LECTURE GIVEN BY B. M. WILSON
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BRUCE C. BERNDT

1. INTRODUCTION

Reproduced verbatim below is a lecture by B. M. Wilson, one of the three editors of Ramanujan’s *Collected Papers* [13], given at the University of Leeds in (probably) May, 1927. It might be noted that L. J. Rogers had been a faculty member at the University of Leeds; he died in 1933. A handwritten copy of this lecture can be found among the papers of G. N. Watson in the library at Trinity College, Cambridge. Although readers familiar with Ramanujan’s contributions to partitions will not find any new material in this lecture, we think that it has interest for historical reasons. After providing the lecture, we offer a few additional remarks at the end of this article. In the sequel, C. P. S. is shorthand for Cambridge Philosophical Society, and MS(S) is an abbreviation for manuscript(s).

2. WILSON’S LECTURE

It is now just over 7 years since Ramanujan died; he left behind not only his published papers (38 in number), but also a large quantity of unpublished material, – note-books in which he had jotted down results and conjectures as they occurred to him, and also the MSS of papers on which he had been engaged towards the time of his death. Of these three sources little need be said about the published papers; they are readily available in periodicals, and will shortly be made more readily available by publication in collected form: the note-books date from his earliest Indian days, and contain much that is definitely wrong; republication of these without annotation would be almost useless, and the labour involved by verification and annotation is a labour from which any mathematician might well shrink, – even where verification is possible. The note-books contain something like 30 chapters, and each chapter contains hundreds of formulae, for which, usually, no indication of proof is given. Professor Hardy has performed the editorial work necessary for the publication of one of these chapters, – that concerned with the generalised hypergeometric series, and published his version “A Chapter from Ramanujan’s Note-Book” in the Proceedings of the C. P. S. He tells me that the preparation of this version cost him 3 or 4 months of hard continuous labour. Again, when Professor Polya was visiting Oxford he borrowed from Hardy 2 of the Note-Books; but after keeping them only a day or two returned them to Hardy in a state almost of panic; for he explained that however long he kept them, he would have to go on attempting to verify formulae there stated, and would never again have time to establish a wholly original result. Ramanujan’s Note-Books are unquestionably formidable; fortunately, except for this passing reference, they do not further concern me now, as it is not these that Ramanujan’s

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work on partitions is to be found. With regard to the third source, – MSS, – I shall speak more particularly of one special MS shortly.

Ramanujan had an unorthodox mathematical education, and remained to his death an unorthodox mathematician. Almost everything he published bears, either in matter or in treatment, the imprint of his personality; his was highly individual work. And of it all none is more characteristic, as Hardy says in his Obituary Notice on Ramanujan, than his work on the theory of partitions and the allied parts of the theories of elliptic functions and continued fractions. It is here, for example, that the famous and singularly beautiful Rogers–Ramanujan identities belong: in the example just quoted Ramanujan must take second place to Professor L. J. Rogers, late of the University whose guests we now are. But in the work to which I shall now turn Ramanujan was wholly original, not only in intention, but also in fact.

If \( n \) is a positive integer we understand by a partition of \( n \) a method of expressing \( n \) as a sum of positive integers (zero excluded). Thus \( 2 + 1 + 1 + 1 \) is a partition of 5; so is \( 2 + 2 + 1 \). Two partitions of the same number \( n \) which differ only in the order of the integers used to give the sum \( n \) are regarded as identical; thus \( 2 + 2 + 1 \) is the same partition of 5 as \( 2 + 1 + 2 \). With this convention we denote by \( p(n) \) the number of distinct partitions of \( n \). Thus,

\[
\begin{align*}
1 &= 1; & p(1) &= 1 \\
2 &= 2 = 1 + 1; & p(2) &= 2 \\
5 &= 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1; & p(5) &= 7.
\end{align*}
\]

The function \( p(n) \) steadily increases with \( n \), and for large values of \( n \) the dominant term in the asymptotic expansion of \( p(n) \) is

\[
\frac{1}{4n\sqrt{3}} \exp \left\{ \sqrt{\frac{2n}{3}} \right\}.
\]

