Problems

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. \( x^2 y'' + 4xy' + 2y = 0 \)

2. \( (x + 1)^2 y'' + 3(x + 1)y' + 0.75y = 0 \)

3. \( x^2 y'' - 3xy' + 4y = 0 \)

4. \( x^2 y'' + 3xy' + 5y = 0 \)

5. \( x^2 y'' - xy' + y = 0 \)

6. \( (x - 1)^2 y'' + 8(x - 1)y' + 12y = 0 \)

7. \( x^2 y'' + 6xy' - y = 0 \)

8. \( 2x^2 y'' - 4xy' + 6y = 0 \)

9. \( x^2 y'' - 5xy' + 9y = 0 \)

10. \( (x - 2)^2 y'' + 5(x - 2)y' + 8y = 0 \)

11. \( x^2 y'' + 2xy' + 4y = 0 \)

12. \( x^2 y'' - 4xy' + 4y = 0 \)

In each of Problems 13 through 16 find the solution of the given initial value problem.

13. \( 2x^2 y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4 \)

14. \( 4x^2 y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3 \)

15. \( x^2 y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3 \)

16. \( x^2 y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1 \)

17. Find all values of \( \alpha \) for which all solutions of \( x^2 y'' + \alpha xy' + (5/2)y = 0 \) approach zero as \( x \to 0 \).

18. Find all values of \( \beta \) for which all solutions of \( x^2 y'' + \beta y = 0 \) approach zero as \( x \to 0 \).

19. Find \( y \) so that the solution of the initial value problem \( x^2 y'' - 2y = 0, y(1) = 1, y'(1) = y \)

is bounded as \( x \to \infty \).

20. Find all values of \( \alpha \) for which all solutions of \( x^2 y'' + \alpha xy' + (5/2)y = 0 \) approach zero as \( x \to \infty \).

21. Consider the Euler equation \( x^2 y'' + \alpha xy' + \beta y = 0 \). Find conditions on \( \alpha \) and \( \beta \) so that
   (a) All solutions approach zero as \( x \to 0 \).
   (b) All solutions are bounded as \( x \to 0 \).
   (c) All solutions approach zero as \( x \to \infty \).
   (d) All solutions are bounded as \( x \to \infty \).
   (e) All solutions are bounded both as \( x \to 0 \) and as \( x \to \infty \).

22. Using the method of reduction of order, show that if \( r_1 \) is a repeated root of \( r(r - 1) + \alpha r + \beta = 0 \), then \( x^{r_1} \) and \( x^{r_1} \ln x \) are solutions of \( x^2 y'' + \alpha xy' + \beta y = 0 \) for \( x > 0 \).

23. Transformation to a Constant Coefficient Equation. The Euler equation \( x^2 y'' + \alpha xy' + \beta y = 0 \) can be reduced to an equation with constant coefficients by a change of the independent variable. Let \( x = e^z \), or \( z = \ln x \), and consider only the interval \( x > 0 \).
   (a) Show that
      \[
      \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{1}{x^2} \frac{d^2 y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}.
      \]
   (b) Show that the Euler equation becomes
      \[
      \frac{d^2 y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0.
      \]

Letting \( r_1 \) and \( r_2 \) denote the roots of \( r^2 + (\alpha - 1)r + \beta = 0 \), show that
   (c) If \( r_1 \) and \( r_2 \) are real and different, then
      \[
      y = c_1 e^{r_1 z} + c_2 e^{r_2 z} = c_1 x^{r_1} + c_2 x^{r_2}.
      \]
   (d) If \( r_1 \) and \( r_2 \) are real and equal, then
      \[
      y = (c_1 + c_2 z)x^{r_1} = (c_1 + c_2 \ln x)x^{r_1}.
      \]
5.6 Series Solutions near a Regular Singular Point, Part I

(e) If \( r_1 \) and \( r_2 \) are complex conjugates, \( r_1 = \lambda + i\mu \), then
\[
y = e^{x\lambda}[c_1 \cos(\mu x) + c_2 \sin(\mu x)] = x^\lambda [c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].
\]
In each of Problems 24 through 29 use the method of Problem 23 to solve the given equation for \( x > 0 \).

24. \( x^2 y'' - 2y = 0 \)
25. \( x^2 y'' - 3xy' + 4y = \ln x \)
26. \( x^2 y'' + 7xy' + 5y = x \)
27. \( x^2 y'' - 2xy' + 2y = 3x^2 + 2 \ln x \)
28. \( x^2 y'' + xy' + 4y = \sin(\ln x) \)
29. \( 3x^2 y'' + 12xy' + 9y = 0 \)

30. Show that if \( L[y] = x^2 y'' + axy' + \beta y \), then
\[
L[(-xy)] = (-x)^{y}F(r)
\]
for all \( x < 0 \), where \( F(r) = r(r - 1) + \alpha r + \beta \). Hence conclude that if \( r_1 \neq r_2 \) are roots of \( F(r) = 0 \), then linearly independent solutions of \( L[y] = 0 \) for \( x < 0 \) are \((-x)^{r_1}\) and \((-x)^{r_2}\).

31. Suppose that \( x^{r_1} \) and \( x^{r_2} \) are solutions of an Euler equation, where \( r_1 \neq r_2 \), and \( r_1 \) is an integer. According to Eq. (24) the general solution in any interval not containing the origin is \( y = c_1 x^{r_1} + c_2 x^{r_2} \). Show that the general solution can also be written as
\[
y = k_1 x^{r_1} + k_2 x^{r_2}.
\]

Hint: Show by a proper choice of constants that the expressions are identical for \( x > 0 \), and by a different choice of constants that they are identical for \( x < 0 \).

Complex Coefficients. If the constants \( \alpha \) and \( \beta \) in the Euler equation \( x^2y'' + axy' + \beta y = 0 \) are complex numbers, it is still possible to obtain solutions of the form \( x^r \). However, in general, the solutions are no longer real-valued. In each of Problems 32 through 34 determine the general solution of the given equation.

32. \( x^2 y'' + 2ixy' - iy = 0 \)
33. \( x^2 y'' + (1 + i)xy' + 2y = 0 \)
34. \( x^2 y'' + xy' - 2iy = 0 \)

5.6 Series Solutions near a Regular Singular Point, Part I

We now consider the question of solving the general second order linear equation
\[
P(x)y'' + Q(x)y' + R(x)y = 0
\]
in the neighborhood of a regular singular point \( x = x_0 \). For convenience we assume that \( x_0 = 0 \). If \( x_0 \neq 0 \), the equation can be transformed into one for which the regular singular point is at the origin by letting \( x = x_0 t \) equal \( t \).

The fact that \( x = 0 \) is a regular singular point of Eq. (1) means that \( xQ(x)/P(x) = xp(x) \) and \( x^2 R(x)/P(x) = x^2 q(x) \) have finite limits as \( x \to 0 \), and are analytic at \( x = 0 \). Thus they have power series expansions of the form
\[
 xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n,
\]
which are convergent for some interval \( |x| < \rho, \rho > 0 \), about the origin. To make the quantities \( xp(x) \) and \( x^2 q(x) \) appear in Eq. (1), it is convenient to divide Eq. (1) by \( P(x) \) and then to multiply by \( x^2 \), obtaining
\[
x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0,
\]