**Abstract:** In his lost notebook, Ramanujan recorded two identities involving double series of Bessel functions that are closely connected with the classical, unsolved circle and divisor problems. In a series of papers with A.~Zaharescu, the authors proved these identities under various interpretations, as well as Riesz mean analogues. In this paper, logarithmic mean analogues, also involving double series of Bessel functions, are established. Weighted divisor sums involving characters play a central role.
LOGARITHMIC MEANS AND DOUBLE SERIES OF BESSEL FUNCTIONS

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1. INTRODUCTION

If \( a(n) \) is an arithmetic function, in bounding the error term in an asymptotic formula for \( \sum_{n \leq x} a(n) \), it may be more convenient to examine the weighted sum \( \sum_{n \leq x} a(n)(x-n)^a \), for certain \( a > 0 \). Then one can apply a method of finite differences, originally due to E. Landau, to the latter sum to gain information about the former sum [12, Theorem 4.1, pp. 106–111]. The sums \( \sum_{n \leq x} a(n)(x-n)^a \) are sometimes called Riesz sums, with the power \( (x-n)^a \) a “smoothing factor.” We emphasize that the logarithmic sums \( \sum_{n \leq x} a(n) \log^a (x/n) \) could also be used in the study of the average order of certain arithmetic functions, since \( \log^a (x/n) \) has a “smoothing” effect similar to that of \( (x-n)^a \), for in each case, when \( n \) is small, the contributions of these factors are “large,” while when \( n \) is large, the contributions of these factors are “small.” Generally, for “small” \( a \), the simplicities of the identities for \( \sum_{n \leq x} a(n)(x-n)^a \) and \( \sum_{n \leq x} a(n) \log^a (x/n) \) are comparable, but for “large” \( a \), the identities for the former sum are usually more elegant than those for the latter sum. It is likely for this reason that Riesz sums have been employed more frequently than logarithmic sums.

In [7] and [10], the present authors and A. Zaharescu derived Riesz sum analogues for two fascinating identities from Ramanujan’s lost notebook [16]. We refrain from recording these Riesz sum identities here, but we do now offer Ramanujan’s identities from [16, p. 335], which are intimately connected, respectively, with the classical unsolved circle and divisor problems. To state Ramanujan’s claims, we first define

\[
F(x) = \begin{cases} 
[x], & \text{if } x \text{ is not an integer,} \\
\frac{x - \frac{1}{2}}{2}, & \text{if } x \text{ is an integer.} 
\end{cases}
\] (1.1)

Secondly, let \( J_\nu(x) \) denote the ordinary Bessel function of order \( \nu \), let \( Y_\nu(x) \) denote the second solution of Bessel’s differential equation of order \( \nu \) [18, p. 64, equation (1)], and let \( K_\nu(x) \) denote the modified Bessel function of order \( \nu \) [18, p. 78, equation (6)]. For brevity,
after Ramanujan, define
\[ I_\nu(z) := -Y_\nu(z) + \frac{2}{\pi} \cos(\pi \nu) K_\nu(z). \] (1.2)

**Entry 1.1** (p. 335). Let \( F(x) \) be defined by (1.1). If \( 0 < \theta < 1 \) and \( x > 0 \), then
\[ \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n \theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4} \cot(\pi \theta) \] (1.3)
\[ + \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ J_1\left(4\pi \sqrt{m(n+\theta)}x\right) - J_1\left(4\pi \sqrt{m(n+1-\theta)}x\right) \right\} \] (1.3)

**Entry 1.2** (p. 335). Let \( F(x) \) be defined by (1.1). Then, for \( x > 0 \) and \( 0 < \theta < 1 \),
\[ \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n \theta) = \frac{1}{4} - x \log(2 \sin(\pi \theta)) \] (1.4)
\[ + \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ I_1\left(4\pi \sqrt{m(n+\theta)}x\right) + I_1\left(4\pi \sqrt{m(n+1-\theta)}x\right) \right\} \] (1.4)

where \( I_1(z) \) is defined in (1.2).

For proofs of these identities under different interpretations of the double sums, and for their connections with the circle and divisor problems, see the papers [5], [8], [9] by the present two authors and Zaharescu, and the paper by the first named author and Zaharescu [11].