The complete asymptotic formula for \( p(n) \) was studied in a very detailed manner by Hardy and Ramanujan in a paper published in 1918; with these (asymptotic) properties of \( p(n) \) I am not now concerned, and I mention the paper only because at the end of it there is given a table of values of \( p(n) \) for all values of \( n \) from 1 to 200 [\( p(200) = 3,972,999,029,388 \), i.e., nearly 4 million millions]. This table was calculated by Major MacMahon and Ramanujan, the purpose being to illustrate the agreement between the accurately computed values of \( p(n) \) with the approximate values given by Hardy and Ramanujan’s asymptotic formula for the function; and the closeness of the agreement is, in fact, quite remarkable; for Hardy and Ramanujan show that their formula can be used to find the value of \( p(n) \) not merely approximately, but exactly. Ramanujan, studying this table of 200 entries, noticed that so far as it extended, the values of \( p(n) \) appeared to satisfy a remarkable series of congruences, and was thus led, in a way very characteristic of his initial methods, to conjecture the truth of the following theorem:

**Theorem 2.1.** If \( \delta = 5^a 7^b 11^c \), \( 24\lambda \equiv 1 \pmod{\delta} \), and \( n \equiv \lambda \pmod{\delta} \), then

\[
p(n) \equiv 0 \pmod{\delta}.
\]
This theorem is supported by all the available evidence; but no general proof of it has been obtained, either by Ramanujan or by Darling, Rogers, or Mordell all of whom have subsequently investigated these congruence properties of the function $p(n)$. Proofs of certain special cases of the theorem (i.e., cases arising by attributing special values of $a$, $b$ and $c$) have, however, been obtained, and it is concerning the present state of our knowledge on this subject that I wish to report.

Taking the admissible values of $\delta$ in increasing order, the general theorem gives rise to the following series of special theorems:

1. $a = 1, b = c = 0$  
   \[ p(4, 9, 14, 19, \ldots) \equiv 0 \pmod{5} \]
2. $a = c = 0, b = 1$  
   \[ p(5, 12, 19, 26, \ldots) \equiv 0 \pmod{7} \]
3. $a = b = 0, c = 1$  
   \[ p(6, 17, 28, 39, \ldots) \equiv 0 \pmod{11} \]
4. $a = 2, b = c = 0$  
   \[ p(24, 49, 74, 99, \ldots) \equiv 0 \pmod{25} \]
5. $a = b = 1, c = 0$  
   \[ p(19, 54, 89, 124, \ldots) \equiv 0 \pmod{35} \]
6. $a = c = 0, b = 2$  
   \[ p(47, 96, 145, 194, \ldots) \equiv 0 \pmod{49} \]
7. $a = c = 1, b = 0$  
   \[ p(39, 94, 149, \ldots) \equiv 0 \pmod{55} \]
8. $a = 0, b = c = 1$  
   \[ p(61, 138, \ldots) \equiv 0 \pmod{77} \]
9. $a = b = 0, c = 2$  
   \[ p(116, 237, \ldots) \equiv 0 \pmod{121} \]
10. $a = 3, b = c = 0$  
    \[ p(99, 224, \ldots) \equiv 0 \pmod{125} \]

During his life time Ramanujan published, in connection with the above results only a short paper (4 pages) in the Proceedings of the C. P. S. (1919), and an abstract in the Proceedings of the L. M. S. (1920). In the C. P. S. paper he gave proofs of 1, 2, 4, 5, 6 of the above table; in the L. M. S. Abstract he stated that he possessed another method whereby he could prove in addition 3 and 9. He continued these and allied investigations further, and after his death there came into Hardy’s hands a MS which was of the nature of the first draft of a further paper which he proposed to publish. Hardy writes of it “The MS contains a large number of further results [i.e., results in addition to 1, 2 and 3]. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it the three simplest and most striking results, as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time.” Hardy’s edition of a small section of this MS was published in the Math. Zeitschrift (1921), and contains alternative proofs of 1 and 2 and a proof of 3. The rest of the MS has remained untouched.

With regard to the special theorems tabulated we notice at once that 5 would follow immediately from 1 and 2; for $19 \equiv 4 \pmod{5}$ and $19 \equiv 5 \pmod{7}$. Similarly 7 is a corollary of 1 and 3; 8 of 2 and 3; and so on. Generally, if Ramanujan’s conjecture were established for two co-prime values of $\delta$, say $\delta = 5^a$, $\delta' = 7^b$ its truth would follow for $\delta'' = 5^a 7^b$. For if $24\lambda \equiv 1 \pmod{5^a 7^b}$, i.e.,

