Section 3.1, #23 \[ y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0 \]
\[ \lambda^2 - (2\alpha - 1)\lambda + \alpha(\alpha - 1) = (\lambda - 1)^2 = 0 \]
Characteristic roots are \(\lambda, \lambda = 1\). Thus, a general solution is
\[ y = c_1 e^{t \lambda} + c_2 \]
If \(\alpha > 1\), then \(e^{t \lambda} \to \infty\) as \(t \to \infty\).
If \(\alpha < 0\), then \(e^{t \lambda} \to 0\) as \(t \to \infty\).

Section 3.2, #25 \[ y'' - 2y' + y = 0 \]
\[ \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \]
Characteristic roots are \(1, 1\). Thus, \(y_1(t) = e^t\) is a solution.
Let \(y_2(t) = t e^t\). \(y_2'(t) = e^t + t e^t\), \(y_2''(t) = e^t + e^t + t e^t = t e^t + 2e^t \)
\[ L(y_2) = te^t + 2e^t - 2(te^t e^t) + te^t = 0 \]
Thus, \(y_2\) is a solution.
\[ W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = te^t + e^t - te^t = e^t \neq 0 \]
Thus, \(e^t, te^t\) form a fundamental set.

Section 3.2, #24
\[ L(x) = (1-x \cot x) \cdot 0 - x \cdot 1 + x = 0 \quad \text{\(x\) is a solution} \]
\[ L(\sin x) = (1-x \cot x)(-\sin x) - x \cos x + \sin x \]
\[ = -\sin x + x \cos x - x \cos x + \sin x = 0 \quad \sin x \text{ is a solution} \]
\[ W(x, \sin x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x \cos x - \sin x \]
At \(x = \frac{\pi}{2}\), \(W(x, \sin x) = 0 - 1 = -1 \neq 0 \)
Thus, \(x, \sin x\) form a fundamental set.
Section 3.6, p. 20 (a) \( r^2 + 2ar + a^2 = (r+a)^2 \Rightarrow r = -a, -a. \)

Thus, \( e^{-at} \) is a solution.

(b) \( W(y_1, y_2) = c \exp \left[ -\int 2adx \right] = ce^{-2at}, \ c \ \text{constant} \)

\[ = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = y_1 \dot{y}_2 - y_2 \dot{y}_1 \ \text{by definition} \]

(c) Thus,

\[ ce^{-at} y_2' + ae^{-at} y_2 = ce^{-at} \]
\[ y_2' + ay_2 = ce^{-at} \] \hspace{1cm} (X)

Since we are more familiar with finding integrating factors for differential equations in \( x, y \), we rewrite (X) in the form

\[ ay - ce^{-ax} + y' = 0, \]

which is not exact. Here, \( M(x, y) = ay - ce^{-ax}, N(x, y) = 1 \).

We find an integrating \( \mu(x) \). Thus,

\[ \frac{d\mu}{dx} = \frac{My - Nx}{N} \mu = \frac{a - 0}{1} \mu = a \mu \]

\[ \therefore \frac{d\mu}{dx} = ax \Rightarrow \log \mu = ax \Rightarrow \mu(x) = e^{ax} \]

\[ \therefore e^{ax} (y') + ce^{ax} (ay - ce^{-ax}) = 0 \]
\[ e^{ax} y' + e^{ax} ay = c \]
\[ \frac{d}{dx} (e^{ax} y) = c \]
\[ e^{ax} y = cx + c' \]
\[ y = cx e^{-ax} + ce^{-ax} \]

Thus, converting to the original notation,

\[ y_2(t) = te^{-at} \]

is a solution, since \( e^{-at} \) was previously known as a solution.