In view of our discussion above, it is natural to ask if Entries 1.1 and 1.2 have logarithmic mean analogues. The purpose of this paper is to prove such analogues. However, we concentrate only on the case \( a = 1 \). The methods employed in this special case indeed can be applied more generally to the cases \( a > 1 \). However, the identities for increasing \( a \) become more unwieldy, and so this is why we confine ourselves only to the case \( a = 1 \).

In order to accomplish our goals, we first need to derive identities for weighted divisor sums. For a character \( \chi \), define the weighted divisor sum
\[ d_\chi(n) := \sum_{d|n} \chi(d). \] (1.5)

To derive the requisite identities for (1.5), we appeal to a general theorem about logarithmic sums or means [2, p. 365, Theorem 1]. Let \( \{\lambda_n\} \) and \( \{\mu_n\} \) be sequences of positive numbers such that \( \lambda_n, \mu_n \to \infty \) as \( n \to \infty \). Let \( \{a(n)\} \) and \( \{b(n)\} \), \( 1 \leq n < \infty \), be sequences of complex numbers such that
\[ \phi(s) := \sum_{n=1}^{\infty} a(n) \lambda_n^{-s} \quad \text{and} \quad \psi(s) := \sum_{n=1}^{\infty} b(n) \mu_n^{-s} \] (1.6)
have finite abscissae of absolute convergence \( \sigma_\phi \) and \( \sigma_\psi \), respectively. We assume that \( \phi(s) \) and \( \psi(s) \) have analytic continuations to the entire complex plane, with the only possible singularities being poles that belong to a compact set in the \( s \)-plane. We also assume that for
some positive integer \( m \) and positive number \( r \), \( \phi(s) \) and \( \psi(s) \) satisfy a functional equation of the form
\[
\Gamma^m(s)\phi(s) = \Gamma^m(r - s)\psi(r - s).
\] (1.7)
For a complete description of the necessary properties of \( \phi(s) \) and \( \psi(s) \), see [2, pp. 361–362]. For each positive integer \( a \), define, for \( x > 0 \), the logarithmic mean or sum
\[
S(x; a) := \frac{1}{a!} \sum_{\lambda_n \leq x} a(n) \log^a(x/\lambda_n).
\]

Let
\[
s(u, v; a) := \sum_{j=0}^{a} (-1)^j \frac{\log^j u \log^{a-j} v}{j!(a-j)!}.
\]
Lastly, let \( J_\nu(z) \) denotes the ordinary Bessel function of order \( \nu \) of the first kind, and let \( k \) and \( m \) be integers with \( k \geq 0 \) and \( m > 0 \). Define the \( m \)-fold integral
\[
K(x; \mu, \nu; k, m) := \int_0^\infty u_1^{\mu - 1} J_\mu(u_1) \cdots \int_0^\infty u_m^{\mu - 1} J_\mu(u_m) \, du_m \cdots \int_0^\infty u_k^{\mu - 1} J_\mu(u_k) \, du_k
\]
\[
\cdot \int_0^\infty u_{k-1}^{\nu - 1} J_\nu(u_{k-1}) \cdots \int_0^\infty u_1^{\nu - 1} J_\nu(u_1) \, du_1 \int_{\alpha} J_\nu(u_1 / u_2 \cdots u_m) \, du_1,
\]
where if \( m = 1 \),
\[
K(x; \mu, \nu; -1) = J_\nu(x).
\] (1.8)

**Theorem 1.3.** Let \( \phi(s) \) and \( \psi(s) \) be Dirichlet series satisfying the functional equation (1.6) and the conditions outlined above. Suppose that \( x > 0 \), \( a \) is a positive integer, \( a > 2m\sigma_a - mr - \frac{1}{2} \), and \( \xi_n = 2^m \sqrt{\mu_n x} \).

(i) Assume that \( a < m \). If \( m \geq 2 \), assume \( r > -\frac{1}{2} \). Then
\[
S(x; a) = R(x; a) + 2^a \sum_{n=1}^\infty b(n) \left( \frac{x}{\mu_n} \right)^{r/2} K_r(\xi_n; r - 1, r; k, m),
\] (1.9)
where the series on the right-hand side of (1.9) converges absolutely and where
\[
R(x; a) := \frac{1}{2\pi i} \int_{C} \frac{\phi(s)x^s}{s^{a+1}} \, ds,
\] (1.10)
where \( C \) is a positively oriented curve encircling the poles of the integrand.

