RAMANUJAN'S CONGRUENCES FOR THE PARTITION FUNCTION MODULO 5, 7, AND 11

BRUCE C. BERNDT[1]

To my teacher and friend, Marvin Knopp

Abstract. Using Ramanujan's differential equations for Eisenstein series and an idea from Ramanujan's unpublished manuscript on the partition function $p(n)$ and the tau function $\tau(n)$, we provide simple proofs of Ramanujan's congruences for $p(n)$ modulo 5, 7, and 11.

Key Words: partition congruences, Eisenstein series, Ramanujan's lost notebook.

2000 Mathematical Reviews Classification Numbers: Primary, 11P83; Secondary, 11P82.

1. Introduction

In his paper [4], [7, pp. 210–213], Ramanujan gave proofs, for the first time, of the the first two of his three famous congruences for the partition function $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.3}$$

where n is any nonnegative integer. The congruence (1.3) was stated for the first time by Ramanujan in [5], [7, p. 230]. After Ramanujan died in 1920, G. H. Hardy [6], [7, pp. 232–238] extracted proofs of (1.1)–(1.3) from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$. This manuscript was published for the first time in handwritten form in a volume [8] containing Ramanujan's lost notebook. An expanded and annotated version was prepared by the author and K. Ono [1]. The proofs in [6] employ Eisenstein series.

Our objective in this paper is to give simple proofs of (1.1)–(1.3) that also utilize Eisenstein series. The principal idea behind the new proofs is actually due to Ramanujan in the aforementioned manuscript. To describe this idea, we need to define Ramanujan's Eisenstein series. Let

$$P := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{1.4}$$

$$Q := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \tag{1.5}$$

[1]Research partially supported by grant MDA904-00-1-0015 from the National Security Agency.

and

$$R := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \tag{1.6}$$

where here and in the sequel $|q| < 1$. These Eisenstein series satisfy Ramanujan's famous differential equations [3, eq. (30)]

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{1.7}$$

Moreover, Q and R satisfy the well-known discriminant relation [3, eq. (44)]

$$Q^3 - R^2 = 1728q(q;q)_\infty^{24} =: 1728 \sum_{n=1}^{\infty} \tau(n) q^n. \tag{1.8}$$

In his manuscript, Ramanujan proves or states further congruences for $p(n)$. In particular, he proves that $p(13n-7) \equiv 11\tau(n) \pmod{13}$. To prove this congruence, Ramanujan establishes the "identity"

$$(q^{13};q^{13})_\infty \sum_{n=1}^{\infty} p(13n-7)q^n = 11 \sum_{n=1}^{\infty} \tau(n) q^n + 13J. \tag{1.9}$$

He then writes, "It is not necessary to know all the details above in order to prove (1.9). The proof can be very much simplified as follows; using (1.7) and ... we can show that

$$(Q^3 - R^2)^7 = q\frac{dJ}{dq} + 3(Q^3 - R^2) + 13J.\text{"} \tag{1.10}$$

Here $J = J(q)$ is a power series in q with integral coefficients. We use this notation of Ramanujan throughout the paper and emphasize that J is not necessarily the same at each occurrence. Apparently, Ramanujan realized this simplification at precisely this juncture while writing his paper, for he did not return to his proofs of (1.1)–(1.3) to utilize this observation and thereby simplify them. In his unpublished doctoral dissertation, J. M. Rushforth [9] used this idea to simplify Ramanujan's proof of (1.3), as extracted by Hardy for [6]. In this paper, we use Ramanujan's observation above along with (1.7) and (1.8) to give simplified proofs of (1.1)–(1.3), with the proof of (1.3) being precisely that of Rushforth. Another approach employing Eisenstein series to prove congruences for partition functions was developed by A. D. Forbes [2].

For the proofs of (1.2) and (1.3), it is still necessary to use three further basic Eisenstein series from Table I of Ramanujan's paper [3], [7, p. 141] (but not nearly as many identities as used in [6]), namely,

$$Q^2 = 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}, \tag{1.11}$$

$$QR = 1 - 264 \sum_{n=1}^{\infty} \frac{n^9 q^n}{1-q^n}, \tag{1.12}$$

and

$$441Q^3 + 250R^2 = 691 + 65520\sum_{n=1}^{\infty}\frac{n^{11}q^n}{1-q^n}. \tag{1.13}$$

2. Proofs of (1.1)–(1.3)

Theorem 2.1. *For each nonnegative integer n,*

$$p(5n+4) \equiv 0 \pmod 5. \tag{2.1}$$

Proof. From the definitions (1.5) and (1.6), respectively,

$$Q = 1 + 5J \qquad \text{and} \qquad R = P + 5J, \tag{2.2}$$

since $n^5 \equiv n \pmod 5$ by Fermat's little theorem. It follows from (2.2) and (1.7) that

$$\begin{aligned} Q^3 - R^2 = Q(1+5J)^2 - (P+5J)^2 =& Q - P^2 + 5J \\ =& -12q\frac{dP}{dq} + 5J = 3q\frac{dP}{dq} + 5J. \end{aligned} \tag{2.3}$$

But, by (1.8) and the binomial theorem,

$$Q^3 - R^2 = 1728q(q;q)_\infty^{24} = 3q\frac{(q;q)_\infty^{25}}{(q;q)_\infty} + 5J = 3q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} + 5J. \tag{2.4}$$

Combining (2.3) and (2.4) and using the generating function for $p(n)$, we find that

$$q(q^5;q^5)_\infty^5\sum_{n=0}^{\infty}p(n)q^n = q\frac{dP}{dq} + 5J. \tag{2.5}$$

