Rudiments of the theory of Dirichlet series

by

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Let \( \{a_n\} \) be a sequence of complex numbers, and let \( \{\lambda_n\} \) be a strictly increasing sequence of real numbers tending to \( \infty \). Then a Dirichlet series is a series of the form

\[
\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.
\]

Here, and in the sequel, \( s = \sigma + it \) with \( \sigma \) and \( t \) both real. If \( \lambda_n = \log n \), the resulting series

\[
\sum_{n=1}^{\infty} a_n n^{-s}
\]

is called an ordinary Dirichlet series. The simplest examples of ordinary Dirichlet series are finite series

\[
\sum_{n=1}^{N} a_n n^{-s},
\]

which are called Dirichlet polynomials. The simplest, infinite ordinary Dirichlet series is the Riemann zeta-function.
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1). \]

Dirichlet series were first introduced by, not surprisingly, Dirichlet, who was primarily interested in their application to problems in number theory. The series were also used by Dedekind in number theoretical studies. Both Dirichlet and Dedekind considered Dirichlet series as functions of only a real variable. Jensen and Cahen were the first mathematicians to study Dirichlet series of a complex variable.

The study of Dirichlet series is much more complex (pardon the pun) than the study of power series. Even the theory of Dirichlet polynomials, which have lately found increasing importance in modern analytic number theory, is not particularly well developed. Analysts and number theorists would like to have much more information about the Riemann zeta-function. In the theory of power series, the circles of convergence, absolute convergence, and analyticity of a function all coincide. As we shall see below, a Dirichlet series converges in a half-plane. The half-planes of convergence, absolute convergence, and analyticity may all be distinct. Lastly, there are considerably fewer functions that have representations by Dirichlet series than by power series.
Theorem 1. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converge for $s = s_0$. Then $f(s)$ converges uniformly in the angular region,

$$|\arg (s-s_0)| < \frac{1}{2}\pi - \delta,$$

for any $\delta$, $0 < \delta < \frac{1}{2}\pi$.

Proof. Without loss of generality, we may assume that $s_0 = 0$. For if not, set

$$a'_n = a_n e^{-\lambda_n s_0} \quad \text{and} \quad s' = s - s_0.$$

Then, $\sum_{n=1}^{\infty} a'_n e^{-\lambda_n s'}$ converges for $s' = 0$.

We shall show that the Cauchy criterion for uniform convergence holds.

Since $f(s)$ converges at $s = 0$, $\sum_{n=1}^{\infty} a_n$ converges.

Put

$$R(N) = \sum_{n=N+1}^{\infty} a_n$$

and

$$S(N) = \sum_{n=1}^{N} a_n e^{-\lambda_n s}.$$
Then, if \( N > M, \)

\[
(1) \quad S(N) - S(M) = \sum_{n=M+1}^{N} a_n e^{-\lambda_n s} = \sum_{n=M+1}^{N} \{ R(n+1) - R(n) \} e^{-\lambda_n s} \\
= \sum_{n=M+1}^{N} R(n) \{ e^{-\lambda_{n+1} s} - e^{-\lambda_n s} \} + R(M) e^{-\lambda_{M+1} s} - R(N) e^{-\lambda_{N+1} s}.
\]

Now if \( \sigma > 0, \)

\[
(2) \quad |e^{-\lambda_{n+1} s} - e^{-\lambda_n s}| = \left| - \int_{\lambda_n}^{\lambda_{n+1}} s e^{-us} du \right| \\
\leq |s| \int_{\lambda_n}^{\lambda_{n+1}} e^{-us} du \\
= \frac{|s|}{\sigma} (e^{-\lambda_n \sigma} - e^{-\lambda_{n+1} \sigma}).
\]

