

DIOPHANTINE APPROXIMATION WITH PARTIAL SUMS OF POWER SERIES

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1. INTRODUCTION AND STATEMENTS OF RESULTS

Consider the exponential generating function of a sequence $\{a_n\}$, $n \geq 0$,

$$G(s) := \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n, \quad s \in \mathbb{C}. \quad (1.1)$$

For each nonzero rational number r , in [1], we considered the problem of approximating $G(r)$ with partial sums of the series (1.1). In the case that $a_n \equiv 1$ and $s = 1$, we asked how well one can approximate e by the partial sums $\sum_{\ell=0}^n \frac{1}{\ell!}$. J. Sondow [6] conjectured that exactly two of these partial sums are also convergents to the continued fraction of e . Among several results, Sondow and K. Schalm [7], proved that for almost all positive integers n , the partial sum $\sum_{\ell=0}^n \frac{1}{\ell!}$ is not a convergent to the continued fraction of e . Thus, the probability of obtaining a convergent to the continued fraction of e upon randomly choosing one of the first n partial sums of the power series of e tends to zero as $n \rightarrow \infty$. Knowledge of the continued fraction of $e^{2/a}$, where a is a nonzero integer, and the best possible diophantine approximation of $e^{2/a}$, discovered by S. Ramanujan [5] and rediscovered by C. S. Davis [4], enabled the authors to prove in [2] that at most $O_a(\log M)$ of the first M convergents to the continued fraction of $e^{2/a}$ are also partial sums of the corresponding power series. In [1], we considered general hypergeometric functions ${}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; r)$ and showed that among their first N partial sums, no more than $O(\log N)$ are convergents to the continued fraction of ${}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; r)$. Observe that this result includes $e^{2/a}$ as a special case and that this particular corollary is a dual of the aforementioned result established in [2]. Moreover, when a_n is a real Dirichlet character or when $\{a_n\}$, $n > 0$, are the coefficients of an L -series attached to an elliptic curve without complex multiplication, we proved similar theorems in [1]. Lastly, we remark that in [2], we, in fact, proved Sondow's conjecture.

At the focal point of our study in [1] and in this paper are the following two definitions. For any rational number $\mu = a/b$, with $(a, b) = 1$, consider the height $H(\mu)$ of μ , given by $H(\mu) = \max\{|a|, |b|\}$. For any real number α , and any positive real number δ , denote

$$A_{\alpha, \delta} := \left\{ \mu \in \mathbb{Q} : |\alpha - \mu| < \frac{1}{H(\mu)^{1+\delta}} \right\}. \quad (1.2)$$

In the present paper, we greatly expand our outlook in two different directions. First, we complement the results from [1] with a positivity argument. For sequences $\{a_n\}_{n \in \mathbb{N}}$ of nonnegative integers, we first establish a general result with the accuracy of the results described in our opening paragraph. For each such sequence, we define the *gap function*

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$i(n)$ to be the number of consecutive elements equal to zero immediately following a_n , that is,

$$i(n) := \min\{m : m \geq n + 1, a_m \neq 0\} - n - 1. \quad (1.3)$$

Thus, for sequences in which $a_n > 0$, $n \geq 1$, the gap function is identically equal to zero. In the sequel, we consider sequences for which the gap function satisfies

$$\lim_{n \rightarrow \infty} \frac{i(n)}{n} = 0. \quad (1.4)$$

We shall allow the size of a_n to be quite large. More precisely, we require that $a_n = O_\epsilon(n^{\epsilon n})$, for any $\epsilon > 0$, which we write in the form

$$\lim_{n \rightarrow \infty} \frac{\log(1 + a_n)}{n \log n} = 0. \quad (1.5)$$

We then have the following result.

Theorem 1.1. *Let $\{a_n\}$, $n \in \mathbb{N}$, be a sequence of nonnegative integers satisfying (1.4) and (1.5), and consider the entire function*

$$G(s) := \sum_{n=1}^{\infty} \frac{a_n}{n!} s^n. \quad (1.6)$$

Then for any positive rational number r and any real number $\delta > 0$, there is a constant $C_1 > 0$, depending only on δ , and a constant $C_2 > 0$, depending on r , δ , and the given sequence $\{a_n\}$, $n \in \mathbb{N}$, such that for all positive integers N ,

$$\# \left\{ n \leq N : a_n \neq 0 \text{ and } \sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell \in A_{G(r), \delta} \right\} \leq C_1 \log N + C_2. \quad (1.7)$$

Theorem 1.1 is applicable to a large variety of sequences of interest in number theory. We offer a few classical examples obtained from the Riemann zeta function $\zeta(s)$ and L -functions: $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s}$, where $d_k(n)$ denotes the number of ways of writing n as a product of k positive integers;

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s},$$

where $\varphi(n)$ denotes Euler's φ -function; and, for any two quadratic Dirichlet characters χ_1 and χ_2 , $\zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_1\chi_2)$, a product which plays a key role in Siegel's theorem on possible exceptional (Siegel) zeros; see Davenport's text [3, Chapter 21].

We performed substantial calculations in support of our conjecture that, in each case, values of partial sums coincide with partial quotients of the continued fraction only a finite number of times. In particular, for integral powers of $\zeta(s)$ up to the 20th, we calculated the first 2000 partial sums and partial quotients. The table below provides a list of coalescing partial sums and partial quotients of continued fractions that we found in our calculations for powers of $\zeta(s)$. We conjecture that for any positive integral power of $\zeta(s)$, there are only a finite number of values assumed by both partial sums and partial quotients. Moreover, in each of the cases below, we conjecture that we have found all possible matchings. Observe

that for powers $k > 6$, only when $k = 15$ do we find matching partial sums and partial quotients. Is it true that for $k > 15$, there are no further common values?

k	Common Values	k	Common Values
1	$\{1, \frac{5}{3}\}$	11	$\{\}$
2	$\{2\}$	12	$\{\}$
3	$\{3, \frac{1243}{378}\}$	13	$\{\}$
4	$\{\}$	14	$\{\}$
5	$\{5\}$	15	$\{16, \frac{27649}{1680}\}$
6	$\{5, \frac{27545}{4608}\}$	16	$\{\}$
7	$\{\}$	17	$\{\}$
8	$\{\}$	18	$\{\}$
9	$\{\}$	19	$\{\}$
10	$\{\}$	20	$\{\}$

Next, for a real Dirichlet character χ , we consider

$$G(s) = G_\chi(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n!} s^n.$$

In [1], we conjectured that for any real Dirichlet character χ and any nonzero rational number r , there exist only finitely many positive integers n for which the partial sum $\sum_{n=1}^m \frac{\chi(n)}{n!} r^n$ is a convergent to the continued fraction of $G_\chi(s)$. In connection with this conjecture, we showed in [1] that at most $O_{\chi,r}(\log N)$ of the first N partial sums are also convergents to the continued fraction of $G_\chi(r)$. We also raised the natural problem to estimate, for large x , the largest number of the common values of the partial sums $\sum_{n=1}^m \frac{\chi(n)}{n!} r^n$ and the convergents to the continued fraction of $G_\chi(r)$ as χ runs over all real characters of conductor $\leq x$. In relation to this, we prove the following result.

