SOLVING BESSEL'S DIFFERENTIAL EQUATIONS

1. Introduction

Recall that Bessel's differential equation of order p is defined by

$$L[y] := x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{1.1}$$

Actually, p can be any complex number, but we shall always assume that $p \geq 0$. The roots of the indicial equation are $\pm p$. We have shown that, for any real nonnegative number p, one solution of (1.1) is given by the Bessel function of order p, namely,

$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \tag{1.2}$$

where $\Gamma(s)$ denotes the Gamma function. If p is not a nonnegative integer, then a second solution of (1.1) is given by the Bessel function of order p is given by $J_{-p}(x)$, namely,

$$J_{-p}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-p+n+1)} \left(\frac{x}{2}\right)^{2n-p}.$$

Thus, the general solution of (1.1) is given by

$$y(x) = c_1J_p(x) + c_2J_{-p}(x), \tag{1.3}$$

where c_1 and c_2 are arbitrary constants. When the indicial roots $\pm p$ differ by a positive integer, then we know from the theory of second order differential equations with a regular singular point that a logarithm may be involved in the second solution. There are two cases. First, when $p - (-p) = 2p$ is an odd positive integer, then we saw that, in fact, the second solution did not involve a logarithm, and hence that (1.3) gives the general solution in this case. The second case is when $p - (-p) = 2p$ is an even nonnegative integer, i.e., p is a nonnegative integer. When $p = 0$, we found that a second solution of (1.1) is

$$Y_o(x) := \sum_{n=0}^{\infty} \frac{(-1)^n H_n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} + \log x J_0(x), \tag{1.4}$$

where H_n is the harmonic sum

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n \geq 1; H_0 = 1. \tag{1.5}$$

2. The Second Solution When p Is a Positive Integer

We have yet to consider the case when $p > 0$ is an integer. The second solution of (1.1) is of the form

$$\begin{aligned} y(x) &= x^{-p}\sum_{n=0}^{\infty} a_n x^n + C\log x J_p(x) \\ &= \sum_{n=0}^{\infty} a_n x^{n-p} + C \log x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}, \end{aligned} \tag{2.1}$$

since $\Gamma(n) = n!$ when n is a positive integer, and where C is a constant yet to be determined. We will put (2.1) in Bessel's differential equation (1.1) of order p. Thus, by the product rule,

$$y'(x) = \sum_{n=0}^{\infty} a_n(n-p)x^{n-p-1} + \frac{C}{x}J_p(x) + C\log x J_p'(x) \tag{2.2}$$

and

$$y''(x) = \sum_{n=0}^{\infty} a_n(n-p)(n-p-1)x^{n-p-2} + \frac{2C}{x}J_p'(x) - \frac{C}{x^2}J_p(x) + C\log x J_p''(x). \tag{2.3}$$

Hence, putting (2.1), (2.2), and (2.3) into (1.1), we find that

$$\begin{aligned}
L[y] &= \sum_{n=0}^{\infty} a_n(n-p)(n-p-1)x^{n-p} + 2Cx\,J_p'(x) - CJ_p(x) + Cx^2\log x J_p''(x) \\
&\quad + \sum_{n=0}^{\infty} a_n(n-p)x^{n-p} + CJ_p(x) + Cx\log x J_p'(x) \\
&\quad + \sum_{n=0}^{\infty} a_n x^{n-p+2} + Cx^2\log x J_p(x) - p^2\sum_{n=0}^{\infty} a_n x^{n-p} - Cp^2\log x J_p(x) \\
&= \sum_{n=0}^{\infty}[a_n(n-p)(n-p-1) + a_n(n-p) - p^2 a_n]x^{n-p} + \sum_{n=0}^{\infty} a_n x^{n-p+2} \\
&\quad + C\log x[x^2J_p''(x) + xJ_p'(x) + (x^2-p^2)J_p(x)] + 2CxJ_p'(x) \\
&= \sum_{m=0}^{\infty}[a_m(m-p)(m-p-1) + a_m(m-p) - p^2 a_m]x^{m-p} + \sum_{m=2}^{\infty} a_{m-2}x^{m-p} \\
&\quad + 2C\sum_{n=0}^{\infty}\frac{(-1)^n(2n+p)}{n!(n+p)!}\left(\frac{x}{2}\right)^{2n+p},
\end{aligned} \tag{2.4}$$