Given \( \epsilon > 0, \) clearly there is a number \( N_0 = N_0(\epsilon), \) such that for \( n \geq N_0, \) \(|R(n)| < \epsilon. \) Thus, from (1) and (2) we have for \( \sigma > 0 \) and \( M, N \geq N_0, \)

\[
(3) \quad |S(N) - S(M)| \leq \frac{\epsilon |s|}{\sigma} \sum_{n=M+1}^{N} \left( e^{-\lambda_n \sigma} - e^{-\lambda_{n+1} \sigma} \right) + \epsilon e^{-\lambda_{M+1} \sigma} + \epsilon e^{-\lambda_{N+1} \sigma} \\
\leq \frac{2\epsilon |s|}{\sigma} + 2\epsilon,
\]
where we have, without loss of generality, assumed that
\( \lambda_{M+1} \geq 0 \). If \( |\arg s| \leq \frac{1}{2} \pi - \delta \), we have

\[
\frac{|s|}{\sigma} = \sec(\arg s) \leq \sec(\frac{1}{2} \pi - \delta) = \csc \delta.
\]

Hence, from (3) we conclude that for \( |\arg s| \leq \frac{1}{2} \pi - \delta \)
and \( M,N \geq N_0 \),

\[
|S(N)-S(M)| \leq 2\varepsilon (\csc \delta + 1).
\]

Thus, by Cauchy's criterion \( f(s) \) converges uniformly in the sector \( |\arg s| \leq \frac{1}{2} \pi - \delta \).

**Corollary 2.** If \( s_0 = \sigma_0 + i t_0 \) is as in Theorem 1, then \( f(s) \) converges for \( \sigma > \sigma_0 \).

**Proof.** If \( s \) is fixed with \( \sigma > \sigma_0 \), choose \( \delta > 0 \)
small enough so that \( |\arg (s-s_0)| \leq \frac{1}{2} \pi - \delta \). The result now follows from Theorem 1.

**Definition.** Let \( \sigma_0 = \inf \{ \sigma : f(\sigma+it) \text{ converges} \} \). Then
\( \sigma_0 \) is called the abscissa of convergence of \( f \). The line \( \sigma = \sigma_0 \)
is called the line of convergence of \( f \).

Three possibilities arise. It could happen that
\( f(s) \) converges for all \( s \), i.e., \( \sigma_0 = -\infty \). An example for this
occurrence is the series \( \sum_{n=1}^\infty n^{-s}/n! \). Secondly, \( f(s) \) may
converge for no values of \( s \), i.e., \( \sigma_0 = +\infty \). The series
\[
\sum_{n=1}^{\infty} \frac{n!}{n^s} \quad \text{is an example of such a series. Of course,} \quad \sigma_0 \quad \text{may be finite; for} \quad \zeta(s), \quad \sigma_0 = 1.
\]

On the line of convergence \( \sigma = \sigma_0 \), \( f(s) \) may converge or diverge. Thus, \( \zeta(s) \) converges at no point on the line \( \sigma = 1 \). On the other hand, \( \sum_{n=2}^{\infty} (\log n)^{-2} n^{-s} \) converges at all points on the line \( \sigma = 1 \).

**Corollary 3.** \( f(s) \) is analytic for \( \sigma > \sigma_0 \).

**Proof.** Note that \( a_n e^{-\lambda_n s} \) is an entire function of \( s \) for every integer \( n, n \geq 1 \). If \( \sigma > \sigma_0 \), \( s \) lies in some region \( S \) of uniform convergence by Theorem 1. Since
\[
\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}
\]
converges uniformly on \( S \), \( f(s) \) is analytic on \( S \).

**Definition.** We say that \( f(s) \) converges absolutely if \( \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n \sigma} \right| \) converges. The abscissa of absolute convergence of \( f(s) \) is the abscissa of convergence of \( \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n s} \right| \) and is denoted by \( \sigma_a \). The line \( \sigma = \sigma_a \) is called the line of absolute convergence of \( f \).
It may happen that \( \sigma_0 = \sigma_a \), which is the case for \( \zeta(s) \). On the other hand,

\[
\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1-2^{1-s}) \zeta(s)
\]

has abscissa of convergence \( \sigma_0 = 0 \) and abscissa of absolute convergence \( \sigma_a = 1 \). Furthermore, observe that

\[
g(s) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n^{\frac{1}{2}} (\log n)^s}
\]

converges for all values of \( s \), i.e., \( \sigma_0 = -\infty \). But, \( g(s) \) converges absolutely for no values of \( s \), i.e., \( \sigma_a = +\infty \).

However, for ordinary Dirichlet series we have the following result.

