

WHAT IS A q -SERIES?

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Abstract. Historically, research in q -series has not always been appreciated, and those who accomplished it even less acknowledged.

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As we demonstrate below, the beautiful and useful theorems in q -series are now cherished, and many of their progenitors greatly admired. It is our fondest wish that the introduction to q -series that follows will inspire formerly uninitiated readers to willingly contract the “ q -series disease,” in the words of Richard Askey, who has “suffered” from this disease for several decades.

1. INTRODUCTION

What is a q -series? The simplest and most manifestly useless definition would be a series with q 's in the summands. Having begun this essay with meaningless drivel, we next admit that there does not exist a “good” definition of a q -series. We might define a q -series to be one with summands containing expressions of the type

$$(a)_n := (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \geq 0, \quad (1.1)$$

where we interpret $(a; q)_0 = 1$. If the *base* q is understood, we often use the notation at the far left-hand side of (1.1), but in this paper, since we are going to use this notation for *rising factorials* or *shifted factorials*, to avoid confusion, we shall always write $(a; q)_n$. This definition of a q -series is also not completely satisfactory, because often in the theory of q -series, we let parameters in the summands tend to 0 or to ∞ , and consequently it may happen that no factors of the type $(a; q)_n$ remain in the summands. In some cases, what may remain is a theta function. Following the lead of Ramanujan, we shall define a general theta function $f(a, b)$ by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (1.2)$$

In the theory of q -series, theta functions also frequently arise in identities satisfied by series with products (1.1) in their summands. Thus, for these reasons, theta functions are an integral part of the theory and are also considered to be q -series. As we shall see in the sequel, infinite q -products

$$(a; q)_{\infty} := \lim_{n \rightarrow \infty} (a; q)_n, \quad |q| < 1,$$

arise both in product representations of theta functions and, more generally, in identities for q -series.

In the theory of q -series, there is an important class of series

$${}_{p+1}\phi_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; q, t \right) := \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_{p+1}; q)_n}{(b_1; q)_n (b_2; q)_n \cdots (b_p; q)_n (q; q)_n} t^n, \quad |t| < 1, \quad (1.3)$$

which are called *basic hypergeometric series*. Observe that the quotient of successive terms in (1.3) is a rational function of q^n , and conversely every such series with this property is of the form (1.3). The parameters $a_1, a_2, \dots, a_{p+1}, b_1, b_2, \dots, b_p$ are arbitrary complex numbers, except for the restrictions $b_j \neq q^{-n}$, $1 \leq j \leq p$, $n \geq 0$. Note that the number of parameters in the numerator is one more than the number of parameters in the denominator. This restriction is not necessary, but in the more general case it is best to slightly modify the definition (1.3). If the number of parameters is “small,” then we write them on one line, instead of two; e.g., we usually write ${}_2\phi_1(a, b; c; q, t)$.

In Section 2, we give a brief history of q -series indicating a few of the prominent theorems in the subject and describing some of the mathematicians who played leading roles in its development. In the following Section 3, we indicate some of the primary tools that are used to prove theorems about q -series. We concentrate on the use of combinatorial methods, because of exciting recent activity and because they often give new, fascinating insights into the combinatorial nature of q -series identities. Theorems about basic hypergeometric series are often q -analogues of results about ordinary or generalized hypergeometric series, and so in Section 4, we describe this symbiosis and offer a few examples in illustration. In case readers think that theta functions might arise only trivially or coincidentally in the theory of q -series, in Section 5, we provide a few examples where theta functions arise in some of Ramanujan’s beautiful theorems, in particular, on mock theta functions. Lastly, by the time readers reach the end of Section 5, if we have been successful in stimulating them to action, they will want dig more deeply into the subject, and so we indicate in Section 6 some of the primary sources from which one can study and learn more about the properties and applications of q -series.

2. IMPORTANT THEOREMS AND FIGURES IN THE HISTORY OF q -SERIES

One could claim that the theory of q -series began with certain famous theorems of L. Euler [36, Chapter 16] and C.F. Gauss [40]. We begin non-chronologically with Gauss, who is generally considered to be the founder of the theory of theta functions. In Ramanujan’s notation, perhaps the three most important special cases of (1.2) are defined by

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}, \quad (2.1)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (2.2)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}, \quad (2.3)$$

where the three product representations in (2.1)–(2.3) are special cases of the Jacobi triple product identity

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad (2.4)$$

which is arguably the most useful and celebrated theorem in the theory of theta functions, and which was actually first discovered by Gauss. The identity (2.3) is called Euler's pentagonal number theorem, which, in a less abbreviated notation, is given by

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \prod_{n=1}^{\infty} (1 - q^n). \quad (2.5)$$

To discern its combinatorial interpretation, we first define the partition function $p(n)$ to be the number of ways the positive integer n can be written as a sum of positive integers, with the order of these positive integers immaterial. For example, $p(5) = 7$, because there are 7 ways to write 5 as a sum of positive integers, namely, 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1. Observe that $\prod_{n=1}^{\infty} (1 + q^n)$ generates the number of partitions of a positive integer into *distinct* parts. With this in mind, we offer a combinatorial interpretation of (2.5).

Corollary 2.1 (Combinatorial Version of Euler's Pentagonal Number Theorem). *Let $D_e(n)$ denote the number of partitions of n into an even number of distinct parts, and let $D_o(n)$ denote the number of partitions of n into an odd number of distinct parts. Then*

$$D_e(n) - D_o(n) = \begin{cases} (-1)^j, & \text{if } n = j(3j \pm 1)/2, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

We next give perhaps the most important and useful theorem in which a q -product $(a; q)_n$ appears in the summands.