\[ 24\lambda = 5^a 7^b k + 1, \]
then
\[ 24\lambda \equiv 1 \pmod{5^a} \quad \text{and} \quad 24\lambda \equiv 1 \pmod{7^b}. \]
Also if
\[ n \equiv \lambda \pmod{5^a7^b}, \quad \text{i.e.,} \quad n = 5^a7^bk + \lambda \]
then
\[ n \equiv \lambda \pmod{5^a} \quad \text{and} \quad n \equiv \lambda \pmod{7^b}. \]
Hence by hypothesis
\[ p(n) \equiv 0 \pmod{5^a} \quad \text{and} \quad p(n) \equiv 0 \pmod{7^b}, \]
and, therefore, since \( 5^a \) and \( 7^b \) are co-prime,
\[ p(n) \equiv 0 \pmod{5^a7^b}. \]
Thus for complete proof of Ramanujan’s conjecture we should have to prove it only for the 3 cases \( 5^a, 7^b, 11^c \). And in any case we need concern ourselves only with cases such as 1, 2, 3, 4, 6, 9, 10, . . . where \( \delta \) is one of these 3 forms (powers of 5, 7 or 11). Thus next in interest after cases 1, 2, and 3 (two of \( a, b, c \) zero; the third unity) are cases 4, 6 and 9 (two of \( a, b, c \) zero; the third = 2). Of these last 3 cases 4 and 6 are disposed of in the C. P. S. paper; no proof of 9 has, so far as I know, ever been published, but one is to be found in Ramanujan’s unpublished MS. In this MS Ramanujan also considers prime moduli other than 5, 7, 11; in particular 13, 17, 19 and 23; but in these cases no equally simple and striking results appear to exist.

A word should now be said about method[s]. One verifies immediately that, if \( |x| < 1 \),
\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \sum_{n=1}^{\infty} p(n)x^n
\]
\[ = \sum_{n=0}^{\infty} p(n)x^n, \]
if for simplicity of notation we agree to define \( p(0) = 1 \). Thus the natural approach to a study of \( p(n) \) is by way of the above generating function, and this puts at once at our disposal all the known machinery of the theory of elliptic functions, in particular transformation theory, the theory of \( q \)-series and products, and the theory of elliptic modular functions. It is on these lines, for example, that Professor Mordell proceeded in his investigations. So also does Ramanujan in his C. P. S. paper. But in the unpublished MS he bases himself rather on results which he had himself independently established in the memoir “On certain arithmetical functions” (Transactions C. P. S., 1916); though I believe it would be correct to say that all the results actually used could also be deduced from the theory of the elliptic modular functions.

For the values 5 and 7 the dependence of Ramanujan’s results on formulae in the theory of \( \vartheta \)-functions can be shown quite briefly. For writing \( q = e^{\pi i\tau} \) where \( \Im(\tau) > 0 \), and

\[ q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) = \sum_{\nu=-\infty}^{\infty} (-1)^\nu q^{\nu(3\nu+1)}, \]
\[ 2\pi q_0^3 q^{\frac{1}{4}} = \vartheta_1'(0) = 2\pi \sum_{\mu=0}^{\infty} (-1)^\mu (2\mu + 1) q^{(\mu + \frac{1}{2})^2}. \]

This gives, on writing \( q^2 = x \),

\[ x \left\{ (1 - x)(1 - x^2)(1 - x^3) \cdots \right\}^4 = \sum_{\mu=0}^{\infty} \sum_{\nu=-\infty}^{\infty} (-1)^{\mu+\nu} (2\mu + 1) x^{1 + \frac{1}{2} \mu(\mu+1) + \frac{1}{2} \nu(3\nu+1)}. \quad (2.1) \]

Now suppose \( 1 + \frac{1}{2} \mu(\mu + 1) + \frac{1}{2} \nu(3\nu + 1) \equiv 0 \pmod{5} \); it follows that \((2\mu + 1)\) and \((\nu + 1)\) are both multiples of 5; i.e., the coefficient of \( x^{5n} \) in (2.1) is a multiple of 5.