**Theorem 4.** If \( f(s) \) is an ordinary Dirichlet series,

\[
\sigma_a - \sigma_0 \leq 1.
\]

**Proof.** We show that \( f(s) \) converges absolutely for \( s = \sigma_0 + 1 + \delta + it, \delta > 0 \). This then implies that \( \sigma_a \leq \sigma_0 + 1 \).

Let \( s' = \sigma_0 + \epsilon + it, 0 < \epsilon < \delta \). Since \( \sum_{n=1}^{\infty} a_n n^{-s'} \) converges,

\( a_n n^{-s'} \) tends to 0 as \( n \) tends to \( \infty \). In particular, there exists a number \( M > 0 \) such that for all \( n \geq 1 \),
\[ |a_n n^{-s}| \leq M. \]

Thus,

\[ \left| \sum_{n=1}^{\infty} a_n n^{-(\sigma_0 + \delta - it)} \right| \leq \sum_{n=1}^{\infty} |a_n| n^{-\sigma_0 - \epsilon - 1 - \delta + \epsilon} \leq M \sum_{n=1}^{\infty} n^{-\epsilon + \delta} < \infty, \]

since \( \epsilon < \delta \), and we are done.

**Theorem 5.** Let \( S(N) = \sum_{n=1}^{N} a_n \) and

\[ \alpha = \lim_{n \to \infty} \frac{\log |S(n)|}{\lambda_n}. \]

Then, if \( \sum_{n=1}^{\infty} a_n \) is divergent, \( \sigma_0 = \alpha \).

**Proof.** We first show that \( \alpha \leq \sigma_0 \). If \( \sigma_0 = \infty \), there is nothing to prove. Hence, assume that \( \sigma_0 < \infty \). Now choose a fixed real number \( s \) with \( s > \sigma_0 \). Note that \( s > 0 \). Let

\[ A(N) = \sum_{n=1}^{N} a_n e^{-\lambda_n s}. \]
Let $M$ be chosen so that for every $n \geq 1$, $|A(n)| \leq M$.

Then,

$$|S(N)| = \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} e^{\lambda_n s} \right|$$

$$= \left| \sum_{n=1}^{N} \{A(n) - A(n-1)\} e^{\lambda_n s} \right|$$

$$= \left| \sum_{n=1}^{N-1} A(n) \{e^{\lambda_n s} - e^{\lambda_{n+1} s}\} + A(N) e^{\lambda_N s} \right|$$

$$\leq M \sum_{n=1}^{N-1} \{e^{\lambda_{n+1} s} - e^{\lambda_n s}\} + M e^{\lambda_N s}$$

$$= M \left( e^{\lambda_N s} - e^{\lambda_1 s} \right) + M e^{\lambda_N s}$$

$$< 2Me^{\lambda_N s}.$$

Thus,

$$\log |S(N)| < \log(2M) + \lambda_N s.$$ 

It follows that $\alpha \leq s$, and hence that $\alpha \leq \sigma_0$.

We next show that $\sigma_0 \leq \alpha$. If $\alpha = \infty$, obviously, $\sigma_0 \leq \alpha$. Thus, assume that $\alpha$ is finite. Let $\delta$ be any fixed positive number. We shall show that the series is convergent for $s = \alpha + \delta$. It then follows that $\sigma_0 \leq \alpha$. For the remainder of the proof, $s = \alpha + \delta$. 
Choose $\epsilon$ so that $0 < \epsilon < \delta$. By definition of $\alpha$, there exists a number $N_0$ such that for all $N \geq N_0$,

$$(4) \quad \log |S(N)| < (\alpha + \delta - \epsilon) \lambda_N'$$

Let $N \geq N_0$. By partial summation,

$$\sum_{n=1}^{N} a_n e^{-\lambda_ns} = \sum_{n=1}^{N-1} S(n) \{ e^{-\lambda_n^s} - e^{-\lambda_{n+1}^s} \} + S(N) e^{-\lambda_N^s}.$$  

Now from (4),

$$|S(N) e^{-\lambda_N^s}| \leq e^{(\alpha + \delta - \epsilon) \lambda_N} e^{-\epsilon \lambda_N},$$

which tends to 0 as $N$ tends to $\infty$. Hence, from (4), (5), and the above, it suffices to show the convergence of

$$\sum_{n=1}^{\infty} e^{(s-\epsilon) \lambda_n} \{ e^{-\lambda_n^s} - e^{-\lambda_{n+1}^s} \}.$$ 