Theorem 2.2. *For $|q|, |z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}. \quad (2.7)$$

Proof. Note that the product on the right side of (2.7) converges uniformly on compact subsets of $|z| < 1$ and so represents an analytic function on $|z| < 1$. Thus, we may write

$$F(z) := \frac{(az; q)_{\infty}}{(z; q)_{\infty}} = \sum_{n=0}^{\infty} A_n z^n, \quad |z| < 1. \quad (2.8)$$

From the product representation in (2.8), we can readily verify that

$$(1 - z)F(z) = (1 - az)F(qz). \quad (2.9)$$

Equating coefficients of z^n , $n \geq 1$, on both sides of (2.9), we find that

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1},$$

or

$$A_n = \frac{1 - aq^{n-1}}{1 - q^n} A_{n-1}, \quad n \geq 1. \quad (2.10)$$

Iterating (2.10) and using the value $A_0 = 1$, which is readily apparent from (2.8), we deduce that

$$A_n = \frac{(a; q)_n}{(q; q)_n}, \quad n \geq 0. \quad (2.11)$$

Using (2.11) in (2.8), we complete the proof of (2.7). \square

Corollary 2.3. For $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1, \quad (2.12)$$

and

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q; q)_n} = (z; q)_{\infty}, \quad |z| < \infty. \quad (2.13)$$

Theorem 2.2 is called the *q-analogue of the binomial theorem*, and in Section 4 we relate in detail why this name is given to this theorem. In its general form, Theorem 2.2 is due to A. Cauchy [30], while the special cases in Corollary 2.3 are due to Euler [36, Chapter 16]. The proof of (2.7) given above follows along the lines of that given by Euler and Cauchy and has been copied from the author's book [24, p. 8].

Although certain important theorems were found by Euler, Gauss, and Cauchy, the systematic development of the theory of *q*-series began with a paper by E. Heine [43] in 1847. In particular, the following theorem, called Heine's transformation, is perhaps his signature theorem.

Theorem 2.4. For $|t|, |b| < 1$,

$${}_2\phi_1(a, b; c; q, t) = \frac{(b; q)_{\infty}(at; q)_{\infty}}{(c; q)_{\infty}(t; q)_{\infty}} {}_2\phi_1(c/b, t; at; q, b). \quad (2.14)$$

In his lost notebook [57], S. Ramanujan, who independently discovered Heine's transformation [56, Chapter 16, Entry 6], [21, p. 15], found many applications of it [10, Chapter 1].

Two English mathematicians, F.H. Jackson and L.J. Rogers, at the end of the 19th and beginning of the 20th centuries devoted most of their mathematical careers to further developing the theory of *q*-series, but their efforts were not appreciated by their contemporary researchers. We first discuss Rogers.

L.J. Rogers, who was born on March 30, 1862 and died on September 12, 1933, was recognized during his lifetime for only a handful of his contributions to mathematics. As a mathematics student, he was elected to a Scholarship at Balliol College, Oxford in 1879, but he also earned a Bachelor of Music degree in 1884. In 1888, he became Professor of Mathematics at Yorkshire College, now called the University of Leeds. It appeared that music, in fact, took precedence over mathematics in his professional life. A.L. Dixon wrote one of the two obituaries that we have of Rogers; let us quote from that obituary [35].

Rogers was a man of extraordinary gifts in many fields, and everything he did, he did well. Besides his mathematics and music he had many interests; he was a born linguist and phonetician, a wonderful mimic who delighted to talk broad Yorkshire, a first-class skater, and a maker of rock gardens.

He did things, and did them well, because he liked doing them; but he had nothing of the professional outlook, and his knowledge of other

people's work was vague. He had very little ambition or desire for recognition.

Nonetheless, the Rogers–Ramanujan identities (given below in (2.15) and (2.16)), the Rogers–Ramanujan continued fraction, and the Rogers–Szegő polynomials are important discoveries that are named after him. Moreover, as pointed out by G.H. Hardy, J.E. Littlewood, and G. Pólya [42, p. 25], Rogers [58] discovered Hölder's famous inequality one year before O. Hölder [45], and in a more symmetrical form.

During his lifetime, Rogers was singularly known for the two identities

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}, \quad (2.15)$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}, \quad (2.16)$$

which he discovered in 1894 [59], but which were entirely ignored until Ramanujan rediscovered them about 20 years later. In the words of the writer of Rogers' obituary [61], "Rogers' reputation as a mathematician rests almost entirely on a single incident." In other words, if it had not been for Ramanujan's rediscovery of these now famous identities, in the collective opinion of Rogers' contemporary mathematicians, his name would have been completely forgotten. For a history of these identities, see, for example, Ramanujan's *Collected Papers* [55, pp. 344–346], the monograph [4] by G.E. Andrews, Andrews's survey article [6], or the survey paper written by the author, Y.-S. Choi, and S.-Y. Kang [25].