Theorem 1.2. *For a real Dirichlet character χ , let $\ell(\chi)$ be the number of values to simultaneously be partial sums of the power series and convergents to the continued fraction of $G_\chi(s)$. Then,*

$$\max_{q \leq x} \max_{\substack{\chi \bmod q \\ \chi \text{ real}}} \ell(\chi) \gg \log \log x.$$

In another direction, instead of studying the approximation of one particular real value of the exponential function, i.e., $e^{2/a}$, by rational numbers, we examine the more general and delicate problem of approximating complex numbers of the form $\alpha_1 e^{\beta_1} + \dots + \alpha_r e^{\beta_r}$, where $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$ are in $\overline{\mathbb{Q}}$, the algebraic closure of \mathbb{Q} in \mathbb{C} , by partial sums of their power series. Before commencing this study, we need to describe the setting and fix notation.

For any finite field extension K of \mathbb{Q} contained in $\overline{\mathbb{Q}}$, we denote by \mathcal{O}_K the ring of integers of K . For any ideal J we let $\|J\|$ denote its norm. In what follows, we use the following notion of height of an algebraic number. Given $\theta \in \overline{\mathbb{Q}}$, there exists a unique polynomial

$P(X) = C_0X^d + C_1X^{d-1} + \cdots + C_d \in \mathbb{Z}[X]$, irreducible over \mathbb{Q} , with $C_0 > 0$ and $\gcd(C_0, C_1, \dots, C_d) = 1$ such that $P(\theta) = 0$. We let

$$H(\theta) := \max\{C_0, |C_1|, \dots, |C_d|\}. \quad (1.8)$$

In particular, if $\theta = a/q$ is rational, with $a, q \in \mathbb{Z}, q > 0, (a, q) = 1$, then $P(X) = qX - a$ and $H(\theta) := \max\{q, |a|\}$. We will also use the following terminology. Given a vector $\bar{\beta} = (\beta_1, \dots, \beta_r)$ with $\beta_1, \dots, \beta_r \in \overline{\mathbb{Q}}$, we will say that $\bar{\beta}$ is admissible if the numbers β_1, \dots, β_r are nonzero and distinct, and if two components β_i and β_j have the same largest absolute value, then the ratio β_i/β_j is a root of unity.

Now let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r), \bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \overline{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible, and consider the complex number

$$\rho = \alpha_1 e^{\beta_1} + \cdots + \alpha_r e^{\beta_r}. \quad (1.9)$$

We know from the Lindemann-Weierstrass theorem that ρ is transcendental over \mathbb{Q} . We write ρ in the form

$$\rho = \sum_{\ell=0}^{\infty} \frac{a_\ell}{\ell!}, \quad (1.10)$$

where for any ℓ ,

$$a_\ell = \sum_{j=1}^r \alpha_j \beta_j^\ell, \quad (1.11)$$

and consider approximations of ρ by the partial sums $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$. To assess the quality of such an approximation, we consider the height $H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right)$ and the distance from $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$ to ρ .

For any complex number z and any positive real number τ , we consider the set

$$A_{z,\tau} = \left\{ \theta \in \overline{\mathbb{Q}} : |z - \theta| \leq \frac{1}{H(\theta)^\tau} \right\}. \quad (1.12)$$

Note that by (1.10) and (1.11),

$$\left| \rho - \sum_{\ell=0}^n \frac{a_\ell}{\ell!} \right| \leq \frac{1}{(n!)^{1-\epsilon}}, \quad (1.13)$$

for any fixed $\epsilon > 0$ and n large enough in terms of $\bar{\alpha}, \bar{\beta}$, and ϵ . Also, at least in the case when $\bar{\alpha}$ and $\bar{\beta}$ have rational components,

$$H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right) \leq (n!)^{1+\epsilon}, \quad (1.14)$$

for any fixed $\epsilon > 0$ and n sufficiently large in terms of $\bar{\alpha}, \bar{\beta}$, and ϵ . By (1.13) and (1.14), it follows that if $\bar{\alpha}$ and $\bar{\beta}$ have rational components, then each of the sets $A_{\rho,\tau}$, $0 < \tau < 1$, contains all but finitely many partial sums $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$. In the following, we focus on the case when $\tau > 1$. Our main theorem below shows that in this case very few partial sums $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$ belong to $A_{\rho,\tau}$. More precisely, we prove the following result.

Theorem 1.3. *Let r be a positive integer, let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \overline{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible, and let τ be a real number > 1 . Then, for large x ,*

$$\#\left\{n \in \mathbb{N} : n \leq x, \sum_{\ell=0}^n \frac{a_\ell}{\ell!} \in A_{\rho, \tau}\right\} = O_{\bar{\alpha}, \bar{\beta}, \tau}(\log x), \quad (1.15)$$

where ρ and the a_ℓ 's are given by (1.9) and (1.11). Here, the constant implied by the $O_{\bar{\alpha}, \bar{\beta}, \tau}$ -symbol is effectively computable.

We conjecture that in each nontrivial instance of our theorem, the number of matchings of partial sums with partial quotients of the corresponding continued fractions is actually finite. To that end, we performed extensive calculations for the cases $r = 2$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = a/b$, and $\beta_2 = c/d$, with $1 \leq |a|, |b|, |c|, |d| \leq 50$. In particular, we examined both partial sums and partial quotients up to 10,000 terms. The number of common values that we found ranged from 0 to 3. In the following table, we list only a few representative findings.