where in the first two sums of the second equality we set $n = m$, in the second sum we set $n = m-2$, and lastly we used the fact that $J_p(x)$ is a solution of Bessel's equation (1.1) of order p . Note that the second series begins with $m = 2$, while the first series begins with $m = 0$. In the last series of (2.4), we set $n = m - p$. Thus,

$$\begin{aligned}
L[y] &= a_0[-p(-p-1) - p - p^2]x^{-p} + a_1[(1-p)(-p) + (1-p) - p^2]x^{-p+1} \\
&\quad + \sum_{m=2}^{\infty}[a_m(m-p)(m-p-1) + a_m(m-p) - a_m p^2 + a_{m-2}]x^{m-p} \\
&\quad + 2C\sum_{m=p}^{\infty}\frac{(-1)^{m-p}(2m-p)}{(m-p)!m!}\left(\frac{x}{2}\right)^{2m-p} = 0.
\end{aligned} \tag{2.5}$$

Since we want $y(x)$ to be a solution of Bessel's equation, we equate coefficients to 0 in (2.5). First, we find that

$$a_0 \cdot 0 = 0,$$
$$a_1[1-2p] = 0.$$

The first equality above reflects the fact that p is an indicial root. Thus, a_0 is arbitrary, and $a_1 = 0$ for all integers $p \geq 1$. Secondly, for $2 \leq m < 2p$,

$$a_m[(m-p)(m-p-1)+(m-p)-p^2]+a_{m-2}=0,$$

or, for $2 \leq m < 2p$,

$$a_m[(m-p^2)-p^2] = -a_{m-2}, \tag{2.6}$$

or, for $2 < m < 2p$,

$$a_m = -\frac{a_{m-2}}{m(m-2p)}. \tag{2.7}$$

Since $a_1 = 0$, we see from (2.7) that $a_3 = \alpha_5 = \cdots = a_{2p-1} = 0$. Thus, since only those coefficients with even coefficients are not equal to 0, we replace m by $2m$ in (2.7) to find that, for $1 \leq m < p$,

$$a_{2m} = -\frac{a_{2m-2}}{4m(m-p)}. \tag{2.8}$$

Now let $m \geq p$. We see from (2.5) that (except for the factor x^{-p}) there are no powers of x that are odd in the last sum. Thus, since $a_{2m-1} = 0$ for $m < p$, we see that all of the odd indexed coefficients are equal to 0. Hence,

$$a_{2m}4m(m-p)+a_{2m-2}+2C\frac{(-1)^{m-p}(2m-p)}{(m-p)!m!2^{2m-p}}=0,$$

or, for $m \geq p$,

$$a_{2m}4m(m-p) = -a_{2m-2}-2C\frac{(-1)^{m-p}(2m-p)}{(m-p)!m!2^{2m-p}}. \tag{2.9}$$

Let $m = n+p$ in (2.9). Then, for $n \geq 0$,

$$a_{2n+2p}4(n+p)n = -a_{2n+2p-2}-2C\frac{(-1)^n(2n+p)}{n!(n+p)!2^{2n+p}}. \tag{2.10}$$

Let $n = 0$ in (2.10). Then,

$$a_{2p}\cdot 0 = -a_{2p-2}-\frac{2Cp}{p!2^p} = -a_{2p-2}-\frac{C}{(p-1)!2^{p-1}}. \tag{2.11}$$

Thus,

$$C = -(p-1)!2^{p-1}a_{2p-2}. \tag{2.12}$$

Observe that a_{2p} is arbitrary. Thus, we have 2 arbitrary coefficients, a_0 and a_{2p}. If we set $a_0 = a_{2p} = 0$, we will find that $y(x) \equiv 0$. Of course, it is obvious from the start that this is a solution and is of no value to us. Thus, at least one of a_0 and a_{2p} is not equal to 0. Suppose that $a_0 = 0$. Then, from (2.7), we see that $a_2 = a_4 = \cdots = a_{2p-2} = 0$. Hence, from (2.12), we also see that $C = 0$. Thus, (2.10) takes the simpler form

