2 t e^{3t} + c_3 \cos t + c_4 \sin t$

$\pm i$ are roots of order 1. Thus, $m$(multiplicity) $= 1$
$t^2$ has degree 2.
Thus, $$y_{p_1}(t) = t\left[(At^2+Bt+C)\cos t + (Dt^2+Et+F)\sin t\right]$$
1 is not a root
$t^2$ has degree 2
Thus, $$y_{p_2}(t) = (Gt^2+Ht+I)e^t$$
3 is a double root
$t$ has degree 1.
Thus, $$y_{p_3}(t) = t^2(Jt+K)e^{3t}$$
Thus, $$y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t) \qquad (*)$$
with $y_{p_1}, y_{p_2}, y_{p_3}$ as above. The general sol. of the d.e. is thus,
$$y = c_1 e^{3t} + c_2 t e^{3t} + c_3 \cos t + c_4 \sin t + y_p(t)$$
where $y_p(t)$ is given by $(*)$.