ρ	Common Values
$e^{2/19} + e^{-3/19}$	$\{2, \frac{112}{57}\}$
$e^{-9/10} + e^{4/5}$	$\{2, \frac{21}{8}\}$
$e^{-1/30} + e^{-1/15}$	$\{2, \frac{19}{10}, \frac{137}{72}\}$
$e^{1/21} + e^{2/21}$	$\{2, \frac{15}{7}, \frac{737}{343}\}$
$e^{1/45} + e^{2/45}$	$\{2, \frac{31}{15}, \frac{335}{162}\}$

For the exponential sums in the table above, we conjecture that we have found all the instances where a partial quotient matches a partial sum. We noticed that for $a = -c = 1$ and $b = d$, i.e., when we examined $2 \cosh(1/b)$, we always found two common values for the partial sums and partial quotients. In fact, it is easy to prove that for $\cosh(1/b)$, we always have an intersection of $\{2, 2 + 1/b^2\}$ of partial sums and partial quotients. We conjecture that these are the only coalescing values. Further observations led us to the following theorem, which we do not prove here. It is likely that the two common values in Theorem 1.4 below are the only ones.

Theorem 1.4. *For any positive integers a and c , and any n that is a multiple of $a + c$ and satisfies the inequality*

$$n > \left\{ a + c, \frac{4(a^3 + c^3)}{21ac} \right\},$$

both 2 and $2 + (a + c)/n$ are common values of partial sums and partial quotients for

$$\rho = e^{a/n} + e^{c/n}.$$

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 follows roughly along the same lines as the proofs of the results from [1] that we delineated in the Introduction.

Let $\{a_n\}$ be a sequence of nonnegative integers satisfying (1.4) and (1.5), and let $G(s)$ be given by (1.6). Fix a positive rational number r and a real number $\delta > 0$. Fix a small $\eta > 0$, whose precise value will depend on δ only. For a fixed large integer N , let $S_{G,r,\delta,\eta,N}$ denote the set of integers in the interval $[N, (1 + \eta)N]$ for which $a_n \neq 0$ and

$$\sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell \in A_{G(r),\delta}.$$

Next, take any nonempty subset S in $S_{G,r,\delta,\eta,N}$ and denote $v = \#(S)$. By the positivity of r and the nonnegativity of a_n , $n \geq 1$, it follows that for any integer n ,

$$G(r) - \sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell = \sum_{\ell=n+1}^{\infty} \frac{a_\ell}{\ell!} r^\ell \geq \frac{a_{n+1+i(n)}}{(n+1+i(n))!} r^{n+1+i(n)}. \quad (2.1)$$

By (2.1), (1.4), Stirling's formula, and the fact that $a_{n+1+i(n)} \geq 1$, we see that, uniformly for all integers $n \in [N, (1 + \eta)N]$,

$$G(r) - \sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell > \frac{1}{n^{n(1+o(1))}}. \quad (2.2)$$

For those $n \in S_{G,r,\delta,\eta,N}$, we then find from (2.2) that

$$H\left(\sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell\right) < \left(G(r) - \sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell\right)^{-1/(1+\delta)} < n^{\frac{1+o(1)}{1+\delta}n} < N^{\frac{(1+\eta)(1+o(1))}{1+\delta}N} < N^{(1-\frac{1}{2}\delta)N}, \quad (2.3)$$

for $0 < \delta < 1$, η sufficiently small in terms of δ , and N sufficiently large.

Next, let $r = a/b$, $a, b \in \mathbb{Z}$, $b \geq 1$, $(a, b) = 1$, and for any $n \in S$, let A_n be the positive integer for which

$$\sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell = \frac{A_n}{b^n n!}. \quad (2.4)$$

Denote $D_n := (A_n, b^n n!)$. Let $n, m \in S$ with $n < m$. Then

$$\sum_{\ell=1}^n \frac{a_\ell}{\ell!} r^\ell < \sum_{\ell=1}^m \frac{a_\ell}{\ell!} r^\ell,$$

and

$$A_m - b^{m-n} m(m-1) \cdots (n+1) A_n \in \mathbb{N}.$$

Uniformly, for all n, m as above, we can conclude that

$$\begin{aligned} (D_n, D_m) &\leq (A_n, A_m) \leq A_m - b^{m-n} m(m-1) \cdots (n+1) A_n \\ &= b^m m! \sum_{\ell=n+1}^m \frac{a_\ell a^\ell}{\ell! b^\ell} \leq (m-n) \frac{m!}{n!} H(r)^m \max_{n < \ell \leq m} a_\ell \\ &= O\left(\frac{m!}{n!} H(r)^m m^{o(m)}\right). \end{aligned} \quad (2.5)$$

If we let L_S denote the least common multiple of the D_n 's, with $n \in S$, then, by Stirling's formula, we find that

$$\frac{\prod_{n \in S} D_n}{\prod_{\substack{n, m \in S \\ n < m}} (D_n, D_m)} \leq L_S \leq b^{[(1+\eta)N]} [(1+\eta)N]! \ll_{\eta} \left(\frac{2bN}{e}\right)^{(1+\eta)N}. \quad (2.6)$$

Now, by the definition of D_n ,

$$H\left(\sum_{\ell=1}^n \frac{a_{\ell}}{\ell!} r^{\ell}\right) = H\left(\frac{A_n}{b^n n!}\right) = \max\left\{\frac{|A_n|}{D_n}, \frac{b^n n!}{D_n}\right\} \sim \frac{b^n n!}{D_n} \max\{1, |G(r)|\}, \quad (2.7)$$

as $n \rightarrow \infty$, and so by (2.7), (2.3), and Stirling's formula,

$$D_n \geq \frac{b^n n!}{N^{(1-\frac{1}{2}\delta)N}} \geq \left(\frac{b}{e}\right)^N N^{\delta N/2}. \quad (2.8)$$

By (2.5)–(2.8), we find that

$$\begin{aligned} \left(\frac{b}{e}\right)^{vN} N^{\delta v N/2} &\ll_{\eta, v} \left(\frac{2bN}{e}\right)^{(1+\eta)N} \prod_{\substack{n, m \in S \\ n < m}} \frac{m!}{n!} H(r)^m m^{o(m)} \\ &\ll_{\eta, v} \left(\frac{2bN}{e}\right)^{(1+\eta)N} H(r)^{\frac{1}{2}(1+\eta)Nv(v-1)} N^{o(v^2 N)} \prod_{\substack{n, m \in S \\ n < m}} \frac{m!}{n!}. \end{aligned} \quad (2.9)$$

Observe that the product on the right side of (2.9) can be written in the simplified form

$$\prod_{\substack{n, m \in S \\ n < m}} \frac{m!}{n!} = \prod_{n \in S} (n!)^{v-1-2\#\{m \in S : m > n\}}. \quad (2.10)$$

