Hermite's Differential Equation

Sect. 5.2, #21 Here λ is any real number

$$L[y] = y'' - 2x\,y' + \lambda y = 0.$$

There are no singular points. Thus, $\exists$ 2 linearly independent power series solutions around $x_0 = 0$ and they converge everywhere. Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$L[y] = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 2\sum_{n=1}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

For $n = 0$, $\quad 2a_2 + \lambda a_0 = 0$ or $a_2 = -\frac{\lambda a_0}{2}$.

For $n \geq 1$, $\quad a_{n+2}(n+2)(n+1) - 2a_n n + \lambda a_n = 0$

or $$a_{n+2} = \frac{(2n-\lambda)a_n}{(n+2)(n+1)}, \quad n \geq 1. \qquad (*)$$

Observe that this recurrence relation also holds for $n = 0$.

Let $a_0 = 1, a_1 = 0$. From $(*)$, $a_3 = 0$, etc., i.e. $a_{2n+1} = 0$, $n \geq 0$.

Let $n = 2$ in $(*)$. Then

$$a_4 = \frac{(4-\lambda)a_2}{4.3} = \frac{(4-\lambda)(-\lambda)}{4!}$$

Let $n = 4$ in $(*)$. Then

$$a_6 = \frac{(8-\lambda)a_4}{6.5} = \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6!} = \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6!}$$

In general,

$$a_{2n} = \frac{((4n-4)-\lambda)((4n-8)-\lambda)\cdots(4-\lambda)(0-\lambda)}{(2n)!}$$

Thus,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{((4n-4)-\lambda)((4n-8)-\lambda)\cdots(4-\lambda)(0-\lambda)}{(2n)!}x^{2n} \qquad (1)$$

Let $a_0 = 0, a_1 = 1$. From $(*)$, $a_2 = 0$, etc., i.e., $a_{2n} = 0$, $n \geq 0$.

Let $n = 1$ in $(*)$. Then

$$a_3 = \frac{(2-\lambda)\cdot 1}{3!} = \frac{(2-\lambda)}{3!},$$

Let $n = 2$ in $(*)$. Then

$$a_5 = \frac{(6-\lambda)a_3}{5\cdot 4} = \frac{(6-\lambda)(2-\lambda)}{5!}.$$

In general, for $n \geq 1$,

$$a_{2n+1} = \frac{((4n-2)-\lambda)((4n-6)-\lambda)\cdots(2-\lambda)}{(2n+1)!}$$

Thus,

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{((4n-2)-\lambda)((4n-6)-\lambda)\cdots(2-\lambda)}{(2n+1)!} x^{2n+1}. \quad (2)$$

Return to (1). Note that if $\lambda = 4k$, $k = 0, 1, 2, \ldots$, we get a polynomial solution. In particular, note that if $n = k+1$, $(4n-4)-\lambda = 4k+4-4-4k = 0$. Thus, $y_1(x)$ is a polynomial of degree $2k$. Return to (2). Note that if $\lambda = 4k-2$, $k = 1, 2, \ldots$, we get a polynomial solution. In particular, note that if $n = k$, $(4n-2)-\lambda = 4k-2-(4k-2) = 0$. Thus, $y_2(x)$ is a polynomial of degree $2k-1$.

We calculate a few polynomial solutions.

$\underline{\lambda=4}$, $y_1(x) = 1 - \frac{4}{2}x^2 = 1-2x^2$

$\underline{\lambda=2}$, i.e, $k=1$, $y_2(x) = x$

$\underline{\lambda=8}$, i.e. $k=2$, $y_1(x) = 1 - \frac{8}{2}x^2 + \frac{(-4)(-8)}{4!}x^4 = 1-4x^2+\frac{4}{3}x^4$

$\underline{\lambda=6}$, i.e, $k=2$, $y_2(x) = x + \frac{(-4)}{3!}x^3 = x - \frac{2}{3}x^3$.

We note that for all the polynomial solutions, $\lambda = 2n$, where n is even for $y_1(x)$ and n is odd for $y_2(x)$. For the Hermite polynomial solutions $H_n(x)$ the coefficient of x^n is 2^n. Thus, we must multiply each polynomial by a constant to make the solution have coefficient of x^n equal to 2^n.

$H_2(x) = -2(1-2x^2) = -2+4x^2$. Note $H_0(x) = 1$

$H_1(x) = 2x$

$H_4(x) = 12(1-4x^2+\frac{4}{3}x^4) = 12-48x^2+2^4x^4$

$H_3(x) = -12(x-\frac{2}{3}x^3) = -12x+2^3x^3$

(Recall that any constant times a solution is still a solution.)

If λ is not an even positive integer, use the ratio test to prove that the radius of convergence is ∞ for all $y_1(x)$, $y_2(x)$.