Now,

$$e^{(s-\epsilon) \lambda_n} \{ e^{-\lambda_n^s} - e^{-\lambda_{n+1}^s} \} \leq e^{\alpha \lambda_n^s} - e^{\epsilon u} \leq e^{\alpha \lambda_n^s} - e^{-\epsilon u} \leq \epsilon \int_{\lambda_n}^{\lambda_{n+1}} e^{-\epsilon u} \, du.$$ 

since $\lambda > 0$. For the only way $\alpha < 0$ is if $\log |S(n)| < 0$ for all $n$. Claim if this is the case then $\alpha = 0$, but not. Thus, $S(n) \rightarrow -\infty$. Thus, $S(n)$ diverges! This contradicts fact that $\sum a_n$ diverges.
Thus,

\[ \sum_{n=1}^{\infty} e^{(s-\varepsilon)\lambda_n} \left( e^{-\lambda_n s} - e^{-\lambda_{n+1} s} \right) \]

\[ \leq s \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\lambda_n} e^{-\varepsilon u} \, du = s \int_{\lambda_1}^{\infty} e^{-\varepsilon u} \, du < \infty. \]

The proof is complete.

Theorem 5 enables us to generalize Theorem 4.

**Theorem 6.** We have

\[ \sigma_a - \sigma_0 \leq \lim_{n \to \infty} \frac{\log n}{\lambda_n}. \]

Proof. Without loss of generality, we may assume that \( \sigma_0 > 0 \). For if not, let \( s = \sigma^* + s' \), where \( \sigma^* < \sigma_0 \). Then the Dirichlet series in \( s' \) has an abscissa of convergence greater than 0. Also note that \( \sigma_a - \sigma_0 \) is invariant under the change of variable.

From Theorem 5, given \( \varepsilon > 0 \), there exists a number \( N_0 \) such that for \( n \geq N_0 \).

\[ |S(n)| < e^{(\sigma_0 + \varepsilon)\lambda_n}, \]
Choose $N_0$ also large enough so that $e^{\varepsilon \lambda_n} \geq 2$ for $n \geq N_0$.

Then,

$$|a_n| = |S(n) - S(n-1)| < 2e^{(\sigma_0 + \varepsilon)\lambda_n} \leq e^{(\sigma_0 + 2\varepsilon)\lambda_n}.$$ 

Let

$$S^*(N) = \sum_{n=1}^{N} |a_n|.$$ 

Then from (6), for $N > N_0$

$$S^*(N) = \sum_{n=1}^{N_0} |a_n| + \sum_{n=N_0+1}^{N} |a_n| \leq S^*(N_0) + \sum_{n=N_0+1}^{N} e^{(\sigma_0 + 2\varepsilon)\lambda_n} \leq S^*(N_0) + Ne^{(\sigma_0 + 2\varepsilon)\lambda_N}$$

for $N$ sufficiently large. Hence, from Theorem 5 and (7) for $N$ sufficiently large,

$$\sigma_a = \lim_{N \to \infty} \frac{\log S^*(N)}{\lambda_N} \leq \frac{\log N + (\sigma_0 + 3\varepsilon)\lambda_N}{\lambda_N},$$

$$\frac{\log S^*(N)}{\lambda_N} \leq \frac{\log N + (\sigma_0 + 3\varepsilon)\lambda_N}{\lambda_N},$$
\[ \sigma_a - \sigma_0 \leq \frac{\log N}{\lambda_N} + 3\epsilon, \]

and the result follows.

Note that if \( f(s) \) is an ordinary Dirichlet series, Theorem 6 reduces to Theorem 4. Also observe that if \( \log n = c(\lambda_n) \) as \( n \) tends to \( \infty \), then \( \sigma_0 = \sigma_a \). In particular, if \( \lambda_n = n \), we have the classical case of a power series in \( z = e^{-s} \). Theorem 5 then yields a modified version of Cauchy's formula for the radius of convergence.