On the right-hand side of (2.10), we use the inequality $n! \geq N!$ for those $n \in S$ for which $v-1-2\#\{m \in S : m > n\} < 0$ and the inequality $n! \leq [(1+\eta)N]!$ for those $n \in S$ for which $v-1-2\#\{m \in S : m > n\} > 0$. It follows that

$$\begin{aligned} \prod_{\substack{n, m \in S \\ n < m}} \frac{m!}{n!} &\leq \begin{cases} \left(\frac{[(1+\eta)N]!}{N!}\right)^{(v^2-1)/4}, & \text{if } v \text{ is odd,} \\ \left(\frac{[(1+\eta)N]!}{N!}\right)^{v^2/4}, & \text{if } v \text{ is even,} \end{cases} \\ &\leq \left(\frac{[(1+\eta)N]!}{N!}\right)^{v^2/4}. \end{aligned} \quad (2.11)$$

By (2.9), (2.11), and Stirling's formula, we find that

$$\begin{aligned} \frac{\delta v N}{2} \log N &\leq (1+\eta)N \log \frac{2bN}{e} + \frac{(1+\eta)Nv(v-1)}{2} \log H(r) \\ &\quad + \frac{v^2}{4} \left((1+\eta)N \log \frac{(1+\eta)N}{e} - N \log \frac{N}{e} \right) + o(v^2 N \log N). \end{aligned} \quad (2.12)$$

We now let $N \rightarrow \infty$ while keeping δ , η , and v fixed. It follows that

$$\frac{\delta v}{2} \leq 1 + \eta + \frac{v^2 \eta}{4} + o(v^2). \quad (2.13)$$

We see that the inequality (2.13) fails to hold if N is large enough, because, for a fixed $\delta > 0$ and a given positive integer v , we can choose η small enough so that

$$\frac{\delta v}{2} > 1 + \eta + \frac{v^2 \eta}{4},$$

which contradicts (2.13) for N sufficiently large.

In conclusion, for such a fixed v and any N sufficiently large, $S_{G,r,\delta,\eta,N}$ cannot have v elements or more. Lastly, we take a sufficiently large integer N , subdivide the interval $[1, N]$ into intervals of the form $[T, (1 + \eta)T]$, and use the upper bound obtained above for each such interval, provided that T is sufficiently large. We thus readily obtain the bound (1.7) to complete the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

Let x be large, and let p_n , $n \geq 1$, be the n -th prime. Define r by

$$2 \cdot 5 \cdot 7 \cdots p_r \leq x < 2 \cdot 5 \cdot 7 \cdots p_r \cdot p_{r+1}. \quad (3.1)$$

Then, by (3.1) and the prime number theorem, we have

$$\log x \sim \sum_{\substack{p \leq p_{r+1} \\ p \text{ prime}}} \log p \sim p_{r+1}. \quad (3.2)$$

Define

$$q = 2 \prod_{n=3}^r p_n.$$

By (3.1), we have $q \leq x$. Now, choose χ to be the principal character modulo q . Then,

$$\chi(n) = \begin{cases} 1, & \text{if } n = 1, 3, 3^2, 3^3, \dots, \\ 0, & \text{if } n < p_{r+1} \text{ and } n \neq 1, 3, 3^2, 3^3, \dots, \end{cases}$$

and

$$\alpha := \sum_{n \geq 1} \frac{\chi(n)}{n!} = 1 + \frac{1}{3!} + \frac{1}{3^2!} + \frac{1}{3^3!} + \cdots + \sum_{n \geq p_{r+1}} \frac{\chi(n)}{n!}.$$

We now let k be a nonnegative integer such that $3^{k+1} < p_{r+1}$, and we define a_k as follows:

$$1 + \frac{1}{3!} + \frac{1}{3^2!} + \frac{1}{3^3!} + \cdots + \frac{1}{(3^k)!} = \frac{a_k}{3^k!}.$$

Then,

$$\left| \alpha - \frac{a_k}{3^k!} \right| < \sum_{n \geq (3^{k+1})!} \frac{1}{n!} \leq \frac{2}{(3^{k+1})!} < \frac{1}{2(3^k!)^2}. \quad (3.3)$$

Recalling that for a rational number a/b , if $|\alpha - a/b| < 1/(2b^2)$, then a/b is a convergent for the continued fraction of α , we can see that $a_k/(3^k!)$ is a convergent and a partial sum of α . Thus,

$$\ell(\chi) \gg \log p_{r+1} \sim \log \log x,$$

by (3.2), which completes the proof.

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is based on several lemmas.

Lemma 4.1. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \mathbb{Q} - \{0\}$ and $\bar{\beta}$ admissible. There exists a positive integer $N_{\bar{\alpha}, \bar{\beta}}$, depending only on $\bar{\alpha}$ and $\bar{\beta}$, with the following property. For any integers m and n satisfying*

$$\min\{m, n\} \geq N_{\bar{\alpha}, \bar{\beta}} \quad \text{and} \quad |m - n| \geq r, \quad (4.1)$$

the partial sums $\sum_{\ell=0}^m \frac{a_\ell}{\ell!}$ and $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$, with a_ℓ given by (1.11), are distinct.

Proof of Lemma 4.1. Let $\bar{\alpha}$ and $\bar{\beta}$ be as in the statement of the lemma, and fix a $j_0 \in \{1, \dots, r\}$ for which

$$|\beta_{j_0}| = \max\{|\beta_1|, |\beta_2|, \dots, |\beta_r|\}. \quad (4.2)$$

Let

$$\mathcal{S} = \{j \in \{1, \dots, r\} : |\beta_j| = |\beta_{j_0}|\}. \quad (4.3)$$

Since $\bar{\beta}$ is admissible, we know that for any $j \in \mathcal{S}$, β_j/β_{j_0} is a root of unity. Therefore, the function given by

$$\ell \mapsto \sum_{j \in \mathcal{S}} \frac{\alpha_j}{\alpha_{j_0}} \left(\frac{\beta_j}{\beta_{j_0}} \right)^\ell \quad (4.4)$$

takes only finitely many values as ℓ runs over \mathbb{N} . Hence, there exists a constant $\epsilon_{\bar{\alpha}, \bar{\beta}} \geq 0$ depending only on $\bar{\alpha}$ and $\bar{\beta}$, such that for any $\ell \in \mathbb{N}$,

$$\sum_{j \in \mathcal{S}} \frac{\alpha_j}{\alpha_{j_0}} \left(\frac{\beta_j}{\beta_{j_0}} \right)^\ell = 0 \quad \text{or} \quad \left| \sum_{j \in \mathcal{S}} \frac{\alpha_j}{\alpha_{j_0}} \left(\frac{\beta_j}{\beta_{j_0}} \right)^\ell \right| \geq \epsilon_{\bar{\alpha}, \bar{\beta}}. \quad (4.5)$$