We now equate those terms on both sides of (2.5) whose powers are of the form q^{5n} to find that

$$(q^5;q^5)_\infty^5\sum_{n=0}^{\infty}p(5n+4)q^{5n+5} = 5J. \tag{2.6}$$

Ramanujan's congruence (2.1) follows immediately from (2.6). □

Theorem 2.2. *For each nonnegative integer n,*

$$p(7n+5) \equiv 0 \pmod 7. \tag{2.7}$$

Proof. The first two steps of our proof are the same as those of Ramanujan and Hardy [6], [7, p. 235]. From the definition of R, it is obvious that

$$R = 1 + 7J. \tag{2.8}$$

Using (1.11), Fermat's little theorem, and the definition (1.4), we also find that

$$Q^2 = P + 7J. \tag{2.9}$$

Hence, from (2.8), (2.9), and (1.7),

$$\begin{aligned} (Q^3 - R^2)^2 =& (PQ - 1 + 7J)^2 \\ =& P^2Q^2 - 2PQ + 1 + 7J \\ =& P(P^2 - Q) - PQ + R + 7J \end{aligned}$$

$$=5Pq\frac{dP}{dq} - 3q\frac{dQ}{dq} + 7J$$
$$=6q\frac{d}{dq}P^2 - 3q\frac{dQ}{dq} + 7J = q\frac{dJ}{dq} + 7J. \tag{2.10}$$

On the other hand, by (1.8) and the binomial theorem,

$$(Q^3 - R^2)^2 = q^2\frac{(q;q)_\infty^{49}}{(q;q)_\infty} = q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} + 7J = (q^7;q^7)_\infty^7\sum_{n=0}^\infty p(n)q^{n+2} + 7J. \tag{2.11}$$

We now equate the right sides of (2.10) and (2.11) and then extract those terms involving q^7. Equating these terms, we find that

$$(q^7;q^7)_\infty^7\sum_{n=0}^\infty p(7n+5)q^{7n+7} = 7J,$$

from whence (2.7) immediately follows. □

Theorem 2.3. *For each nonnegative integer n,*

$$p(11n+6) \equiv 0 \pmod{11}. \tag{2.12}$$

Proof. The beginning of the proof is identical to that of Ramanujan as related by Hardy [6], [7, pp. 235–236]. Let it suffice to say that using (1.13), one can easily show that

$$Q^3 - 3R^2 = -2P + 11J, \tag{2.13}$$

and, from (1.12), it is trivial that

$$QR = 1 + 11J. \tag{2.14}$$

Using (2.13) and (2.14), Ramanujan and Hardy [6] then prove that

$$(Q^3 - R^2)^5 = P^5 - 3P^3Q - 4P^2R + 6QR + 11J. \tag{2.15}$$

We refer readers to the details in [6], [7, p. 236] demonstrating (2.15).

Invoking Ramanujan's differential equations (1.7), we find that

$$\begin{aligned}P^5 - 3P^3Q - 4P^2R + 6QR =&(P^3 + 3PQ + 5R)(P^2 - Q) - 5P^2(PQ - R)\\ &- 3P(PR - Q^2) + 11J\\ =&12(P^3 + 3PQ + 5R)q\frac{dP}{dq} - 15P^2q\frac{dQ}{dq} - 6Pq\frac{dR}{dq} + 11J\\ =&(P^3 + 3PQ + 5R)q\frac{dP}{dq} - 4P^2q\frac{dQ}{dq} + 5Pq\frac{dR}{dq} + 11J\\ =&3q\frac{d}{dq}P^4 - 4q\frac{d}{dq}(P^2Q) + 5q\frac{d}{dq}(PR) + 11J\\ =&q\frac{dJ}{dq} + 11J.\end{aligned} \tag{2.16}$$

Now, by (1.8), the binomial theorem, (2.15), and (2.16),

$$(Q^3 - R^2)^5 = q^5 \frac{(q;q)_\infty^{121}}{(q;q)_\infty} = q^5 (q^{11};q^{11})_\infty^{11} \sum_{n=0}^{\infty} p(n) q^n = q\frac{dJ}{dq} + 11J. \tag{2.17}$$

Equating the terms involving the powers q^{11n} on both sides of (2.17), we easily deduce that

$$(q^{11};q^{11})_\infty^{11} \sum_{n=0}^{\infty} p(11n+6) q^{11n+11} = 11J,$$

from which (2.12) is apparent. □

The author is grateful to Professor Norrie Everitt, the School of Mathematics, and the Library at the University of Birmingham for providing a copy of Rushforth's thesis.

References

[1] B. C. Berndt and K. Ono, *Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary*, Sém. Lotharingien de Combinatoire **42** (1999), 63 pp.; in *The Andrews Festschrift*, D. Foata and G.–N. Han, eds., Springer–Verlag, Berlin, 2001, pp. 39–110.

[2] A. D. Forbes, *Congruence properties of functions related to the partition function*, Pacific J. Math. **158** (1993), 145–156.

[3] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184.

[4] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. **19** (1919), 207–210.

[5] S. Ramanujan, *Congruence properties of partitions*, Proc. London Math. Soc. **18** (1920), xix.

[6] S. Ramanujan, *Congruence properties of partitions*, Math. Z. **9** (1921), 147–153.

[7] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[8] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[9] J. M. Rushforth, *Congruence Properties of the Partition Function and Associated Functions*, Doctoral Thesis, University of Birmingham, 1950.

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: berndt@math.uiuc.edu