Theorem 5 was proved under the hypothesis that \[ \sum_{n=1}^{\infty} a_n \] diverges. In the general case, unless \( \sigma_0 = -\infty \), it is no loss of generality to assume this divergence, for if not, as pointed out in the proof of Theorem 6, we may make a simple change of variable to obtain another Dirichlet series whose sum of coefficients diverges. Nonetheless, the following theorem is a helpful one. Although we shall prove the result for ordinary Dirichlet series only, a similar theorem holds in the more general case.

**Theorem 7.** Suppose that \( \sum_{n=1}^{\infty} a_n \) converges. Put

\[ R(N) = \sum_{n=N+1}^{\infty} a_n \]
and

\[ \beta = \lim_{n \to \infty} \frac{\log |R(n)|}{\log n} . \]

Then \( \sigma_0 = \beta \).

Proof. We first show that \( \beta \leq \sigma_0 \). Let \( s \leq 0 \) be fixed and chosen so that \( \sum_{n=1}^{\infty} a_n n^{-s} \) converges. Then in the notation of the proof of Theorem 5,

\[ |R(N)| = \left| \sum_{n=N+1}^{\infty} a_n n^{-s} n^s \right| \]

\[ = \left| \sum_{n=N+1}^{\infty} [A(n) - A(n-1)] n^s \right| \]

\[ = \left| \sum_{n=N}^{\infty} A(n) \{n^s - (n+1)^s\} - A(N) n^s \right| \]

\[ \leq M \sum_{n=N}^{\infty} \{n^s - (n+1)^s\} + MN^s \]

\[ = 2MN^s . \]

Note that the series on the far right side of (8), indeed, does converge. If \( s = 0 \), this is obvious. If \( s \neq 0 \),
\[(9) \quad n^s -(n+1)^s = s \int_{n}^{n+1} u^{s-1} \, du = o(n^{s-1}),\]

and the assertion follows since \( s \leq 0 \). From (8) it follows readily that \( \beta \leq \sigma_0 \).

We now show that \( \beta \geq \sigma_0 \). From the definition of \( \beta \), it is easily seen that \( R(n) = o(n^{\beta+\epsilon}) \) for every \( \epsilon > 0 \) as \( n \) tends to \( \infty \). Fix \( s \) with \( s > \beta + \epsilon \). Then using partial summation and (9), we find that for \( N > M \),

\[
\sum_{n=M+1}^{N} a_n n^{-s} = \sum_{n=M+1}^{N} (R(n) - R(n-1)) n^{-s} \\
= \sum_{n=M+1}^{N} R(n) [(n+1)^{-s} - n^{-s}] + R(M+1)^{-s} - R(N)(N+1)^{-s} \\
= o(\sum_{n=M+1}^{N} n^{\beta+\epsilon} (\{ (n+1)^{-s} - n^{-s} \})) + o(M^{\beta+\epsilon} - s) + o(N^{\beta+\epsilon} - s) \\
= o(\sum_{n=M+1}^{N} n^{\beta+\epsilon} - s - 1) + o(N^{\beta+\epsilon} - s) \\
= o(N^{\beta+\epsilon} - s) + o(N^{\beta+\epsilon} - s) \\
= o(N^{\beta+\epsilon} - s) \\
= \mathcal{O}(1),
\]
since \( s > \beta + \epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( \sum_{n=1}^{\infty} a_n n^{-s} \) converges for every \( s > \beta \) by the Cauchy criterion. It follows that \( \beta \geq \sigma_0 \), and the proof is complete.

The circle of convergence of a power series always possesses at least one singularity. On the contrary, the line of convergence of a Dirichlet series may not contain any singularities. However, if all of the coefficients are non-negative, there is at least one singularity as the following theorem shows.

**Theorem 8.** Let \( a_n \geq 0 \) for every \( n \geq 1 \). Then the point \( s = \sigma_0 \) is a singularity of \( f \).