In other words, for any $\ell \in \mathbb{N}$,

$$\sum_{j \in \mathcal{S}} \alpha_j \beta_j^\ell = 0 \quad \text{or} \quad \left| \sum_{j \in \mathcal{S}} \alpha_j \beta_j^\ell \right| \geq \epsilon_{\bar{\alpha}, \bar{\beta}} |\alpha_{j_0}| |\beta_{j_0}|^\ell. \quad (4.6)$$

In the case \mathcal{S} does not coincide with the entire set $\{1, \dots, r\}$, fix a $j_1 \in \{1, \dots, r\} - \mathcal{S}$ for which

$$|\beta_{j_1}| = \max\{|\beta_j| : j \in \{1, \dots, r\} - \mathcal{S}\}. \quad (4.7)$$

Then, for all ℓ ,

$$\left| \sum_{j \in \{1, \dots, r\} - \mathcal{S}} \alpha_j \beta_j^\ell \right| = O_{\bar{\alpha}}(|\beta_{j_1}|^\ell). \quad (4.8)$$

Since $|\beta_{j_1}| < |\beta_{j_0}|$, the upper bound on the right side of (4.8) is exponentially small compared to the lower bound on the right side of (4.6) as $\ell \rightarrow \infty$. By (1.11), (4.6), and (4.8), for all ℓ sufficiently large,

$$|a_\ell| = O_{\bar{\alpha}}\left(|\beta_{j_1}|^\ell\right) \quad \text{or} \quad |a_\ell| \geq \frac{1}{2}\epsilon_{\bar{\alpha},\bar{\beta}}|\alpha_{j_0}||\beta_{j_0}|^\ell. \quad (4.9)$$

Next, let us remark that the first alternative in (4.6) cannot happen for $\#(\mathcal{S})$ consecutive values of ℓ . Indeed, assume that, for some m ,

$$\sum_{j \in \mathcal{S}} \alpha_j \beta_j^\ell = 0, \quad \text{for } \ell = m+1, m+2, \dots, m + \#(\mathcal{S}). \quad (4.10)$$

The associated Vandermonde determinant is nonzero, since the β_j 's are distinct; hence the system (4.10) can only have the trivial solution $\bar{\alpha} = \bar{0}$, which is not the case here. Therefore, since $\#(\mathcal{S}) \leq r$, in any set of r consecutive values of ℓ , at least one of them satisfies the second alternative in (4.6). It follows that the second alternative in (4.9) holds at least once in any r consecutive large values of ℓ .

Now let m and n be large positive integers with $n \geq m+r$. Let ℓ_0 be the smallest value of ℓ in $\{m+1, \dots, n\}$ for which the second alternative in (4.9) holds. Thus, $\ell_0 \leq m+r$. It follows that

$$|a_\ell| = O_{\bar{\alpha},\bar{\beta}}\left(|\beta_{j_1}|^\ell\right), \quad \text{for } m+1 \leq \ell < \ell_0, \quad (4.11)$$

and

$$|a_{\ell_0}| \geq \frac{1}{2}\epsilon_{\bar{\alpha},\bar{\beta}}|\alpha_{j_0}||\beta_{j_0}|^{\ell_0}. \quad (4.12)$$

Also, trivially from (1.11),

$$|a_\ell| = O_{\bar{\alpha}}\left(|\beta_{j_0}|^\ell\right), \quad \text{for all } \ell > \ell_0. \quad (4.13)$$

By (4.13) and (4.12),

$$\sum_{\ell_0 < \ell \leq n} \frac{|a_\ell|}{\ell!} = O_{\bar{\alpha}}\left(\sum_{\ell=\ell_0+1}^n \frac{|\beta_{j_0}|^\ell}{\ell!}\right) = O_{\bar{\alpha},\bar{\beta}}\left(\frac{|\beta_{j_0}|^{\ell_0}}{m(\ell_0)!}\right) = O_{\bar{\alpha},\bar{\beta}}\left(\frac{|a_{\ell_0}|}{m(\ell_0)!}\right). \quad (4.14)$$

Also, by (4.11) and (4.12),

$$\sum_{m < \ell < \ell_0} \frac{|a_\ell|}{\ell!} = O_{\bar{\alpha},\bar{\beta}}\left(\frac{|\beta_{j_1}|^{\ell_0}}{(m+1)!}\right) = O_{\bar{\alpha},\bar{\beta}}\left(m^{\ell_0-m} \left(\frac{\beta_{j_1}}{\beta_{j_0}}\right)^m \frac{|a_{\ell_0}|}{(\ell_0)!}\right). \quad (4.15)$$

For m large, the far right-hand sides of (4.14) and (4.15) are negligible compared to $|a_{\ell_0}|/(\ell_0)!$; therefore

$$\begin{aligned} \left| \sum_{\ell=0}^n \frac{a_\ell}{\ell!} - \sum_{\ell=0}^m \frac{a_\ell}{\ell!} \right| &= \left| \sum_{\ell=m+1}^n \frac{a_\ell}{\ell!} \right| \geq \frac{a_{\ell_0}}{(\ell_0)!} - \sum_{\substack{m < \ell < n \\ \ell \neq \ell_0}} \frac{a_\ell}{\ell!} \\ &\geq \frac{a_{\ell_0}}{2(\ell_0)!} \geq \frac{\epsilon_{\bar{\alpha},\bar{\beta}}|\alpha_{j_0}||\beta_{j_0}|^{\ell_0}}{4(\ell_0)!}. \end{aligned} \quad (4.16)$$

This shows that the partial sums $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$ and $\sum_{\ell=0}^m \frac{a_\ell}{\ell!}$ are distinct, and completes the proof of the lemma. \square

Lemma 4.2. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \bar{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible, and let ρ be given by (1.9). There exist a positive integer $N'_{\bar{\alpha}, \bar{\beta}}$ and an $\epsilon'_{\bar{\alpha}, \bar{\beta}} > 0$, depending only on $\bar{\alpha}$ and $\bar{\beta}$, such that for all $m \geq N'_{\bar{\alpha}, \bar{\beta}}$,*

$$\left| \rho - \sum_{\ell=0}^m \frac{a_\ell}{\ell!} \right| \geq \frac{\epsilon'_{\bar{\alpha}, \bar{\beta}} (\max\{|\beta_1|, \dots, |\beta_r|\})^m}{m^r m!}. \quad (4.17)$$