$$a_{2n+2p}4(n+p)n = -a_{2n+2p-2}. \tag{2.13}$$

Let $n = 1$ in (2.13). Thus,

$$a_{2+2p}4(1+p) = -a_{2p}. \tag{2.14}$$

Remember that a_{2p} is arbitrary, except that we cannot let $a_{2p} = 0$. Let us set

$$a_p := \frac{1}{2^p\Gamma(p+1)} = \frac{1}{2^p p!}.$$

Hence, from (2.14),

$$a_{2+2p}4(1+p) = -\frac{1}{2^p p!},$$

or

$$a_{2+2p} = -\frac{1}{2^{p+2}(p+1)!}.$$

Set $n = 2$ in (2.13) to deduce that

$$a_{4+2p} = -\frac{a_{2+2p}}{4 \cdot 2(p+2)} = \frac{1}{2^{p+4}2!(p+2)!}.$$

Set $n = 3$ in (2.13) to deduce that

$$a_{6+2p} = -\frac{a_{4+2p}}{4 \cdot 3(p+3)} = -\frac{1}{2^{p+6}3!(p+3)!}.$$

From our calculations above, we can see a pattern evolving, namely, for $n \geq 0$,

$$a_{2n+2p} = \frac{(-1)^n}{2^{p+2n}n!(n+p)!}.$$

Hence, or solution is

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-p}}{2^{p+2n}n!(n+p)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p} = J_p(x). \end{aligned}$$

Grown! This is useless. We already know that $J_p(x)$ is a solution of (1.1). Our goal was to find a second linearly independent solution of Bessel's differential equation. Thus, if $a_0 = 0$, we have failed. Since we cannot take $a_0 = 0$, let us take $a_0 = 1$ and $a_{2p} = 0$. Let $m = 1, 2, 3$ in (2.8) to find that

$$\begin{aligned} a_2 &= -\frac{a_0}{4(1-p)} = -\frac{1}{2^2(1-p)}, \\ a_4 &= -\frac{a_2}{4 \cdot 2(2-p)} = \frac{1}{2^4 2!(1-p)(2-p)}, \\ a_6 &= -\frac{a_4}{4 \cdot 3(3-p)} = -\frac{1}{2^6 3!(1-p)(2-p)(3-p)}. \end{aligned}$$

Thus, for $1 \leq n < p$, it appears that

$$a_{2n} = \frac{(-1)^n}{2^{2n}n!(1-p)(2-p)\cdots(n-p)}, \tag{2.15}$$

which is easily proved by induction on n. In particular, if we let $n = p - 1$ in (2.15), we find that

$$\begin{aligned} a_{2p-2} &= \frac{(-1)^{p-1}}{2^{2p-2}(p-1)!(1-p)(2-p)\cdots(-1)} \\ &= \frac{1}{2^{2p-2}((p-1)!)^2}. \end{aligned} \tag{2.16}$$

Now return to (2.12) and use the value of a_{2p-2} from (2.16) to deduce that

$$C = -\frac{(p-1)!2^{p-1}}{2^{2p-2}((p-1)!)^2} = -\frac{1}{2^{p-1}(p-1)!}.$$

Our recurrence relation (2.10) then takes the form, for $n \geq 1$,

$$\begin{aligned} a_{2n+2p}4n(n+p) &= -a_{2n+2p-2} + \frac{2(2n+p)(-1)^n}{2^{p-1}(p-1)!n!(n+p)!2^{2n+p}} \\ &= -a_{2n+2p-2} + \frac{(2n+p)(-1)^n}{2^{2n+2p-2}(p-1)!n!(n+p)!}. \end{aligned} \tag{2.17}$$

The recurrence relation also holds for $n = 0$. Thus, if we set $n = 0$ in (2.17) and employ (2.16), we find that

$$\begin{aligned} a_{2p} &= -\frac{1}{2^{2p-2}((p-1)!)^2} + \frac{p}{2^{2p-2}(p-1)!p!} \\ &= -\frac{1}{2^{2p-2}((p-1)!)^2} + \frac{1}{2^{2p-2}((p-1)!)^2} = 0, \end{aligned}$$

which is correct since we had previously set $a_{2p} = 0$. The recurrence (2.17) can be used to calculate the coefficients of our second solution for $n \geq 1$. The general formula involves 2 harmonic sums (1.5), and we will not proceed further.