The Rogers–Ramanujan continued fraction, also first studied by Rogers in [59], is defined by

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \quad |q| < 1. \quad (2.17)$$

The continued fraction (2.17) is intimately connected with the Rogers–Ramanujan functions by the beautiful theorem

$$R(q) = q^{1/5} \frac{H(q)}{G(q)} = q^{1/5} \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}, \quad (2.18)$$

by (2.15) and (2.16), which was also first proved by Rogers [59] and rediscovered by Ramanujan. Although Rogers established several of its properties, Ramanujan proved many more theorems about $R(q)$, most of which are found in his lost notebook [57]. For an account of a sizeable portion of Ramanujan's discoveries about $R(q)$, see the book by Andrews and the author [9].

Let us quote from a second obituary [61] of Rogers; more extensive quotations may be found in Andrews's absorbing monograph [4].

Of course the rest of Rogers' work was carefully studied; but, though everything he wrote was marked by a certain distinction, nothing else of first-rate importance was discovered. He was elected a fellow of the Royal Society in 1924, and relapsed into his former obscurity.

To all who knew him but slightly, that incident must appear typical of Rogers. Fine abilities, they would say, wasted by freakishness and inconstancy! . . . Rogers' major abilities can seldom have been excelled in extent or variety; and his minor accomplishments ranged from knitting to skating. . . . Certainly he might have won great fame in music, either as scholar or executant; and surely no position in diplomacy would have been unattainable to one endowed with his easy mastery of languages, his quick intelligence, his sparkling wit, his fine presence, his athletic grace, his courtly charm that no woman could resist. Yet of what the world counts success he achieved practically nothing.

Of course, the first sentence of the quote above is complete nonsense. However, in defense of those in his time who might have attempted to understand Rogers' papers, it is to be admitted that Rogers wrote in a manner that makes his work very difficult to understand. We have discussed only the most famous of Rogers' discoveries, but in recent years, due to the efforts of Askey, Andrews, and others, considerably more of Rogers' mathematics has been discovered and appreciated than heretofore. However, some of Rogers' papers have yet to be thoroughly examined.

Jackson was contemporaneous with Rogers and even less appreciated for his work. The *Bible* in the theory of basic hypergeometric series is the text [39] by G. Gasper and M. Rahman. Readers of this book will find many results due to Jackson and rightly conclude that he, indeed, is one of the founders of the subject. For example, using Heine's transformation (2.14), one can derive Jackson's transformation formula [39, p. 10]

$${}_2\phi_1(a, b; c; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\phi_2(a, c/b; c, az; q, bz).$$

As another example, Jackson proved that for each positive integer n [39, p. 17],

$${}_3\phi_2(a, b, q^{-n}; c, abc^{-1}q^{1-n}; q, q) = \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n (c/(ab); q)_n}. \quad (2.19)$$

Observe that the series on the left-hand side above terminates.

Jackson, who was born on August 16, 1870 and died on April 27, 1960 at the age of 89, authored 49 papers, most of them on q -series. Although he took the Tripos and was a Wrangler at the early age of 19, he chose the ministry as his profession. As an ordained minister, Jackson served for ten years as a chaplain and instructor in the Royal Navy, and held various positions in the Church of England throughout his career. Let us now quote from T.W. Chaundy's obituary [33] of this "amateur" mathematician.

After a few practice pieces (as was usual in those days) titivating known results, his whole mathematical output was devoted to basic analogues or q -analogues (as they are obscurely called). . . . The problem is very much that of unscrambling an egg; we have to reverse the substitution $q = 1$, and this involves us, especially with series, in inserting, variously, various unexpected q^N (often with N quadratic in n). Perhaps these were not unexpected to Jackson, for he seemed to have an especial flair

for these extensions. . . . He had a ceaseless and vivid imagination which (to this writer) seemed often to tax his powers of exposition. . . . In all he was the enthusiastic amateur with gifts that, given the opportunity, might have led him to a professional status.

As further evidence of the contempt in which his work was regarded, he once read a paper before the London Mathematical Society, to which someone remarked, “Surely, Mr. President, we have heard all this before.” Jackson quickly strode from the room and never darkened the pages of the Society again.

Ramanujan undoubtedly contributed more to q -series than anyone either before or after his time. His most famous theorem is his ${}_1\psi_1$ summation theorem, a theorem about bilateral hypergeometric series, which, in importance and usefulness, plays the same role in the theory of q -bilateral series as the q -binomial theorem plays in unilateral q -series. Our first task then is to extend the definition of $(a; q)_n$ to negative integers. To that end, define, for every integer n ,

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. \quad (2.20)$$

Note that (2.20) is equivalent to (1.1) when $n \geq 0$.

Theorem 2.5 (Ramanujan’s ${}_1\psi_1$ Summation). *For $|b/a| < |z| < 1$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az; q)_\infty (q/(az); q)_\infty (q; q)_\infty (b/a; q)_\infty}{(z; q)_\infty (b/(az); q)_\infty (b; q)_\infty (q/a; q)_\infty}. \quad (2.21)$$

Ramanujan’s ${}_1\psi_1$ summation theorem was first stated by Ramanujan in his notebooks [56, Chapter 16, Entry 17]. It was found there by Hardy, who called it, “a remarkable formula with many parameters” and intimated that it could be established by employing the q -binomial theorem [41, pp. 222–223]. There now exist many proofs of Theorem 2.5, and a list of all known proofs (22 up to the printing of [10]), with a few brief descriptions of proofs, can be found in [10, Chapter 3, pp. 54–56]. In this source, one can also find several identities from the lost notebook that can be proved with the use of the ${}_1\psi_1$ summation theorem. The q -binomial theorem and the Jacobi triple product identity are both corollaries of Ramanujan’s ${}_1\psi_1$ summation theorem. Connections of the q -binomial theorem and the ${}_1\psi_1$ theorem with q -analogues of the gamma and beta functions have been eloquently discussed by Askey [13], [14].