Proof of Lemma 4.2. The computations from the proof of Lemma 4.1 presented above were detailed enough to immediately imply Lemma 4.2 as well. More precisely, using (4.16) in combination with the fact that $\ell_0 \leq m + r$, we find that

$$\left| \sum_{\ell=0}^n \frac{a_\ell}{\ell!} - \sum_{\ell=0}^m \frac{a_\ell}{\ell!} \right| \geq \frac{\epsilon'_{\bar{\alpha}, \bar{\beta}} |\beta_{j_0}|^m}{m^r m!}, \quad (4.18)$$

for some constant $\epsilon'_{\bar{\alpha}, \bar{\beta}} > 0$ depending only on $\bar{\alpha}$ and $\bar{\beta}$, any m large enough, and any $n \geq m + r$. The right side of (4.18) is independent of n . Letting $n \rightarrow \infty$ while keeping m fixed, and recalling the definition of $|\beta_{j_0}|$, we find that (4.17) holds for all m large enough. This completes the proof of Lemma 4.2. \square

Next, let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \bar{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible, and consider the number field

$$K := \mathbb{Q}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r). \quad (4.19)$$

Denote

$$b := \min\{d \in \mathbb{Z} : d \geq 1, d\alpha_1, \dots, d\alpha_r, d\beta_1, \dots, d\beta_r \in O_K\}. \quad (4.20)$$

With the a_ℓ 's defined by (1.11), $b^{\ell+1}a_\ell$ belongs to O_K , for each ρ . Therefore, writing each partial sum of the series from the right side of (1.10) in the form

$$\sum_{\ell=0}^n \frac{a_\ell}{\ell!} = \frac{A_n}{b^{n+1}n!}, \quad (4.21)$$

we have $A_n \in O_K$ for all n . We consider the ideal of O_K generated by A_n and $b^{n+1}n!$,

$$J_n := (A_n, b^{n+1}n!), \quad n = 0, 1, 2, \dots, \quad (4.22)$$

and examine the size of the norm $\|J_n\|$ of J_n . One would naturally expect to have very little cancellation on the right side of (4.21), and $\|J_n\|$ to be relatively small. Notice that if for some n the partial sum $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$ is zero, then $A_n = 0$, and the norm of J_n is as large as it can be, $\|J_n\| = (b^{n+1}n!)^{[K:\mathbb{Q}]}$. This only happens finitely many times, since $\sum_{\ell=0}^n \frac{a_\ell}{\ell!} \rightarrow \rho$ as $n \rightarrow \infty$, and $\rho \neq 0$. Notice also that if $\|J_n\|$ is unusually large for one value of n , it will automatically be large for other nearby values of n that happen to produce the exact same partial sum. By Lemma 4.1, this phenomenon is harmless, since the multiplicity of each partial sum $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$, with n sufficiently large, is bounded by r . The following lemma, which generalizes to number fields a key result from [1], shows that with at most a logarithmic number of exceptions, $\|J_n\|$ is bounded by $n^{\epsilon n}$, for any fixed $\epsilon > 0$, and n sufficiently large.

Lemma 4.3. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \bar{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible. For any $\epsilon > 0$, there exists an effectively computable constant $C'_{\bar{\alpha}, \bar{\beta}, \epsilon} > 0$ such that for all sufficiently large real numbers x ,*

$$\#\{n \in \mathbb{Z} : 1 \leq n \leq x, \|J_n\| \geq n^{\epsilon n}\} \leq C'_{\bar{\alpha}, \bar{\beta}, \epsilon} \log x, \quad (4.23)$$

where J_n is defined via (1.11) and (4.19)–(4.22).

Proof of Lemma 4.3. Let $\bar{\alpha}$ and $\bar{\beta}$ be as in the statement of the lemma and fix an $\epsilon > 0$. We choose a small $\eta > 0$, whose precise value will be given later, and will be allowed to depend on ϵ and on the degree of $K = \mathbb{Q}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r)$ over \mathbb{Q} . It suffices to show that there exists an effectively computable constant $C''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}$ and a positive integer $N''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}$, such that for any integer $N \geq N''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}$,

$$\#\{n \in \mathbb{Z} : N \leq n \leq N(1 + \eta), \|J_n\| \geq n^{\epsilon n}\} \leq C''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}. \quad (4.24)$$

Then (4.23) follows immediately from (4.24), by writing the interval $[N''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}, x]$ as a union of at most $\left\lceil \frac{\log x - \log N''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}}{\log(1 + \eta)} \right\rceil$ intervals of the form $[N, N(1 + \eta)]$. In order to prove (4.24), consider such an interval $[N, N(1 + \eta)]$ with N large. By requiring $N''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}$ to be larger than the positive integer $N_{\bar{\alpha}, \bar{\beta}}$ from the statement of Lemma 4.1, we may assume in what follows that $N \geq N_{\bar{\alpha}, \bar{\beta}}$. Consider the exceptional set

$$\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N} := \{n \in \mathbb{Z} : N \leq n \leq N(1 + \eta), \|J_n\| \geq n^{\epsilon n}\}, \quad (4.25)$$

and construct a subset $\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$ of $\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$ as follows. Let n_1 be the smallest element of $\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$. Let n_2 be the smallest element of $\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$ that is $\geq n_1 + r$, let n_3 be the smallest element of $\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$ that is $\geq n_2 + r$, and so on, until this procedure stops. Then set

$$\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N} = \{n_1, n_2, n_3, \dots\}. \quad (4.26)$$

By the foregoing construction, $\#(\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}) \leq r \#(\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N})$, and so (4.24) will be proved if we show that there is an effectively computable constant $C'''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}$ such that, uniformly for all N sufficiently large,

$$\#(\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}) \leq C'''_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta}. \quad (4.27)$$

Fix now a positive integer k satisfying

$$k \leq \#(\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}). \quad (4.28)$$

As a side remark, we emphasize that the argument below would not work if instead of (4.28) we let k equal $\#(\mathcal{E}^*_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N})$ directly. The reason is that in the argument below k needs to be in a suitable range; if k is too small or too large, we do not obtain a contradiction.

