BESSEL'S DIFFERENTIAL EQUATION OF ORDER 0

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1. Review of Bessel's Differential Equation of Order p

Recall that Bessel's differential equation of order $p \geq 0$ is given by

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{1.1}$$

It is easily checked that $x = 0$ is a regular singular point. Since

$$x \cdot \frac{1}{x} = 1, \qquad x^2 \cdot \frac{x^2 - p^2}{x^2} = -p^2 + \cdots,$$

the indicial polynomial is $r(r-1) + r - p^2$, which has the indicial roots $\pm p$. The differential equation (1.1) has no other singularities. Therefore, series solutions will have an infinite radius of convergence, which, when we find solutions, is easily verified by using the ratio test. We shall assume in the sequel that $x > 0$. For any $p \geq 0$, we find that one solution of (1.1) is the Bessel function of order p defined by

$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{1.2}$$

If p is not a nonnegative integer, then a second solution is given by

$$J_{-p}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-p+n+1)} \left(\frac{x}{2}\right)^{2n-p}. \tag{1.3}$$

The general solution of (1.1) in these cases is thus given by

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x), \tag{1.4}$$

where c_1 and c_2 are arbitrary constants. There remains to find a second solution of (1.1) when the indicial root is $0, -1, -2, \ldots$.

Let

$$y = \sum_{n=0}^{\infty} a_n(r) x^{n+r}, \quad x > 0. \tag{1.5}$$

Because r will assume different values, we want to emphasize that the coefficients $a_n(r)$ depend on the parameter r. If we substitute (1.5) into (1.1) and equate coefficients, we find that for $n \geq 2$,

$$a_n(r) = -\frac{a_{n-2}(r)}{(n+r)^2 - p^2}, \qquad n \geq 2. \tag{1.6}$$

2. The case $p = 0$, first method

We shall find the second solution of (1.1) when $p = 0$. In this section, we use a method depending on the differentiation of (1.6) with respect to r. This method can be modified to find the second solution of (1.1) when the indicial root is $-1, -2, \ldots$. In particular, problem #11 in Section 5.7 of our text asks that the second solution be calculated for the indicial root -1. Here, we must use the the authors' description of the method on pages 292–294. See my solution of this problem in the solutions to Homework #11 As you will see, finding this solution will be slightly more difficult than finding the solution for $r = 0$.

We review our basic theorem when the indicial roots are equal, and we shall moreover assume below that $p = 0$. The second solution has the form

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n + J_0(x) \log x, \tag{2.1}$$

where the coefficients b_n are to be determined.

We found a second way to find the second solution of (1.1) by using (1.6). Namely,

$$y_2(x) = \sum_{n=0}^{\infty} a_n'(0) x^n + J_0(x) \log x. \tag{2.2}$$

We emphasize that $a_0(r)$ is always equal to a constant, and so $a_0'(r) = 0$. Also, we saw that when we started to equate coefficients, $a_1(r) = 0$, and so, trivially, $a_1'(r) = 0$. From the recurrence relation, we easily conclude that $a_{2n+1}(r) = 0$, for each $n \geq 1$, and, trivially, $a_{2n+1}'(r) = 0$, $n \geq 0$. Thus, we need only calculate the coefficients of even index.

Let $p = 0$ in (1.6), and so it becomes

$$a_{2n}(r) = -\frac{a_{2n-2}(r)}{(r+2n)^2}, \quad n \geq 1. \tag{2.3}$$

Using (2.3) successively and letting $a_0(r) = 1$, we find that

$$\begin{aligned}
a_2(r) &= -\frac{1}{(r+2)^2}, \\
a_4(r) &= -\frac{a_2(r)}{(r+4)^2} = \frac{1}{(r+2)^2(r+4)^2}, \\
a_6(r) &= -\frac{a_4(r)}{(r+6)^2} = -\frac{1}{(r+2)^2(r+4)^2(r+6)^2}.
\end{aligned}$$

In general,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)(r+4)^2 \cdots (r+2n)^2}, \quad n \geq 1, \tag{2.4}$$

which can easily be proved by induction on n.

We now take a time-out to do an exercise in calculus. Let

$$f(x) := (x+\alpha_1)^{\beta_1}(x+\alpha_2)^{\beta_2} \cdots (x+\alpha_n)^{\beta_n},$$

where $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$ are constants. By the product rule,

$$\begin{aligned} f'(x) = &\beta_1(x+\alpha_1)^{\beta_1-1}(x+\alpha_2)^{\beta_2}\cdots(x+\alpha_n)^{\beta_n} \\ &+ \beta_2(x+\alpha_1)^{\beta_1}(x+\alpha_2)^{\beta_2-1}\cdots(x+\alpha_n)^{\beta_n} \\ &+ \dots \\ &+ \beta_n(x+\alpha_1)^{\beta_1}(x+\alpha_2)^{\beta_2}\cdots(x+\alpha_n)^{\beta_n-1}. \end{aligned}$$

Thus,

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x+\alpha_1} + \frac{\beta_2}{x+\alpha_2} + \cdots + \frac{\beta_n}{x+\alpha_n}. \tag{2.5}$$

Applying (2.5) to (2.4) with $\beta_j = -2$, $1 \le j \le n$, and $\alpha_j = r + 2j$, $1 \le j \le n$, we find that

$$\frac{a'_{2n}(r)}{a_{2n}(r)} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right). \tag{2.6}$$

