1. Introduction

Recall that Bessel’s differential equation of order \( p \) is defined by

\[
L[y] := x^2 y'' + xy' + (x^2 - p^2)y = 0.
\]

(1.1)

Actually, \( p \) can be any complex number, but we shall always assume that \( p \geq 0 \). The roots of the indicial equation are \( \pm p \). We have shown that, for any real nonnegative number \( p \), one solution of (1.1) is given by the Bessel function of order \( p \), namely,

\[
J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p + n + 1)} \left( \frac{x}{2} \right)^{2n+p},
\]

(1.2)

where \( \Gamma(s) \) denotes the Gamma function. If \( p \) is not a nonnegative integer, then a second solution of (1.1) is given by the Bessel function of order \( p \) is given by \( J_{-p}(x) \), namely,

\[
J_{-p}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(-p + n + 1)} \left( \frac{x}{2} \right)^{2n-p}.
\]

Thus, the general solution of (1.1) is given by

\[
y(x) = c_1 J_p(x) + c_2 J_{-p}(x),
\]

(1.3)

where \( c_1 \) and \( c_2 \) are arbitrary constants. When the indicial roots \( \pm p \) differ by a positive integer, then we know from the theory of second order differential equations with a regular singular point that a logarithm may be involved in the second solution. There are two cases. First, when \( p - (-p) = 2p \) is an odd positive integer, then we saw that, in fact, the second solution did not involve a logarithm, and hence that (1.3) gives the general solution in this case. The second case is when \( p - (-p) = 2p \) is an even nonnegative integer, i.e., \( p \) is a nonnegative integer. When \( p = 0 \), we found that a second solution of (1.1) is

\[
Y_0(x) := \sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(n!)^2} \left( \frac{x}{2} \right)^{2n} + \log x J_0(x),
\]

(1.4)

where \( H_n \) is the harmonic sum

\[
H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n \geq 1; H_0 = 1.
\]

(1.5)

2. The Second Solution When \( p \) Is a Positive Integer

We have yet to consider the case when \( p > 0 \) is an integer. The second solution of (1.1) is of the form

\[
y(x) = x^{-p} \sum_{n=0}^{\infty} a_n x^n + C \log x J_p(x)
\]

\[
= \sum_{n=0}^{\infty} a_n x^{n-p} + C \log x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + p)!} \left( \frac{x}{2} \right)^{2n+p},
\]

(2.1)
since \( \Gamma(n) = n! \) when \( n \) is a positive integer, and where \( C \) is a constant yet to be determined. We will put (2.1) in Bessel’s differential equation (1.1) of order \( p \). Thus, by the product rule,

\[
y'(x) = \sum_{n=0}^{\infty} a_n (n-p)x^{n-p-1} + \frac{C}{x} J_p(x) + C \log x J'_p(x) \tag{2.2}
\]

and

\[
y''(x) = \sum_{n=0}^{\infty} a_n (n-p)(n-p-1)x^{n-p-2} + \frac{2C}{x} J'_p(x) - \frac{C}{x^2} J_p(x) + C \log x J''_p(x). \tag{2.3}
\]

Hence, putting (2.1), (2.2), and (2.3) into (1.1), we find that

\[
L[y] = \sum_{n=0}^{\infty} a_n (n-p)(n-p-1)x^{n-p} + 2Cx J'_p(x) - CJ_p(x) + Cx^2 \log x J''_p(x) \\
+ \sum_{n=0}^{\infty} a_n (n-p)x^{n-p} + CJ_p(x) + Cx \log x J'_p(x) \\
+ \sum_{n=0}^{\infty} a_n x^{n-p+2} + Cx^2 \log x J_p(x) - p^2 \sum_{n=0}^{\infty} a_n x^{n-p} - Cp^2 \log x J_p(x) \\
= \sum_{n=0}^{\infty} [a_n (n-p)(n-p-1) + a_n (n-p) - p^2 a_n] x^{n-p} + \sum_{n=0}^{\infty} a_n x^{n-p+2} \\
+ C \log x [x^2 J''_p(x) + x J'_p(x) + (x^2 - p^2) J_p(x)] + 2Cx J'_p(x) \\
= \sum_{m=0}^{\infty} [a_m (m-p)(m-p-1) + a_m (m-p) - p^2 a_m] x^{m-p} + \sum_{m=2}^{\infty} a_{m-2} x^{m-p} \\
+ 2C \sum_{n=0}^{\infty} \frac{(-1)^n (2n+p)}{n!(n+p)!} \left( \frac{x}{2} \right)^{2n+p}, \tag{2.4}
\]