(ii) Assume that \( a \geq m \). Then, if \( a > mr - \frac{1}{2} \),
\[
S(x; a) = R(x; a) - 2^{2a - mr - m}
\]
\[
\times \sum_{n=1}^\infty b(n) \frac{d^{a-m}}{d\mu_n^a} \left( x^{a-m} \int_\xi^\infty \int_0^\infty u^{r-a+m-1} K(u; r, r + a + m; -, m) s(u, \xi_n; a - m) \, du \right),
\] (1.11)
where the series on the right-hand side of (1.11) is absolutely convergent.

In the sequel, \( R_\alpha(f) = R_\alpha \) denotes the residue of a meromorphic function \( f(s) \) at a pole \( s = \alpha \).
2. LOGARITHMIC MEAN IDENTITY FOR ODD PRIMITIVE CHARACTERS

Let \( \chi \) be an odd primitive character modulo \( q \). The generating function for \( d_{\chi}(n) \) is the Dirichlet series \( \zeta(s) L(s, \chi) \), which satisfies the functional equation \([13, \text{pp. 59}, 69], [7, \text{p. 89}]\)

\[
\left( \frac{2\pi}{\sqrt{q}} \right)^{-s} \Gamma(s) L(s, \chi) \zeta(s) = -\frac{i\tau(\chi)}{\sqrt{q}} \left( \frac{2\pi}{\sqrt{q}} \right)^{s-1} \Gamma(1-s) L(1-s, \bar{\chi}) \zeta(1-s),
\]

where \( \tau(\chi) \) denotes the Gauss sum

\[
\tau(\chi) := \sum_{h=1}^{q-1} \chi(h) e^{2\pi i h/q}.
\]

Throughout this paper, we frequently use (without comment) the well-known fact \([4, \text{p. 10, Theorem 1.1.4 (a)}]\)

\[
\tau(\chi) \tau(\bar{\chi}) = \begin{cases} q, & \text{if } \chi \text{ is even} \\ -q, & \text{if } \chi \text{ is odd.} \end{cases}
\]

We now apply Theorem 1.3 (ii) with \( m = 1, r = 1, a = 1, \) and \( \lambda_n = \mu_n = \frac{2\pi n}{\sqrt{q}}, a(n) = d_{\chi}(n), b(n) = -\frac{i\tau(\chi)}{\sqrt{q}} d_{\chi}(n). \)

We calculate

\[
R(x; 1) = \frac{1}{2\pi i} \int_C \frac{(2\pi/\sqrt{q})^{-s} L(s, \chi) \zeta(s) x^s}{s^2} ds,
\]

where \( C \) is a simple closed, positively oriented contour containing the simple pole \( s = 1 \) and the double pole \( s = 0 \) on the interior of \( C \).

We first calculate \( R_0 \). To that end,

\[
\left( \frac{2\pi}{x\sqrt{q}} \right)^{-s} = e^{-s \log(2\pi/x\sqrt{q})} = 1 - \log \left( \frac{2\pi}{x\sqrt{q}} \right) s + \cdots, \tag{2.2}
\]

\[
L(s, \chi) = L(0, \chi) + L'(0, \chi) s + \cdots, \tag{2.3}
\]

and \([17, \text{p. 19, equation (2.4.3)}, \text{p. 20, equation (2.4.5)}]\)

\[
\zeta(s) = \zeta(0) + \zeta'(0) s + \cdots = -\frac{1}{2} - \frac{1}{2} \log(2\pi) s + \cdots. \tag{2.4}
\]

Hence, from (2.2)–(2.4),

\[
R_0 = \frac{1}{2} \log \left( \frac{2\pi}{x\sqrt{q}} \right) L(0, \chi) - \frac{1}{2} \log(2\pi) L(0, \chi) - \frac{1}{2} L'(0, \chi)
\]

\[
= -\frac{1}{2} \log(x\sqrt{q}) L(0, \chi) - \frac{1}{2} L'(0, \chi)
\]

\[
= \frac{i\tau(\chi)}{4\pi} \log(x^2 q) L(1, \bar{\chi}) - \frac{1}{2} L'(0, \chi), \tag{2.5}
\]

where we used an identity from \([5, \text{p. 2072}]\), which is an easy consequence of the functional equation for \( L(s, \chi) \).
Next, since \( \zeta(s) \) has a simple pole at \( s = 1 \) with residue 1, we easily see that

\[
R_1 = \frac{x \sqrt{q}}{2\pi} L(1, \chi). \tag{2.6}
\]