In closing our historical account, we remark that we do not know when it became standard to designate q as the variable. Heine used the letter x in place of q . In the theory of elliptic functions, q has been in use at least since the time of C.G.J. Jacobi, who used it in all of his papers, in particular, in his famous *Fundamenta Nova* [46]. Ramanujan used x as his variable in his earlier notebooks [56] but used q in his published papers [55] and lost notebook [57]. Rogers employed the notation q in all of his papers, and it is possible that the use of q in the theory of q -series and, more precisely, in the theory of basic hypergeometric series, was therefore instigated by Rogers.

3. THE TOOLS USED TO PROVE THEOREMS ABOUT q -SERIES

1. The proof of Theorem 2.2 illustrates the most common approach to proving identities for q -series. In that proof we established a functional equation relating $F(z)$ and $F(qz)$. This yielded a functional equation for the coefficients, which we iterated to complete the proof. Employing q -difference equations, i.e., equations relating functions at different arguments involving q , is the most potent method that we have at our disposal.
2. In the 1940's, W.N. Bailey [15], [16] introduced a powerful method for producing q -identities. One begins by defining a sequence $\{\beta_n\}$ in terms of any given arbitrary sequence $\{\alpha_n\}$. From these two sequences, one can then derive another pair of sequences $\{\alpha'_n\}$ and $\{\beta'_n\}$, each containing the same two free parameters. Of course, one needs to be clever in constructing the two original sequences in order that one can obtain two new meaningful and useful sequences. In recent years, several authors have derived a large number of beautiful identities by ingeniously employing versions of Bailey's Lemma. We now state Bailey's Lemma, as it is given in the text of Andrews, Askey, and Roy [8, p. 584].

Theorem 3.1. *Let $\{\alpha_n\}$, $n \geq 0$, be a sequence of complex numbers, and define also*

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r}(aq; q)_{n+r}}.$$

Define another sequence $\{\alpha'_n\}$ by

$$\alpha'_n := \frac{(\rho_1; q)_n(\rho_2; q)_n(aq; /(\rho_1\rho_2); q)_n\alpha_n}{(aq/\rho_1; q)_n(aq/\rho_2; q)_n},$$

where ρ_1 and ρ_2 are arbitrary non-zero complex numbers. Then

$$\beta'_n = \sum_{r=0}^n \frac{\alpha'_r}{(q; q)_{n-r}(aq; q)_{n+r}},$$

where

$$\beta'_n := \sum_{j=0}^{\infty} \frac{(\rho_1; q)_j(\rho_2; q)_j(aq; /(\rho_1\rho_2); q)_{n-j}\beta_j}{(q; q)_{n-j}(aq/\rho_1; q)_n(aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2} \right)^j.$$

Among those who have fruitfully developed the theory of Bailey pairs are Lucy Slater, Andrews, David Bressoud, S. Ole Warnaar, and Andrew Sills.

3. Using partial fractions is another effective technique for producing q -series identities. In the past few years, Andrews and this author have concluded that Ramanujan likely employed partial fractions far more than was formerly believed. See Chapter 12 in the first book [9] that Andrews and Berndt wrote on the lost notebook, the papers [7] and [23] by Andrews and Berndt, respectively, and S.H. Chan's thesis [31] and paper [32].
4. Combinatorial reasoning is an extremely compelling method for proving q -series identities. Although, arguably the combinatorics of q -series began with Euler's pentagonal number theorem given by (2.5) or (2.6), it lay dormant until it

received a tremendous impetus with the bijections of J.J. Sylvester [66] and F. Franklin [38] in the 1880's, followed by P.A. MacMahon [50] early in the twentieth century. In the last few decades, the combinatorial efforts of several authors, led especially by Andrews, Bressoud, Frank Garvan, Krishnaswami Alladi, Sylvie Corteel, and Ae Ja Yee, have provided not only insightful proofs but have also led to new theorems as well. We briefly discuss some of these contributions.

Recall that prior to Corollary 2.1, we defined the ordinary partition function $p(n)$. Euler first established the generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

and so it is perhaps not surprising that partitions arise ubiquitously in combinatorially proving q -series identities. Euler found two further identities for $1/(q; q)_{\infty}$. First, a trivial consequence of Euler's theorem (2.12) is

$$\sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = \frac{1}{(q; q)_{\infty}}. \tag{3.1}$$

The identity (3.1) can easily be proved combinatorially. Observe that $1/(q; q)_n$ generates partitions into parts with each part less than or equal to n . The numerator q^n generates the part n . Thus, $q^n/(q; q)_n$ generates all partitions with largest part equal to n . The left side of (3.1) thus generates all partitions while sorting them out according to the largest part n . The partition $9 + 9 + 7 + 7 + 3 + 3 + 2$ in the Ferrers diagram depicted below is therefore one of the partitions counted by the summand with $n = 9$ on the left-hand side of (3.1). Euler also established the beautiful identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n^2} = \frac{1}{(q; q)_{\infty}}. \tag{3.2}$$

The identity (3.2) can be proved combinatorially as follows. Consider the Fer-

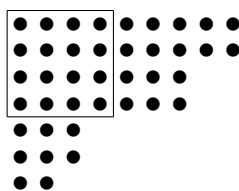


FIGURE 1. Durfee square of side $n = 4$

rrers graph of an arbitrary partition, and extract the largest *Durfee square* of side n . These nodes are then generated by q^{n^2} . The nodes to the right of the square form a partition into parts not exceeding n , while the nodes below the Durfee square also form a partition into parts less than or equal to n . These two partitions are generated by $1/(q; q)_n^2$, and so the generating function on the

left side of (3.2) generates all partitions and separates them according to their largest Durfee square. Thus in Figure 1, the largest Durfee square is of size 4^2 with the partition at the right being $4 + 4 + 4 + 2 + 2$ and the partition below the square being $3 + 3 + 2$. See also Andrews's book [3, pp. 27–28].