where in the first two sums of the second equality we set \( n = m \), in the second sum we set \( n = m - 2 \), and lastly we used the fact that \( J_p(x) \) is a solution of Bessel’s equation (1.1) of order \( p \). Note that the second series begins with \( m = 2 \), while the first series begins with \( m = 0 \). In the last series of (2.4), we set \( n = m - p \). Thus,

\[
L[y] = a_0 [-p(p-1) - p^2] x^{-p} + a_1 [(1-p)(-p) + (1-p) - p^2] x^{-p+1} \\
+ \sum_{m=2}^{\infty} [a_m (m-p)(m-p-1) + a_m (m-p) - a_m p^2 + a_{m-2}] x^{m-p} \\
+ 2C \sum_{m=p}^{\infty} \frac{(-1)^m (2m-p)}{(m-p)!m!} \left( \frac{x}{2} \right)^{2m-p} = 0 \tag{2.5}
\]

Since we want \( y(x) \) to be a solution of Bessel’s equation, we equate coefficients to 0 in (2.5). First, we find that

\[
a_0 \cdot 0 = 0, \\
a_1 [1 - 2p] = 0.
\]
The first equality above reflects the fact that \( p \) is an indicial root. Thus, \( a_0 \) is arbitrary, and \( a_1 = 0 \) for all integers \( p \geq 1 \). Secondly, for \( 2 \leq m < 2p \),
\[
a_m[(m-p)(m-p-1)+(m-p)-p^2]+a_{m-2}=0,
\]
or, for \( 2 \leq m < 2p \),
\[
a_m[(m-p^2)-p^2]=-a_{m-2}, \tag{2.6}
\]
or, for \( 2 < m < 2p \),
\[
a_m=-\frac{a_{m-2}}{m(m-2p)}. \tag{2.7}
\]
Since \( a_1 = 0 \), we see from (2.7) that \( a_3 = a_5 = \cdots = a_{2p-1} = 0 \). Thus, since only those coefficients with even indices are not equal to 0, we replace \( m \) by \( 2m \) in (2.7) to find that, for \( 1 \leq m < p \),
\[
a_{2m} = -\frac{a_{2m-2}}{4m(m-p)}. \tag{2.8}
\]
Now let \( m \geq p \). We see from (2.5) that (except for the factor \( x^{-p} \)) there are no powers of \( x \) that are odd in the last sum. Thus, since \( a_{2m-1} = 0 \) for \( m < p \), we see that all of the odd indexed coefficients are equal to 0. Hence,
\[
a_{2m}4m(m-p)+a_{2m-2}+2C\frac{(-1)^{m-p}(2m-p)}{(m-p)!m!2^{m-p}}=0,
\]
or, for \( m \geq p \),
\[
a_{2m}4m(m-p)=-a_{2m-2}-2C\frac{(-1)^{m-p}(2m-p)}{(m-p)!m!2^{m-p}}. \tag{2.9}
\]
Let \( m = n + p \) in (2.9). Then, for \( n \geq 0 \),
\[
a_{2n+2p}(n+p)n=-a_{2n+2p-2}-2C\frac{(-1)^n(2n+p)}{n!(n+p)!2^{n+p}}. \tag{2.10}
\]
Let \( n = 0 \) in (2.10). Then,
\[
a_{2p} \cdot 0 = -a_{2p-2} - \frac{2Cp}{p!2p} = -a_{2p-2} - \frac{C}{(p-1)!2p+1}. \tag{2.11}
\]
Thus,
\[
C = -(p-1)!2^{p-1}a_{2p-2}. \tag{2.12}
\]
Observe that \( a_{2p} \) is arbitrary. Thus, we have 2 arbitrary coefficients, \( a_0 \) and \( a_{2p} \). If we set \( a_0 = a_{2p} = 0 \), we will find that \( y(x) = 0 \). Of course, it is obvious from the start that this is a solution and is of no value to us. Thus, at least one of \( a_0 \) and \( a_{2p} \) is not equal to 0. Suppose that \( a_0 = 0 \). Then, from (2.7), we see that \( a_2 = a_4 = \cdots = a_{2p-2} = 0 \). Hence, from (2.12), we also see that \( C = 0 \). Thus, (2.10) takes the simpler form
\[
a_{2n+2p}(n+p)n=-a_{2n+2p-2}. \tag{2.13}
\]
Let \( n = 1 \) in (2.13). Thus,
\[
a_{2+2p}(1+p) = -a_{2p}. \tag{2.14}
\]
Remember that \( a_{2p} \) is arbitrary, except that we cannot let \( a_{2p} = 0 \). Let us set
\[
a_p = \frac{1}{2p\Gamma(p+1)} = \frac{1}{2p!},
\]
Hence, from (2.14),
\[
a_{2+2p}(1+p) = -\frac{1}{2p!},
\]
or
\[ a_{2+2p} = -\frac{1}{2^{p+2}(p + 1)!}. \]