**Proof.** Without loss of generality, assume that \( \sigma_0 = 0 \). Suppose that \( f \) is analytic at \( s = 0 \). Since also \( f(s) \) is analytic for \( \sigma > 0 \) by Corollary 3, \( f(s) \) can be expanded in a Taylor series about \( s = 1 \) with a radius of convergence strictly greater than one. Let \( s \) be a negative number for which this Taylor series converges. Then,

\[
f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (s-1)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{(s-1)^k}{k!} \sum_{n=1}^{\infty} (-\lambda_n)^k a_n e^{-\lambda n}
\]
\[ = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n} \]

\[ = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \sum_{k=0}^{\infty} \frac{(1-s)^k \lambda_n^k}{k!}, \]

where the inversion in order of summation is justified since all of the series terms are non-negative. From the above, it follows that

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} e^{(1-s)\lambda_n} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \]

which is a contradiction, since \( s < 0 \) and \( \sigma_0 = 0 \). Hence, \( s = 0 \) must be a singularity of \( f(s) \).

As an example, \( \zeta(s) \) has a singularity at \( s = 1 \).

**Theorem 9.** For a given sequence \( \{\lambda_n\} \), a function possesses at most one Dirichlet series representation.

**Proof.** Suppose that

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} b_n e^{-\lambda_n s} \]

for \( \sigma > c \). Then
\[
\sum_{n=1}^{\infty} (a_n - b_n) e^{-\lambda_n s}
\]
converges uniformly to 0 in some half-plane. Fix \( s \) with \( \sigma > c \). Let \( m \) be the least integer such that \( a_m \neq b_m \), i.e., \( a_n = b_n \) if \( n < m \). Without loss of generality, assume that \( \lambda_n \geq 0 \) for \( n \geq m \). Hence, if \( c < \alpha < \sigma \),

\[
\left| \sum_{n=1}^{\infty} (a_n - b_n) e^{-\lambda_n s} \right| \geq \left| a_m - b_m \right| e^{-\lambda_m \sigma} - \sum_{n=m+1}^{\infty} \left| a_n - b_n \right| e^{-\lambda_n \sigma} \\
\geq \left| a_m - b_m \right| e^{-\lambda_m \sigma} - e^{-\lambda_{m+1} (\alpha - \sigma)} \sum_{n=m+1}^{\infty} \left| a_n - b_n \right| e^{-\lambda_n \alpha} \\
= c_1 e^{-\lambda_m \sigma} - c_2 e^{-\lambda_{m+1} \sigma} > 0,
\]

if \( \sigma \) is large enough, since \( \lambda_{m+1} > \lambda_m \), where \( c_1 = \left| a_m - b_m \right| \) and \( c_2 = e^{\lambda_{m+1} \alpha} \sum_{n=m+1}^{\infty} \left| a_n - b_n \right| e^{-\lambda_n \alpha} \). But

\[
\sum_{n=1}^{\infty} (a_n - b_n) e^{-\lambda_n s} = 0,
\]
and so we have reached a contradiction. Thus, \( a_n = b_n \) for every \( n \geq 1 \).
Corollary 10. Every Dirichlet series, not identically zero, has a half-plane free of zeros.

Proof. Proceed by identically the same argument as in the proof of Theorem 9 with \((a_n - b_n)\) replaced by \(a_n\). For \(\sigma\) large enough we find that

\[
\left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \right| > 0,
\]

and the result follows.

Theorem 11. Let \(f(s) = \sum_{n=1}^{\infty} a_n n^{-s}\) and \(g(s) = \sum_{n=1}^{\infty} b_n n^{-s}\) have abscissae of absolute convergence \(\sigma_{a_1}\) and \(\sigma_{a_2}\), respectively. Then for \(\sigma > \sup \{\sigma_{a_1}, \sigma_{a_2}\}\),

\[
f(s)g(s) = \sum_{n=1}^{\infty} c_n n^{-s},
\]

where

\[
c_n = \sum_{j,k=n} a_j b_k.
\]
Proof. For $\sigma > \text{sup} \{\sigma_{a1}, \sigma_{a2}\}$, both $f(s)$ and $g(s)$ converge absolutely. Thus, when we form the product of $f$ and $g$, rearrangement of terms is justified. Hence, for $\sigma > \text{sup} \{\sigma_{a1}, \sigma_{a2}\},$

$$f(s)g(s) = \sum_{j,k=1}^{\infty} a_j b_k (jk)^{-s}$$

$$= \sum_{n=1}^{\infty} \left( \sum_{jk=n} a_j b_k \right) n^{-s}$$

$$= \sum_{n=1}^{\infty} c_n n^{-s}.$$

References