Probably the simplest partition-theoretic q -identity is Euler's theorem

$$\frac{1}{(q; q^2)_\infty} = (-q; q)_\infty, \quad (3.3)$$

which is equivalent to the assertion that the number of partitions of a positive integer n into odd parts is equal to the number of partitions of n into distinct parts. Sylvester [66] developed a beautiful bijection between partitions with only odd parts and partitions with distinct parts in order to prove (3.3). This bijection is described by Andrews [5] in his interesting paper on Sylvester's work on partitions.

We remarked above that Euler's pentagonal number theorem has an elegant partition-theoretic interpretation given by Theorem 2.1. One might ask if there is a bijective proof of it, and indeed in 1881, Franklin [38] devised what is now widely known as Franklin's bijection to prove Theorem 2.1; see also [3, pp. 10–11]. Moreover, Franklin's bijection, like Sylvester's bijection, has been used on many occasions to give combinatorial proofs of further q -series identities. For example, see papers by Berndt and Yee [28] and Berndt, B. Kim, and Yee [26].

It was not until 1965 that a bijective proof of the Jacobi triple product identity (2.4) was developed by E.M. Wright [69], and since then it has had numerous applications; see, for example, the papers by Warnaar [68] and S. Kim [49].

Leading the modern era in providing combinatorial proofs of q -series identities is Andrews. We cite just one of his many papers on the subject, namely, [2] where a combinatorial proof of the useful Rogers–Fine identity [60], [37, p. 15]

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n}{(\beta; q)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\alpha\tau q / \beta; q)_n \beta^n \tau^n q^{n^2 - n} (1 - \alpha\tau q^{2n})}{(\beta; q)_n (\tau; q)_{n+1}}$$

is given.

The first bijective proof of the q -binomial theorem, Theorem 2.2, was given in 1987 by J.T. Joichi and D. Stanton [47]; see also I. Pak's survey, which provides several combinatorial proofs of partition and q -series identities. A variation of Joichi and Stanton's proof has been devised by Yee [70] to provide another bijective proof of the q -binomial theorem.

Lastly, in 2004, Yee [70] devised a remarkable bijective proof of Ramanujan's ${}_1\psi_1$ summation theorem, Theorem 2.5.

We conclude our discussion of combinatorial proofs by offering a few remarks about an identity

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \sum_{n=0}^{\infty} \frac{(-q; q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2; q^2)_n} \quad (3.4)$$

found on page 28 in Ramanujan's lost notebook [57]. (The claims that we make below about partitions are not easy to establish [26].) Let D_n be the set

of partitions into n distinct parts less than $2n$ such that the smallest part of each partition is 1, and if $2k - 1$ is the largest odd part, then all odd positive integers less than $2k - 1$ occur as parts. Let E_n be the set of partitions into even parts less than or equal to $2n$. In giving a bijective proof of (3.4), Berndt, Kim, and Yee [26] show that the right-hand side of (3.4) generates pairs of partitions (π, σ) with $\pi \in D_n$ and $\sigma \in E_n$, where the exponent of a denotes the number of parts of π minus the number of parts of σ . When we examine the left-hand side of (3.4), we see that the the partitions π and σ “cancel each other out,” except on a very thin set, the squares, where only one single partition survives the cancellation. This example, which has also been studied combinatorially by Yee [71] and Alladi [1], and many other such identities, illustrate why the combinatorial study of q -series identities is fascinating.

5. To complete our list of tools, we briefly mention two theories that have not been successful in establishing q -series identities. Theta functions play a prominent role in the theory of elliptic functions, which reached its peak in the late nineteenth and early twentieth centuries. However, in the past decade or so, Zhi-Guo Liu has written several papers demonstrating the power of elliptic functions in establishing many of Ramanujan’s results on theta functions as well as new theorems in the subject. Theta functions also play a leading role in the theory of modular forms. Although, as we have seen, theta functions are inextricably intertwined with the theory of basic hypergeometric series, the development of the theory of basic hypergeometric series has kept the theories of elliptic functions and modular forms in abeyance standing at the door. Will these theories ever be able to make inroads into basic hypergeometric series?

4. THE SYMBIOSIS OF HYPERGEOMETRIC SERIES AND BASIC HYPERGEOMETRIC SERIES

If $p \geq 0$, let a_1, a_2, \dots, a_{p+1} and b_1, b_2, \dots, b_p be arbitrary complex numbers, except that b_j , $1 \leq j \leq p$, cannot be a non-positive integer. The generalized hypergeometric function ${}_{p+1}F_p$ is defined by

$${}_{p+1}F_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{(b_1)_n (b_2)_n \cdots (b_p)_n n!} z^n, \quad |z| < 1, \quad (4.1)$$

where

$$(a)_0 := 1, \quad (a)_n := a(a+1) \cdots (a+n-1), \quad n \geq 1, \quad (4.2)$$

is called the *rising* or *shifted factorial*. If the number of parameters is “small,” then in place of the left side of (4.1), we write ${}_{p+1}F_p(a_1, a_2, \dots, a_{p+1}; b_1, b_2, \dots, b_p; z)$. Observe that the quotient of successive terms in (4.1) is a rational function of n , and conversely every such series with this property is a hypergeometric series.

