EXACT DIFFERENTIAL EQUATIONS

1. The Main Theorem

Our goal is to solve the differential equation

$$M(x,y) + N(x,y)y' = 0. \tag{1.1}$$

Definition Suppose that there exists a function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x,y). \tag{1.2}$$

Then the differential equation (1.1) is *exact.*

If (1.1) is exact, then, by (1.2),

$$M(x,y) + N(x,y)y' = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\Psi(x,y) = 0. \tag{1.3}$$

Integrating the right side of (1.3), we see that

$$\Psi(x,y) = c, \tag{1.4}$$

where c is an arbitrary constant.

To summarize, if the differential equation (1.1) is exact, then once we find $\Psi(x,y)$, our solutions are simply given by (1.4). How do we determine if (1.1) is exact? If (1.1) is exact, how do we determine $\Psi(x,y)$? The following theorem answers these questions.

Theorem 1.1. *Let* $R = \{(x,y) : \alpha < x < \beta, \gamma < y < \delta\}$. *Let* M, N, M_y, N_x *be continuous on* R. *Then* (1.1) *is exact if and only if*

$$M_y(x,y) = N_x(x,y). \tag{1.5}$$

Proof. First, assume that (1.1) is exact. Thus, there exists a function $\Psi(x,y)$ such that the equations (1.2) hold. Moreover, (1.3) also holds. From (1.2), we find that, respectively,

$$\Psi_{xy}(x,y) = M_y(x,y) \quad \text{and} \quad \Psi_{yx}(x,y) = N_x(x,y). \tag{1.6}$$

By hypothesis, $M_y(x,y)$ and $N_x(x,y)$ are continuous. Hence, by (1.6), $\Psi_{xy}(x,y)$ and $\Psi_{yx}(x,y)$ are respectively continuous. Now we know from calculus that if partial derivatives are continuous, then mixed derivatives are independent of the order in which the partial derivatives are taken. Hence, $\Psi_{xy}(x,y) = \Psi_{yx}(x,y)$. It follows from the foregoing equality and (1.6) that $M_y(x,y) = N_x(x,y)$, which is what we wanted to prove.

Next, we prove the converse. We assume that $M_y = N_x$. Integrate the first equation of (1.2) with respect to x. Recalling that $M(x, y)$ is a function of *2* variables, the constant of integration will depend on y. Hence,

$$\begin{aligned}\Psi(x,y) &:= \int M(x,y)dx + h(y) \\ &:= Q(x,y) + h(y). \end{aligned} \tag{1.7}$$

where $h(y)$ is the constant of integration, and where we are using the notation $Q(x, y)$ for simplicity, instead of writing integral signs throughout the sequel. First, we differentiate the second equality of (1.7) with respect to x using the fundamental theorem of calculus, and then we differentiate the resulting equality with respect to y. Hence,

$$M(x,y) = Q_x(x,y) \quad \text{and} \quad M_y(x,y) = Q_{xy}(x,y). \tag{1.8}$$

Second, differentiate (1.7) with respect to y. Here is a key step. After differentiating, we are now going to define $h'(y)$ by setting the right-hand side equal to $N(x, y)$. Hence,

$$\Psi_y(x,y) = Q_y(x,y) + h'(y) := N(x,y). \tag{1.9}$$

Rearranging the second equality of (1.9) yields

$$h'(y) = N(x,y) - Q_y(x,y). \tag{1.10}$$

At this moment, we do not know if we can find a function $h(y)$ such that (1.10) is valid. If we can find such a function $h(y)$, then first we will need to show that the right-hand side of (1.10) is indeed a function of only y. To do this, we differentiate the right side of (1.10) with respect to x. If we can show that this derivative is equal to 0, then the left side of (1.10), as a function of x, must be a constant. To that end, first, in the second equality below, we use our hypothesis $M_y = N_x$, and then use the second equality in (1.8) to deduce from the continuity of M_y that $Q_{xy} = Q_{yx}$. Second, in the third equality below, we use the second equality of (1.8) again. Thus,

$$\begin{aligned}\frac{\partial}{\partial x}\{N(x,y) - Q_y(x,y)\} &= N_x(x,y) - Q_{yx}(x,y) \\ &= M_y(x,y) - Q_{xy}(x,y) \\ &= M_y(x,y) - M_y(x,y) = 0. \end{aligned} \tag{1.11}$$

Thus, as a function of x, the left side of (1.10) is a constant, because the derivative with respect to x of the right-hand side is equal to 0. In conclusion, our assumption in (1.9) that we can find a function of y such that (1.9) indeed does hold has been justified. We can find a formula for $h(y)$ by integrating (1.10) with respect to y to obtain

$$h(y) = \int N(x,y)dy - Q(x,y). \tag{1.12}$$

In conclusion, now that we have found $h(y)$, we see that from (1.7), $\Psi_x = M$, and that from (1.9), $\Psi_y = N$, i.e., (1.1) is exact.

2. Recipe

From our analysis above, we can develop a recipe for solving an equation of type (1.1).

(1) Check to see that $M_y = N_x$.
(2) Then there exists a function Ψ such that $\Psi_x = M$ and $\Psi_y = N$.
(3) Integrate Ψ_x and remember that the constant of integration is a function of y, i.e., $h(y)$.
(4) Now from above, we have a formula for Ψ. Differentiate it with respect to y.
(5) From our formula above and from $\Psi_y = N$ in item 2 above, solve for $h'(y)$.
(6) Integrate to determine $h(y)$.
(7) Plug this evaluation of $h(y)$ into the formula for Ψ from item 3.
(8) Remember from (1.4) that your solutions are now $\Psi(x, y) = c$, for an arbitrary constant c.