Let $r = 0$ in (2.6) to obtain

$$a'_{2n}(0) = -2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) a_{2n}(0) = -H_n a_{2n}(0), \tag{2.7}$$

where

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad n \ge 1. \tag{2.8}$$

(When we calculate the second solution when $p = 1$, we will see that β_j is not constant. Arising from (2.5), there will be two harmonic series.) But now recall that the coefficients $a_{2n}(0)$, $n \ge 0$, are those in the solution $J_0(x)$. We previously calculated them to be

$$a_{2n}(0) = \frac{(-1)^n}{2^{2n}(n!)^2} \quad n \ge 1. \tag{2.9}$$

Putting (2.9) into (2.7), we conclude that

$$a'_{2n}(0) = -\frac{(-1)^n H_n}{2^{2n}(n!)^2}, \quad n \ge 1.$$

Thus, our second solution of (1.1) with $p = 0$ is given by

$$y_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2^{2n}(n!)^2} + J_0(x) \log x.$$

3. The case $p = 0$, second method

We calculate the coefficients b_n, $n \geq 1$, in (2.1) by substituting (2.1) into (1.1) with $p = 0$. Thus,

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n + J_0(x) \log x, \tag{3.1}$$

$$y_2'(x) = \sum_{n=1}^{\infty} b_n n x^{n-1} + J_0'(x) \log x + \frac{J_0(x)}{x}, \tag{3.2}$$

$$y_2''(x) = \sum_{n=2}^{\infty} b_n (n-1) n x^{n-2} + J_0''(x) \log x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}. \tag{3.3}$$

Substituting (3.1)–(3.3) into (1.1) with $p = 0$, we find that

$$\begin{aligned} L[y_2(x)] = & \sum_{n=2}^{\infty} b_n (n-1) n x^n + x^2 J_0''(x) \log x + 2x J_0'(x) - J_0(x) \\ & + \sum_{n=1}^{\infty} b_n n x^n + x J_0'(x) \log x + J_0(x) + \sum_{n=0}^{\infty} b_n x^{n+2} + x^2 J_0(x) \log x. \end{aligned} \tag{3.4}$$

The terms $\pm J_0(x)$ cancel. If we examine the terms multiplying $\log x$, we find that their sum equals 0, because these terms are simply the left-hand side of (1.1) ($p = 0$) with $y(x)$ replaced by $J_0(x)$, which is a solution of this differential equation! In the last sum on the right-hand side of (3.4), replace n by $n - 2$. With still more simplification in (3.4), we find that

$$b_1 x + \sum_{n=2}^{\infty} \{n^2 b_n + b_{n-2}\} x^n + 2x J_0'(x) = 0.$$

Upon differentiation of (1.2) ($p = 0$), we find that we can write the equation above in the form

$$b_1 x + \sum_{n=2}^{\infty} \{n^2 b_n + b_{n-2}\} x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{(n!)^2 2^{2n}}. \tag{3.5}$$

Equate coefficients on both sides of (3.5). Observe that there are no odd powers of x on the right-hand side of (3.5). Thus, $b_1 = 0$, and for n odd and $n \geq 3$,

$$b_n = -\frac{b_{n-2}}{n^2}.$$

Since $b_1 = 0$, we see that $b_3 = 0$, $b_5 = 0$, etc., i.e., $b_{2n+1} = 0, n \geq 0$. Replacing n by $2n$ and equating coefficients of x^{2n} in (3.5), we find the recurrence relation

$$(2n)^2 b_{2n} + b_{2n-2} = \frac{(-1)^{n+1} n}{(n!)^2 2^{2n-2}}, \quad n \geq 1. \tag{3.6}$$

Note that b_0 is undetermined. We shall take $b_0 = 0$. Letting $n = 1$ in (3.6), we find that

$$2^2 b_2 = 1 \quad \Rightarrow \quad b_2 = \frac{1}{2^2}. \tag{3.7}$$

Letting $n = 2$ in (3.6) and using (3.7), we have

$$4^2 b_4 + b_2 = -\frac{2}{(2!)^2 2^2} \quad \Rightarrow \quad b_4 = \frac{1}{4^2}\left(-\frac{1}{2^2} - \frac{2}{(2!)^2 2^2}\right) = -\frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right). \qquad (3.8)$$

Letting $n = 3$ in (3.6) and using (3.8), we find that

$$6^2 b_6 + b_4 = \frac{3}{(3!)^2 2^4},$$

or

$$b_6 = \frac{1}{6^2}\left(\frac{3}{(3!)^2 2^4} + \frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right)\right)$$
$$= \frac{1}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right). \qquad (3.9)$$

From our calculations (3.7), (3.8), and (3.9), we see that

$$b_{2n} = \frac{(-1)^{n+1}}{2^2 4^2 \cdots (2n)^2} H_n = \frac{(-1)^{n+1}}{(n!)^2 2^{2n}} H_n, \quad n \geq 1, \qquad (3.10)$$

where H_n is defined by (2.8). Hence, from (3.10) and (2.1), our second solution of Bessel's differential equation of order 0 is given by

$$y_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} H_n \left(\frac{x}{2}\right)^{2n} + J_0(x) \log x.$$