Replacing a by q^a in (1.1), where $a > 0$, and using L'Hôpital's rule while letting q tend to 1, we find that

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{(q^a; q)_n}{(1-q)^n} &= \lim_{q \rightarrow 1} \frac{1-q^a}{1-q} \frac{1-q^{a+1}}{1-q} \cdots \frac{1-q^{a+n-1}}{1-q} \\ &= a(a+1) \cdots (a+n-1) = (a)_n. \end{aligned} \quad (4.3)$$

The equality (4.3) demonstrates an intimate relation between basic hypergeometric series defined in (1.3) and hypergeometric series defined in (4.1). In particular, formally,

$$\lim_{q \rightarrow 1^-} {}_{p+1}\phi_p \left(\begin{matrix} q^{a_1}, q^{a_2}, \dots, q^{a_{p+1}} \\ q^{b_1}, q^{b_2}, \dots, q^{b_p} \end{matrix}; q, z \right) = {}_{p+1}F_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; z \right).$$

One could define many other q -series, such that when we let $q \rightarrow 1$, they tend to the same hypergeometric series. Of course, most of these q -series would not play meaningful roles in any theory of q -series. The series and theorems in q -series that have counterparts in the theory of hypergeometric series are called q -analogues. Does every theorem about hypergeometric series have a natural, meaningful q -analogue? Is it possible for a theorem about ordinary hypergeometric series to have more than one q -analogue?

Let us begin with a q -analogue of the binomial coefficient $\binom{n}{m}$. Let n and m denote integers. Then the Gaussian polynomial or q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{cases} \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

If we let q tend to 1 in (4.4) and employ (4.3), we easily see that

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}.$$

We are now able to answer the last question at the close of the penultimate paragraph. It is easy to show by induction on n that, for $n \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}, \quad (4.5)$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ m \end{bmatrix}. \quad (4.6)$$

Both (4.5) and (4.6) are q -analogues of Pascal's familiar recurrence formula for binomial coefficients, which is obtained by letting $q \rightarrow 1$ in either (4.5) or (4.6). Thus, more than one q -analogue may exist!

Next we show that Theorem 2.2 is justified in being christened the q -binomial theorem. In (2.7), replace a by q^a , where we now assume that a is an integer. After simplifying the right-hand side of (2.7) and then letting q tend to 1, we deduce the ordinary binomial theorem in the form

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}. \quad (4.7)$$

If a is not an integer, letting $q \rightarrow 1$ in (2.7) still yields (4.7), but it is considerably more difficult to obtain the right-hand side of (4.7) [8, p. 491, Theorem 10.2.4].

We provide another example in which a famous theorem about ordinary hypergeometric series has an analogue in the theory of basic hypergeometric series. In 1812, Gauss proved that, if $\operatorname{Re}(c - a - b) > 0$ [8, p. 66, Theorem 2.2.2],

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (4.8)$$

The special case

$${}_2F_1(-n, a; c; 1) = \frac{(c - a)_n}{(c)_n}, \quad (4.9)$$

where n is a non-negative integer, is called the Chu–Vandermonde Theorem, because S.–C. Chu in 1303 [34] and A.T. Vandermonde in 1772 [67] had previously discovered (4.9). The q -analogue of (4.8) is due to Heine. For $|c/(ab)| < 1$,

$${}_2\phi_1(a, b; c; q, c/(ab)) = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/(ab); q)_\infty}. \quad (4.10)$$

Note that Euler’s identity (3.2) is the special case $a = b = 0$, $c = q$ of (4.10). To show that (4.10) is a q -analogue of (4.8), it is perhaps best to introduce a q -analogue of the classical gamma function, namely,

$$\Gamma_q(x) := \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad |q| < 1.$$

Then it can be shown that [8, p. 495, Corollary 10.3.4]

$$\lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x). \quad (4.11)$$

If we now replace a , b , and c in (4.10) by q^a , q^b , and q^c , respectively, let q tend to 1, and employ (4.11), we immediately deduce (4.8). Setting $b = q^{-n}$ in (4.10) and reversing the order of summation, we can obtain a q -analogue of the Chu–Vandermonde Theorem (4.9) in the form

$${}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a; q)_n}{(c; q)_n} a^n. \quad (4.12)$$

Observe that Jackson’s theorem (2.19) is a generalization of (4.12), which can be seen by letting $b \rightarrow \infty$ in (2.19). If we had not inverted the order of summation above, we would have obtained a second analogue of the Chu–Vandermonde theorem [39, p. 14].