To proceed, let J denote the least common multiple of the ideals $J_{n_1}, J_{n_2}, \dots, J_{n_k}$. For $1 \leq i < j \leq k$, let J_{ij} denote the greatest common divisor of the ideals J_{n_i} and J_{n_j} . Let \mathcal{P} be an arbitrary prime ideal of O_K , and let e_1, e_2, \dots, e_k denote the exponents of \mathcal{P} in $J_{n_1}, J_{n_2}, \dots, J_{n_k}$, respectively. Then the exponent of \mathcal{P} in J equals $\max\{e_1, e_2, \dots, e_k\}$, the exponent of \mathcal{P} in the product $\prod_{1 \leq j \leq k} J_{n_j}$ equals the sum $\sum_{1 \leq j \leq k} e_j$, and for any $1 \leq$

$i < j \leq k$, the exponent of \mathcal{P} in J_{ij} equals $\min\{e_i, e_j\}$. Note that the nonnegative integers e_1, e_2, \dots, e_k always satisfy the inequality

$$e_1 + e_2 + \dots + e_k \leq \max\{e_1, e_2, \dots, e_k\} + \sum_{1 \leq i < j \leq k} \min\{e_i, e_j\}. \quad (4.29)$$

Indeed, let $s \in \{1, 2, \dots, k\}$ be such that $e_s = \max\{e_1, e_2, \dots, e_k\}$. Then e_s appears as the first term on the right side of (4.29), each term e_i on the left side with $i < s$ also appears on the right side as $\min\{e_i, e_s\}$, and each term e_j on the left side with $j > s$ also appears on the right side as $\min\{e_s, e_j\}$. Also, all the remaining terms on the right side are nonnegative.

Since the previous discussion applies to all prime ideals of O_K , we see that

$$\prod_{1 \leq j \leq k} J_{n_j} \text{ divides } J \prod_{1 \leq i < j \leq k} J_{ij}. \quad (4.30)$$

Since n_1, \dots, n_k belong to the exceptional set $\mathcal{E}_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta, N}$,

$$\prod_{1 \leq j \leq k} \|J_{n_j}\| \geq \prod_{1 \leq j \leq k} n_j^{\epsilon n_j} \geq N^{k\epsilon N}. \quad (4.31)$$

On the other hand, for each j , J_{n_j} divides the principal ideal $(b^{n_j+1}n_j!)$, which in turn divides $(b^{\lfloor N(1+\eta)+1 \rfloor} \lfloor N(1+\eta) \rfloor!)$, and therefore J also divides this ideal. Taking norms, we find that

$$\|J\| \leq b^{[K:\mathbb{Q}][N(1+\eta)+1]} (\lfloor N(1+\eta) \rfloor!)^{[K:\mathbb{Q}]}. \quad (4.32)$$

Combining (4.32) with Stirling's formula, we see that for N sufficiently large,

$$\|J\| \leq N^{N(1+2\eta)[K:\mathbb{Q}]}. \quad (4.33)$$

By (4.30), (4.31), and (4.33), for N sufficiently large,

$$N^{k\epsilon N} \leq N^{N(1+2\eta)[K:\mathbb{Q}]} \prod_{1 \leq i < j \leq k} \|J_{ij}\|. \quad (4.34)$$

Lastly, in order to bound the product on the right side of (4.34), recall that by Lemma 4.1, the partial sums $\sum_{\ell=0}^{n_j} \frac{a_\ell}{\ell!}$, with $1 \leq j \leq k$, are distinct. It follows that for each pair (i, j) , $1 \leq i < j \leq k$, the number $w_{i,j}$ defined by

$$w_{i,j} := b^{n_j+1}n_j! \left(\sum_{\ell=0}^{n_j} \frac{a_\ell}{\ell!} - \sum_{\ell=0}^{n_i} \frac{a_\ell}{\ell!} \right) = A_{n_j} - \frac{b^{n_j-n_i}n_j!A_{n_i}}{n_i!} \quad (4.35)$$

is a nonzero algebraic integer. Moreover, since $A_{n_j} \in J_{n_j} \subseteq J_{ij}$ and $A_{n_i} \in J_{n_i} \subseteq J_{ij}$, by (4.35), we see that $w_{i,j}$ belongs to J_{ij} . Therefore,

$$\|J_{ij}\| \leq |N_{K/\mathbb{Q}}(w_{i,j})|, \quad \text{for } 1 \leq i < j \leq k. \quad (4.36)$$

Next, we need an alternative way to bound the norm of each element $w_{i,j}$. We start by rewriting (4.35) as

$$\begin{aligned} w_{i,j} &= b^{n_j+1}n_j! \sum_{n_i < \ell \leq n_j} \frac{a_\ell}{\ell!} \\ &= b^{n_j+1} (a_{n_j} + n_j a_{n_j-1} + \dots + n_j(n_j-1) \dots (n_i+1) a_{n_i}). \end{aligned} \quad (4.37)$$

For any embedding σ of K into \mathbb{C} ,

$$\sigma(w_{i,j}) = b^{n_j+1} \left(\sigma(a_{n_j}) + n_j \sigma(a_{n_j-1}) + \cdots + n_j(n_j-1) \cdots (n_i+1) \sigma(a_{n_i}) \right). \quad (4.38)$$

By (1.11), for each σ and each ℓ ,

$$\sigma(a_\ell) = \sum_{j=1}^r \sigma(\alpha_j) \sigma(\beta_j)^\ell. \quad (4.39)$$

If we denote

$$\|\bar{\alpha}\| = \max_{\sigma} \max_{1 \leq j \leq r} |\sigma(\alpha_j)|, \quad (4.40)$$

and similarly

$$\|\bar{\beta}\| = \max_{\sigma} \max_{1 \leq j \leq r} |\sigma(\beta_j)|, \quad (4.41)$$

then by (4.39)–(4.41),

$$|\sigma(a_\ell)| \leq r \|\bar{\alpha}\| \|\bar{\beta}\|^\ell, \quad (4.42)$$

for any σ and ℓ . Combining (4.38) and (4.42), we find that

$$|\sigma(w_{i,j})| \leq r \|\bar{\alpha}\| b^{n_j+1} \max\{\|\bar{\beta}\|^{n_i}, \|\bar{\beta}\|^{n_j}\} n_j^{n_j-n_i+1}. \quad (4.43)$$

Since $n_j \leq N(1+\eta)$ and $n_j - n_i \leq \eta N$, we see that for N sufficiently large,

$$|\sigma(w_{i,j})| \leq N^{2\eta N}, \quad (4.44)$$

uniformly for $1 \leq i < j \leq k$ and all embeddings σ of K into \mathbb{C} . Therefore,

$$|N_{K/\mathbb{Q}}(w_{i,j})| = \prod_{\sigma} |\sigma(w_{i,j})| \leq N^{2\eta N [K:\mathbb{Q}]}, \quad (4.45)$$

for $1 \leq i < j \leq k$. Combining (4.34), (4.36), and (4.45), we deduce that

$$k\epsilon \leq (1+2\eta)[K:\mathbb{Q}] + k(k-1)\eta[K:\mathbb{Q}]. \quad (4.46)$$

