Hermite's Differential Equation

Sect. 5.2.424  Here \( \lambda \) is any real number

\[ L[y] = y'' - 2xy' + \lambda y = 0. \]

There are no singular points. Thus, if \( I \) is the linearly independent power series solution around \( x_0 = 0 \), and they converge everywhere. Let

\[ y = \sum_{n=0}^{\infty} a_n x^n. \]

Then

\[ L[y] = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n nx^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0. \]

For \( n = 0 \), \( 2a_2 + \lambda a_0 = 0 \) \( \Rightarrow a_2 = -\frac{\lambda a_0}{2}. \)

For \( n \geq 1 \), \( a_{n+2}(n+2)(n+1)x^n - 2a_n n x^n + \lambda a_n x^n = 0 \)

\[ a_{n+2} = \frac{(2n-\lambda)a_n}{(n+2)(n+1)}, \quad n \geq 1. \tag{1} \]

Observe that this recurrence relation also holds for \( n = 0 \).

Let \( a_0 = 1, a_1 = 0 \). From (1), \( a_2 = 0 \), etc., i.e., \( a_{2n+1} = 0, n \geq 0 \).

Let \( n = 2 \) in (1). Then

\[ a_4 = \frac{(4-\lambda)a_2}{4!} = \frac{(4-\lambda)(-\lambda)}{4!} \]

Let \( n = 4 \) in (1). Then

\[ a_6 = \frac{(8-\lambda)a_4}{6!} = \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6!} = \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6!} \]

in general,

\[ a_{2n} = \frac{(4n-\lambda)(4n-8-\lambda)\ldots(4-\lambda)(0-\lambda)}{(2n)!}. \]

Thus,

\[ y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(4n-\lambda)(4n-8-\lambda)\ldots(4-\lambda)(0-\lambda)}{(2n)!} x^{2n} \tag{1} \]

Let \( a_0 = 0, a_1 = 1 \). From (1), \( a_2 = 0 \), etc., i.e., \( a_{2n} = 0, n \geq 0 \).

Let \( n = 1 \) in (1). Then

\[ a_3 = \frac{(2-\lambda)a_1}{3!} = \frac{(2-\lambda)}{3!} \]

Let \( n = 3 \) in (1). Then

\[ a_5 = \frac{(6-\lambda)a_3}{5!} = \frac{(6-\lambda)(2-\lambda)}{5!}. \]
In general, for \( n \geq 1, \)
\[
a_{2n+1} = \frac{((4n-2)-\lambda)((4n-6)-\lambda) \cdots (2-\lambda)}{(2n+1)!}
\]

Thus,
\[
y_2(x) = x + \sum_{n=1}^{\infty} \frac{((4n-2)-\lambda)((4n-6)-\lambda) \cdots (2-\lambda)}{(2n+1)!} x^{2n+1}
\]  
(2)

Return to (1). Note that if \( \lambda = 4k, \ k = 0, 1, 2, \ldots \), we get a polynomial solution. In particular, note that if \( n = k+1, \)
\[
(4(n-1)-\lambda) = 4k+4-4-4k = 0.
\]
Thus, \( y_1(x) \) is a polynomial of degree \( \lambda \). Return to (2). Note that if \( \lambda = 4k-2, \ k = 1, 2, \ldots \), we get a polynomial solution. In particular, note that if \( n = k, \)
\[
(4(n-2)-\lambda) = 4k-2-(4k-2) = 0.\]  
Thus, \( y_2(x) \) is a polynomial of degree \( 2k-1 \).

We calculate a few polynomial solutions:

\( \lambda = 4, \) \( y_1(x) = 1 - \frac{4}{2} x^2 = 1-2x^2 \)

\( \lambda = 2, \) i.e. \( k = 1, \) \( y_2(x) = x \)

\( \lambda = 8, \) i.e. \( k = 2, \) \( y_1(x) = 1 - \frac{8}{2} x^2 + \frac{(-4)(-8)}{4!} x^4 = 1-4x^2 + \frac{8}{3} x^4 \)

\( \lambda = 6, \) i.e. \( k = 2, \) \( y_2(x) = x + \frac{(-4)}{3!} x^3 = x - \frac{2}{3} x^3. \)

We note that for all the polynomial solutions, \( \lambda = 2n, \) where \( n \) is even for \( y_1(x) \) and \( n \) is odd for \( y_2(x) \). For the Hermite polynomial solutions \( H_n(x) \) the coefficient of \( x^n \) is \( 2^n \). Thus, we must multiply each polynomial by a constant to make the solution have coefficient of \( x^n \) equal to \( 2^n \).

\( H_2(x) = -2(1-2x^2) = -2+4x^2. \) Note \( H_0(x) = 1 \)

\( H_1(x) = 2x \)

\( H_4(x) = 12(1-4x^2 + \frac{4}{3} x^4) = 12-48x^2 + 24x^4 \)

\( H_8(x) = -12(x - \frac{2}{3} x^3) = -12x + 2^3 x^3 \)

(Recall that any constant times a solution is still a solution.)

If \( \lambda \) is not an even positive integer, use the ratio test to prove that the radius of convergence is \( \infty \) for all \( y_1(x), \ y_2(x) \).