Ramanujan derived a multitude of beautiful continued fractions for quotients of Gamma-functions that are found in his notebooks [56]. Proofs of these continued fractions can be found in the author’s books [20, Chapter 12], [22, Chapter 32]. Methods for proving these continued fraction formulas are varied and at times *ad hoc*. Ramanujan evidently had a systematic procedure for proving these continued fractions, but we do not know what it is. As an illustration, we cite Entry 25 from Chapter 12 of Ramanujan’s second notebook [56], [20, p. 140]. Suppose that either n is an odd integer

and x is any complex number, or that n is any complex number and $\operatorname{Re} x > 0$. Then

$$\frac{\Gamma(\frac{1}{4}(x+n+1))\Gamma(\frac{1}{4}(x-n+1))}{\Gamma(\frac{1}{4}(x+n+3))\Gamma(\frac{1}{4}(x-n+3))} = \frac{4}{x} - \frac{n^2-1^2}{2x} - \frac{n^2-3^2}{2x} - \frac{n^2-5^2}{2x} - \dots \quad (4.13)$$

On the other hand, consider another continued fraction, Entry 12 in Chapter 16 of Ramanujan's second notebook [56], [21, p. 24]. Suppose that a , b , and q are complex numbers such that $|ab| < 1$ and $|q| < 1$, or that $a = b^{2m+1}$ for some integer m . Then

$$\frac{(a^2q^3; q^4)_\infty (b^2q^3; q^4)_\infty}{(a^2q; q^4)_\infty (b^2q; q^4)_\infty} = \frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \dots \quad (4.14)$$

At a first glance, it does not appear that (4.14) is a q -analogue of (4.13). However, if we replace a and b by q^a and q^b , respectively, then replace q^4 by q , set $2a = x+n$ and $2b = x-n$, multiply both sides of (4.14) by $(1-q)$, and lastly let $q \rightarrow 1$, we formally obtain (4.13).

We had earlier asked if there were theorems about ordinary hypergeometric series that do not have q -analogues. We provide one example for which q -analogues indeed do exist, but they are not *faithful* analogues, in that they do not have a shape that one would suspect. Consider Clausen's identity [8, p. 116]

$${}_2F_1^2\left(a, b; a+b+\frac{1}{2}; z\right) = {}_3F_2\left(2a, 2b, a+b; a+b+\frac{1}{2}, 2a+2b; z\right). \quad (4.15)$$

This identity was used by Ramanujan in deriving his 17 famous series for $1/\pi$ [54], which were not entirely proved in print until 1987 when Jonathan and Peter Borwein [29, Chapter 5] used (4.15), modular equations, and special values of theta functions to prove all of them. As an example, we offer

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} (6n+1) \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} \frac{1}{4^n}. \quad (4.16)$$

For an expository discussion of proofs of these 17 series representations and many more such series formulas for $1/\pi$, see the survey article by N.D. Baruah, Berndt, and H.H. Chan [17].

Gasper and Rahman [39, p. 232, Eq. (8.8.3); p. 234, Eq. (8.8.12)] have derived two analogues of (4.15). The first, which is a consequence of the second, is given by

$$\left\{ {}_4\phi_3 \left(\begin{matrix} a, b, abz, ab/z \\ ab\sqrt{q}, -ab\sqrt{q}, -ab; q, q \end{matrix} \right) \right\}^2 = {}_5\phi_4 \left(\begin{matrix} a^2, b^2, ab, abz, ab/z \\ a^2b^2, ab\sqrt{q}, -ab\sqrt{q}, -ab; q, q \end{matrix} \right),$$

under the assumption that the series terminate. The more general q -analogue of Gasper and Rahman expresses

$$\left\{ {}_2\phi_1(a, b; aq/b; q, zq/b) \right\}^2$$

as a sum of two ${}_5\phi_4$'s.

We have observed that the q -binomial theorem is an analogue of the binomial theorem. Readers may ask if the corresponding bilateral sum theorem, Ramanujan's ${}_1\psi_1$ summation theorem (2.21), has an analogous forerunner in the theory of hypergeometric series. To answer this question, we should define a bilateral hypergeometric series.

For every integer n , define

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)},$$

which agrees with our previous definition (4.2) of $(a)_n$ if n is a nonnegative integer. The bilateral hypergeometric series ${}_pH_p$ is defined for complex parameters a_1, a_2, \dots, a_p and b_1, b_2, \dots, b_p by

$${}_pH_p \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix}; z \right) := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_p)_n} z^n, \quad (4.17)$$

provided that the parameters are chosen so that poles do not arise. There is only a meager theory of bilateral hypergeometric series, which is mostly covered in Slater's book [65, Chapter 6], primarily because, if they converge, they converge only on $|z| = 1$ under the condition

$$\operatorname{Re}(b_1 + b_2 + \cdots + b_p - a_1 - a_2 - \cdots - a_p) > 1. \quad (4.18)$$

Thus, to answer our question, Ramanujan's ${}_1\psi_1$ summation theorem does not appear to be a q -analogue of any theorem about bilateral hypergeometric series. Bilateral q -series have a much richer theory than the series (4.17).

5. THETA FUNCTIONS, q -SERIES, AND MOCK THETA FUNCTIONS IN RAMANUJAN'S LOST NOTEBOOK

In Section 3, we discussed the combinatorics of (3.4). The series on the left-hand side of (3.4) is a *partial theta function*. We offer now some q -series representations for certain *complete* theta functions that are found in Ramanujan's lost notebook [57, p. 35], [10, pp. 32–35, Entries 1.7.7, 1.7.9, 1.7.6, 1.7.8]. If $f(a, b)$ is defined by (1.2) and $\varphi(q)$ is defined by (2.1), then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q; q)_n (1 - q^{2n+1})} &= qf(q, q^7), \\ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n (1 - q^{2n+1})} &= f(q^3, q^5), \\ \varphi(-q) \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{(n+1)(n+2)/2}}{(q; q)_n (q; q^2)_{n+1}} &= qf(-q, -q^7), \\ \varphi(-q) \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+1)/2}}{(q; q)_n (q; q^2)_{n+1}} &= f(-q^3, -q^5). \end{aligned}$$

As in (3.4), the partitions generated on the left-hand sides above mostly “cancel” each other, as we see from the right-hand sides. The first two are proved combinatorially in [26].