With ϵ and the degree $[K:\mathbb{Q}]$ fixed, if η is chosen small enough, then the function given by

$$t \mapsto (1+2\eta)[K:\mathbb{Q}] + t(t-1)\eta[K:\mathbb{Q}] - t\epsilon \quad (4.47)$$

has two real and positive roots $t_1 < t_2$. Moreover, denoting $d = [K:\mathbb{Q}]$,

$$t_2 - t_1 = \frac{\sqrt{(\epsilon + \eta d)^2 - 4\eta(1+2\eta)d^2}}{\eta d}. \quad (4.48)$$

For fixed d and ϵ , the right-hand side of (4.48) tends to infinity as $\eta \rightarrow 0$. One can choose an effectively computable $\eta = \eta_{d,\epsilon} > 0$, depending only on d and ϵ , such that $(\epsilon + \eta_{d,\epsilon}d)^2 - 4\eta_{d,\epsilon}(1+2\eta_{d,\epsilon})d^2 > 0$ and such that the right-hand side of (4.48) is larger than 1. One can then choose an effectively computable positive integer $k = k_{d,\epsilon}$, depending only on d and ϵ , such that $t_1 < k_{d,\epsilon} < t_2$.

In conclusion, since (4.46) fails for $\eta_{d,\epsilon}$ and $k_{d,\epsilon}$, it follows that (4.28) fails for $\eta_{d,\epsilon}$ and $k_{d,\epsilon}$, for every single sufficiently large N . In other words, (4.27) holds for all N sufficiently large, with $\eta = \eta_{d,\epsilon}$ and $C_{\bar{\alpha}, \bar{\beta}, \epsilon, \eta_{d,\epsilon}}''' = k_{d,\epsilon} - 1$. This completes the proof of the lemma. \square

Lemma 4.4. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\bar{\beta} = (\beta_1, \dots, \beta_r)$, with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r \in \overline{\mathbb{Q}} - \{0\}$ and $\bar{\beta}$ admissible. For any $\epsilon > 0$, there exists an effectively computable constant $C_{\bar{\alpha}, \bar{\beta}, \epsilon} > 0$, such that for all large x ,*

$$\#\left\{n \in \mathbb{Z} : 1 \leq n \leq x, H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right) \leq n^{(1-\epsilon)n}\right\} \leq C_{\bar{\alpha}, \bar{\beta}, \epsilon} \log x, \quad (4.49)$$

where the a_ℓ 's are given by (1.11).

Proof of Lemma 4.4. Let $\bar{\alpha}$ and $\bar{\beta}$ be as in the statement of the lemma. Fix an $\epsilon > 0$ and take a large x . By Lemma 4.3, for all $1 \leq n \leq x$ with at most $C'_{\bar{\alpha}, \bar{\beta}, \epsilon} \log x$ exceptions, we have

$$\|J_n\| \leq n^{\epsilon n}. \quad (4.50)$$

Take such an n , and let d_n denote the degree of $\sum_{\ell=0}^n \frac{a_\ell}{\ell!}$ over \mathbb{Q} . Let

$$P_n(X) = c_{n,0}X^{d_n} + c_{n,1}X^{d_n-1} + \dots + c_{n,d_n} \in \mathbb{Z}[X]$$

be irreducible over \mathbb{Q} , such that $c_{n,0} > 0$, $\gcd(c_{n,0}, c_{n,1}, \dots, c_{n,d_n}) = 1$, and

$$P_n\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right) = 0. \quad (4.51)$$

By (4.21) and (4.51), we derive

$$c_{n,0}A_n^{d_n} + c_{n,1}A_n^{d_n-1}b^{n+1}n! + \dots + c_{n,d_n}b^{(n+1)d_n}(n!)^{d_n} = 0. \quad (4.52)$$

It follows from (4.52) that $c_{n,0}A_n^{d_n}$ is a multiple of $b^{n+1}n!$ in O_K . Let D_n denote the greatest common divisor of the positive integers $c_{n,0}$ and $b^{n+1}n!$, and write

$$c_{n,0} = D_n\mu_n, \quad b^{n+1}n! = D_n\nu_n, \quad (\mu_n, \nu_n) = 1. \quad (4.53)$$

Then $\mu_n A_n^{d_n}$ is a multiple of ν_n in O_K , and since $(\mu_n, \nu_n) = 1$ and ν_n divides $b^{n+1}n!$, it follows that the principal ideal (ν_n) divides the ideal generated by $A_n^{d_n}$ and $b^{n+1}n!$. Since this ideal divides $J_n^{d_n}$, it follows that (ν_n) divides $J_n^{d_n}$. Taking norms and using (4.50), we find that

$$\nu_n^{[K:\mathbb{Q}]} \leq \|J_n^{d_n}\| \leq n^{\epsilon n d_n}. \quad (4.54)$$

The degree d_n is bounded by $[K:\mathbb{Q}]$, and so (4.54) implies that

$$\nu_n \leq n^{\epsilon n}. \quad (4.55)$$

By (4.53) and (4.55), and the definition of height given by (1.8), we derive that

$$H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right) \geq c_{n,0} \geq D_n = \frac{b^{n+1}n!}{\nu_n} \geq \frac{b^{n+1}n!}{n^{\epsilon n}}. \quad (4.56)$$

This completes the proof of the lemma, by an appeal to Stirling's formula. \square

Proof of Theorem 1.3. Let $\bar{\alpha}$, $\bar{\beta}$, and τ be as in the statement of the theorem. Take a large x , and let $n \leq x$ be such that $\sum_{\ell=0}^n \frac{a_\ell}{\ell!} \in A_{\rho, \tau}$, with ρ given by (1.10). Thus,

$$\left| \rho - \sum_{\ell=0}^n \frac{a_\ell}{\ell!} \right| \leq \frac{1}{H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right)^\tau}. \quad (4.57)$$

Combining (4.57) with Lemma 4.2, for $n \geq N'_{\bar{\alpha}, \bar{\beta}}$, we have

$$H\left(\sum_{\ell=0}^n \frac{a_\ell}{\ell!}\right) \leq \frac{n^{r/\tau} (n!)^{1/\tau}}{(\epsilon'_{\bar{\alpha}, \bar{\beta}})^{1/\tau} (\max\{|\beta_1|, \dots, |\beta_r|\})^{n/\tau}}. \quad (4.58)$$

Letting now $\epsilon = (1 - 1/\tau)/2$, we see that the right-hand side of (4.58) is $\leq n^{(1-\epsilon)n}$ for n large enough. Thus, Lemma 4.4 is applicable, and in combination with (4.58) completes the proof of the theorem. \square

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