Further we verify that

\[ \frac{1 - x^5}{(1 - x)^5} \equiv 1 \pmod{5}, \]

and

\[ \frac{(1 - x^5)(1 - x^{10})(1 - x^{15}) \cdots}{\left\{ (1 - x)(1 - x^2)(1 - x^3) \cdots \right\}^5} \equiv 1 \pmod{5}. \]

Multiplying these two results together, the coefficient of \( x^{5n} \) in

\[ x \left\{ (1 - x)(1 - x^2)(1 - x^3) \cdots \right\}^4 \frac{(1 - x^5)(1 - x^{10}) \cdots}{\left\{ (1 - x)(1 - x^2) \cdots \right\}^5} \]

i.e.,

\[ \frac{x(1 - x^5)(1 - x^{10}) \cdots}{\left\{ (1 - x)(1 - x^2) \cdots \right\}} \]

is a multiple of 5. Hence the coefficient of \( x^{5n} \) in

\[ \frac{x}{(1 - x)(1 - x^2) \cdots} \]

is a multiple of 5: i.e., \( p(5n - 1) \equiv 0 \pmod{5} \).

Similarly with \( \delta \) equal to 7 we have to deal with \( q_0^6 \) i.e., with \( \{ \vartheta_1'(0) \}^2 \), leading to the series

\[ \sum_{\mu,\nu=0}^{\infty} (-1)^{\mu+\nu} (2\mu + 1)(2\nu + 1) x^{2 + \frac{1}{2} \mu(\mu+1) + \frac{1}{2} \nu(\nu+1)} \]

and giving \( p(7n - 2) \equiv 0 \pmod{7} \).

All the results so far established (including Ramanujan’s unpublished one involving 121) enable us to assert the truth of Ramanujan’s conjecture for \( \delta = 5^a 7^b 11^c \) where \( a, b, c = 0, 1, \) or 2. Concerning the next simplest value of \( \delta (=125) \) we do not know whether it is true or false.

By a deeper method, depending on the expression of a series of the form

\[ p(\lambda) + p(\lambda + \delta) x + p(\lambda + 2\delta) x^2 + \cdots \]

in terms of \( \theta \) functions Ramanujan obtains the identity

\[ p(4) + p(9)x + p(14)x^2 + \cdots = 5 \frac{(1 - x^5)(1 - x^{10})(1 - x^{15}) \cdots}{\left\{ (1 - x)(1 - x^2)(1 - x^3) \cdots \right\}^6} \quad (2.2) \]

from which it follows at once that \( p(5n - 1) \equiv 0 \pmod{5} \).
Equation (2.2) has been singled out by Hardy and MacMahon as the most beautiful of Ramanujan’s many identities. Ramanujan’s MSS contains many other striking results, which (as usual) he takes care to state in their “snappiest” form. Along with \( p(n) \) he considers the function \( \tau(n) \), defined by the expansion
\[
\sum_{n=1}^{\infty} \tau(n)x^n = x \left\{ (1 - x)(1 - x^2)(1 - x^3) \cdots \right\}^{24},
\]
and of course intimately related with the elliptic modular functions. For the primes other than 5, 7 and 11 the congruence properties of \( \tau(n) \) are simpler than those of \( p(n) \). I will conclude by quoting one example taken almost at random because it well illustrates the air of strangeness that is so characteristic of so many of Ramanujan’s results. An arithmetical function \( f(n) \) is said to possess a certain property for almost all values of \( n \) if, when \( N(n) \) denotes the number of values \( \leq n \) for which it does not possess the property, then
\[
\frac{N}{n} \to 0 \quad \text{as } n \to \infty.
\]
Ramanujan proves that
\[
\tau(n) \equiv 0 \pmod{2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 691}
\]
for almost all values of \( n \).

3. Commentary

Wilson’s claim in the first paragraph that many results in Ramanujan’s earlier notebooks [14] are “definitely wrong” is not true; of the 3200–3300 claims in the notebooks, there are only a handful that are false. Wilson and G. N. Watson began editing the notebooks shortly after the publication of Ramanujan’s Collected Papers, for which Wilson had evidently done the bulk of the preparation, for on the title page his name is in larger print than his co-editors, G. H. Hardy and P. V. Seshu Aiyar. Certainly, if Wilson had given this lecture in the early 1930s after he had carefully examined several chapters, he would not have made this erroneous assertion. Wilson worked on editing Ramanujan’s notebooks until 1935 when he tragically died from an infection that he incurred in a hospital. For more details on Wilson’s contributions toward editing the notebooks, see the author’s book [4].

Ramanujan’s first notebook contains 16 chapters, and his second contains 21 chapters. It is not clear how Wilson deduced that there are 30 chapters.