Set \( n = 2 \) in (2.13) to deduce that
\[ a_{4+2p} = -\frac{a_{2+2p}}{4 \cdot 2(p + 2)} = -\frac{1}{2^{p+4}2!(p + 2)!}. \]

Set \( n = 3 \) in (2.13) to deduce that
\[ a_{6+2p} = -\frac{a_{4+2p}}{4 \cdot 3(p + 3)} = -\frac{1}{2^{p+6}3!(p + 3)!}. \]

From our calculations above, we can see a pattern evolving, namely, for \( n \geq 0, \)
\[ a_{2n+2p} = \frac{(-1)^n}{2^{np}2n!(n + p)!}. \]

Hence, or solution is
\[
y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p+p}}{2^{p+2n}n!(n + p)!} \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + p)!} \left( \frac{x}{2} \right)^{2n+p} = J_p(x). \]

Grown! This is useless. We already know that \( J_p(x) \) is a solution of (1.1). Our goal was to find a second linearly independent solution of Bessel’s differential equation. Thus, if \( a_0 = 0, \) we have failed. Since we cannot take \( a_0 = 0, \) let us take \( a_0 = 1 \) and \( a_{2p} = 0. \) Let \( m = 1, 2, 3 \) in (2.8) to find that
\[
a_2 = -\frac{a_0}{4(1 - p)} = -\frac{1}{2^2(1 - p)}, \\
a_4 = -\frac{a_2}{4 \cdot 2(2 - p)} = -\frac{1}{2^42!(1 - p)(2 - p)}, \\
a_6 = -\frac{a_4}{4 \cdot 3(3 - p)} = -\frac{1}{2^63!(1 - p)(2 - p)(3 - p)}. \]

Thus, for \( 1 \leq n < p, \) it appears that
\[ a_{2n} = \frac{(-1)^n}{2^{2n}n!(1 - p)(2 - p) \cdots (n - p)}, \tag{2.15} \]
which is easily proved by induction on \( n. \) In particular, if we let \( n = p - 1 \) in (2.15), we find that
\[
a_{2p-2} = \frac{(-1)^{p-1}}{2^{2p-2}(p - 1)!(1 - p)(2 - p) \cdots (-1)} \\
= \frac{1}{2^{2p-2}((p - 1)!)^2}. \tag{2.16} \]

Now return to (2.12) and use the value of \( a_{2p-2} \) from (2.16) to deduce that
\[ C = -\frac{(p - 1)!2^{p-1}}{2^{2p-2}((p - 1)!)^2} = -\frac{1}{2^{p-1}(p - 1)!}. \]
Our recurrence relation (2.10) then takes the form, for \( n \geq 1 \),
\[
a_{2n+2p} 4n(n + p) = -a_{2n+2p-2} + \frac{2(2n + p)(-1)^n}{2^{p-1}(p - 1)!n!(n + p)!2^{2n+p}}
= -a_{2n+2p-2} + \frac{(2n + p)(-1)^n}{2^{2n+2p-2}(p - 1)!n!(n + p)!}.
\]
(2.17)
The recurrence relation also holds for \( n = 0 \). Thus, if we set \( n = 0 \) in (2.17) and employ (2.16), we find that
\[
a_{2p} = -\frac{1}{2^{2p-2}((p - 1)!)^2} + \frac{p}{2^{2p-2}(p - 1)!p!}
= -\frac{1}{2^{2p-2}((p - 1)!)^2} + \frac{1}{2^{2p-2}((p - 1)!)^2} = 0,
\]
which is correct since we had previously set \( a_{2p} = 0 \). The recurrence (2.17) can be used to calculate the coefficients of our second solution for \( n \geq 1 \). The general formula involves 2 harmonic sums (1.5), and we will not proceed further.