As another example, we offer [57, p. 36], [9, p. 235, Entry 9.4.7]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(-q^2; q^2)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}). \quad (5.1)$$

The function on the left-hand side is a third order mock theta function, and the function on the right-hand side is a *false theta function*. Note that the latter series is the same as Euler's pentagonal number series (2.5), except that the signs of infinitely many terms have been changed. The identity (5.1) does have a combinatorial proof [26], [51].

We conclude this section with the two identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n} = \frac{(q^5; q^5)_{\infty} (q^5; q^{10})_{\infty}}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}} + 2 - 2 \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q; q^5)_{n+1} (q^4; q^5)_n}, \quad (5.2)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n} = \frac{(q^5; q^5)_{\infty} (q^5; q^{10})_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}} + \frac{2}{q} - \frac{2}{q} \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2; q^5)_{n+1} (q^3; q^5)_n}, \quad (5.3)$$

from Ramanujan's lost notebook [57]. The functions on the left-hand sides of (5.2) and (5.3) are fifth order mock theta functions. The first quotients on the right-hand sides of (5.2) and (5.3) can be expressed in terms of theta functions. On pages 18–20 in his lost notebook [57], Ramanujan stated two classes of five identities of which (5.2) and (5.3) are representatives of the two classes, respectively. Andrews and Garvan [11] proved that if one identity in each group of five could be proved, then the remaining four identities would follow in each case. These became known as the *mock theta conjectures*, and they were proved by D. Hickerson [44] in 1988.

The identities (5.2) and (5.3) have interesting combinatorial interpretations. We state, but not prove, the combinatorial equivalent of (5.2). No combinatorial proof of either (5.2) or (5.3) has ever been given, i.e., there do not currently exist combinatorial proofs of the mock theta conjectures. To explicate the first combinatorial interpretation, it is necessary to make a couple definitions and introduce some notation.

Definition 5.1. *The rank of a partition equals the largest part minus the number of parts.*

For example, the rank of the partition $4 + 1$ is 2. Let $N(a, b, n)$ denote the number of partitions of n with rank congruent to a modulo b .

Definition 5.2. *We define $\rho_0(n)$ to be the number of partitions of n with unique smallest part and all other parts \leq the double of the smallest part.*

For example, $\rho_0(5) = 3$, with the relevant partitions being 5 , $2 + 3$, and $1 + 2 + 2$.

The First Mock Theta Conjecture

$$N(1, 5, 5n) = N(0, 5, 5n) + \rho_0(n).$$

For example, if $n = 5$, then $N(1, 5, 25) = 393$, $N(0, 5, 25) = 390$, and, as observed above, $\rho_0(5) = 3$.

6. CONCLUDING REMARKS

We have seen that initially the theory of basic hypergeometric series was developed more or less in isolation with few connections with other parts of mathematics. For this reason, the subject's development was slow and scarcely appreciated. The Rogers–Ramanujan identities (2.15) and (2.16) perhaps provided the first examples to break

these chains of confinement and link q -series with partitions at a deeper level. Since that time, q -series have not only played a leading role in the theory of partitions, but they have appeared prominently in other areas of number theory and combinatorics, e.g., in sums of squares. See [24] for some of these applications.

Today, as a subject in analysis, the work of Gasper, Rahman, Askey, Mourad Ismail, Michael Schlosser and others has continued the work of the early pioneers, Heine, Rogers, Jackson, and Ramanujan, in giving us a beautiful coherent theory still in the process of active development.

Many other bonds with q -series now exist. For example, q -series have appeared in various ways in physics. In particular, see the work of R.J. Baxter in statistical mechanics, e.g., [19], [18], and papers (either solely or coauthored) by A. Schilling [12], [52], [62], [63].

In this introduction to q -series, we have ignored several topics. First, we have not discussed bibasic hypergeometric series, which possess two independent bases, say q and p [39, pp. 80–88]. Second, we have neglected to mention multiple q -series, a subject to which Steve Milne has made numerous contributions. Third, we have not discussed q -integration, the exclusive topic of the monograph [48]. Fourth, one of the current active topics in q -series are elliptic and theta hypergeometric series [39, Chapter 11]. Very briefly and oversimplified, a theta hypergeometric series has in its summands products of theta functions. Schlosser, Warnaar, and Hjelmar Rosengren are three of the leading researchers in the subject. In particular, see Schlosser's excellent survey [64] elsewhere in this volume.

Where should readers begin to learn the subject of q -series. If one wants to learn how q -series interact with the theory of partitions, readers should read Andrews's book [3, esp., Chapter 2] and N.J. Fine's book [37]. If one wishes to learn how q -series articulate with theta functions and number theory, in particular from the point of view of Ramanujan, read Berndt's book [24]. For a highly motivated introduction to the subject of q -series, one should read Chapter 10 in the book of Andrews, Askey, and Roy [8]. Lastly, if you want to be an expert in q -series, or if you really just want to learn the subject very well, then you must read Gasper and Rahman's text [39].

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