The “third source” on which Wilson focuses for most of his lecture is one of Ramanujan’s most important works, and its value cannot be underestimated; it has had an enormous influence on the development of the theory of partitions. The manuscript remained unpublished until it was published in its original handwritten form with Ramanujan’s lost notebook in early 1988 [15]. In the meanwhile, some of its theorems had been rediscovered by others. Prior to its publication, R. A. Rankin had access to this manuscript, which is devoted exclusively to the partition and tau functions, and published a fascinating article on its contents in 1977 [16]. In 1999, the author and K. Ono [7] prepared a complete version with extensive commentary for a volume honoring George Andrews on his 60th birthday. The commentary was significantly revised for inclusion in the third book that Andrews and Berndt have written on the lost notebook [1, Chapter 5]. For more on this manuscript’s history, content,
and influence, readers should consult [1]. Many of Ramanujan’s arguments in this manuscript are quite accessible, and they can be found in Berndt’s book [3], written for advanced undergraduates and beginning graduate students.

The asymptotic estimate for \( p(n) \) quoted by Wilson is the first term of an asymptotic series for \( p(n) \), proved by Hardy and Ramanujan using their celebrated circle method [11].

It was first observed by S. Chowla [8], [9, p. 353] in 1934 that Ramanujan’s Theorem 2.1 needs to be corrected when the modulus includes a power of 7. In such cases, the congruence should be given by

\[
p(7^b n + \lambda) \equiv 0 \pmod{7^{[b/2]+1}},
\]

where \( 24\lambda \equiv 1 \pmod{7^b} \). Evidently, Wilson (and many others after him as well) had failed to notice that in this unpublished manuscript on \( p(n) \) and \( \tau(n) \) Ramanujan gave a complete proof of his conjecture when \( \delta = 5^a \). He had also started to give a complete proof for the case \( \delta = 7^b \), but did not complete his argument. If he had had time to finish his proof, and it is clear that he certainly could have done so, he would have observed in the course of his proof that he would need to modify his conjecture, as indicated above. G. N. Watson, who had access to this unpublished manuscript, published the first proofs of Ramanujan’s conjecture when \( \delta = 5^a \) and \( \delta = 5^b \) (in corrected form) [18]. Watson’s proofs are precisely those indicated by Ramanujan in his unpublished manuscript, but with considerably more details. It is regrettable that nowhere in his paper does Watson acknowledge that his work is an elaboration of the proofs in Ramanujan’s manuscript. A. O. L. Atkin [3] provided the first proof of Ramanujan’s conjecture when \( \delta = 11^c \).

The method employed in Wilson’s lecture to prove the congruences \( p(5n + 4) \equiv 0 \pmod{5} \) and \( p(7n + 5) \equiv 0 \pmod{7} \) originated in Ramanujan’s paper [12]. Hardy reproduced Ramanujan’s proofs in his book [10, pp. 87, 88], and the present author also provided these proofs in his book [5, pp. 31, 32; 39, 40]. Ramanujan’s ideas were elegantly generalized by G. E. Andrews and R. Roy in their paper [2].

Page 182 in Ramanujan’s lost notebook [15] is the fifth page of a manuscript that has otherwise been lost, although it is possible that this page arises from an early draft of [12], in which Ramanujan would have had many further results on partitions. On this isolated page, Ramanujan studies the more general “colored” partition function \( p_r(n) \), defined by

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_{n=0}^{\infty} p_r(n)x^n, \quad |x| < 1.
\]

In particular, Ramanujan records some general congruences that can be proved by an extension of Ramanujan’s elementary methods from [12]. In thoroughly examining this page, the author, C. Gugg, and S. Kim [6] extended Ramanujan’s ideas and those of Andrews and Roy [2] as well. See also the third book that Andrews and the author have written on Ramanujan’s lost notebook [1, pp. 195–198].

In regard to the beautiful identity (2.2), Wilson paraphrased Hardy’s proclamation in his biography of Ramanujan in the Collected Papers [13, p. xxxv], viz., “if I had to select one formula from all Ramanujan’s work, I would agree with Major MacMahon in selecting a formula . . . .” Many authors have quoted Hardy to edify their own work, either in providing another proof of (2.2) or in proving another identity that is to be regarded as an equally beautiful analogue of (2.2).
For a discussion of the final result of Wilson’s lecture, see [7] or [1, pp. 177, 178]. For a deeper discussion of the more general theorem in which the exponents of 2, 3, 5, 7, 23, and 691 are replaced by any positive integers, see H. P. F. Swinnerton-Dyer’s lovely survey article [17]. Ramanujan’s elementary proofs for the congruences satisfied by $\tau(n)$ can also be found in Berndt’s book [5].

REFERENCES