From (1.8), we note that \( K(u; \mu, 1; -1, 1) = J_1(u) \). Since \([18, p. 45]\) \( J_1(u) = -J'_0(u) \), we find that

\[
\int_{\xi_n}^\infty u^0 J_1(u) du = J_0(\xi_n) = J_0 \left( 2 \left( \frac{2\pi x}{\sqrt{q}} \right)^{1/2} \right). \tag{2.7}
\]

Utilizing (2.5)–(2.7) in (1.11), we find that

\[
\sum_{\lambda_n \leq x} d_\chi(n) \log \left( \frac{x}{\lambda_n} \right) = \frac{i\tau(\chi)}{4\pi} \log(x^2 q) L(1, 1, \chi) - \frac{1}{2} L'(0, \chi) + \frac{x \sqrt{q}}{2\pi} L(1, \chi)
+ \frac{i\tau(\chi)}{\sqrt{q}} \sum_{n=1}^\infty \frac{d_\chi(n)}{\mu_n} J_0 \left( 2 \left( \frac{2\pi x}{\sqrt{q}} \right)^{1/2} \right). \tag{2.8}
\]

If we replace \( x \) by \( 2\pi x / \sqrt{q} \) in (2.8), we derive a simpler form, which we state as a theorem.

**Theorem 2.1.** If \( \chi \) denotes an odd primitive character modulo \( q \), then

\[
\sum_{n \leq x} d_\chi(n) \log \left( \frac{x}{n} \right) = \frac{i\tau(\chi)}{2\pi} \log(2\pi x) L(1, 1, \chi) - \frac{1}{2} L'(0, \chi) + x L(1, \chi)
+ \frac{i\tau(\chi)}{2\pi} \sum_{n=1}^\infty \frac{d_\chi(n)}{n} J_0(4\pi \sqrt{n x / q}). \tag{2.9}
\]

3. **Logarithmic Mean Analogue for Ramanujan’s Identity, Entry 1.1**

**Theorem 3.1.** Let \( x > 0 \) and \( 0 < \theta < 1 \). Then

\[
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r\mid n} \sin(2\pi r \theta)
= -\frac{\log(4\pi^2 x)}{4} + \frac{1}{\cot(\pi \theta) + \pi x \left( \frac{1}{2} - \theta \right)} + \frac{1}{4\pi} (\gamma_1(\theta) - \gamma_1(1 - \theta))
- \frac{1}{4\pi} \sum_{m \geq 1} \sum_{n \geq 0} \left\{ \frac{J_0 \left( 4\pi \sqrt{m(n + \theta) x} \right)}{(m(n + \theta))} - \frac{J_0 \left( 4\pi \sqrt{m(n + 1 - \theta) x} \right)}{(m(n + 1 - \theta))} \right\}, \tag{3.1}
\]

where \( \gamma_1(\theta) \) and \( \gamma_1(1 - \theta) \) are the Laurent series coefficients of the Hurwitz zeta function \( \zeta(s, a) \), also called generalized Stieltjes constants, defined by \([3, p. 152]\)

\[
\zeta(s, a) = \frac{1}{s - 1} + \sum_{n=0}^\infty \gamma_n(a)(s - 1)^n. \tag{3.2}
\]
We offer several remarks before embarking upon our proof. The generalized Stieltjes constants have the representation [3, p. 152, Theorem 1]

\[
\gamma_n(a) = \frac{(-1)^n}{n!} \lim_{m \to \infty} \left( \sum_{k=0}^{m} \frac{\log^n(k + a)}{k + a} - \frac{\log^{n+1}(m + a)}{n + 1} \right). \tag{3.3}
\]

Second, as \(x \to \infty\) [18, p. 199, equation (1)],

\[
J_0(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} \cos(x - \frac{x}{4}). \tag{3.4}
\]

Thus, the series on the right side of (3.1) converges absolutely and uniformly in \(x\) on compact subsets of \((0, \infty)\), and also absolutely and uniformly in \(\theta\) on compact subsets of \((0, 1)\). Third, because of the uniform convergence in \(\theta\) of the double series in (3.1), this series represents a continuous function for \(0 < \theta < 1\). It therefore suffices to prove (3.1) for rational numbers \(\theta = a/q\), where \(q\) is prime and \(0 \leq a < q\). Thus, (3.1) is equivalent to the following identity

\[
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r | n} \sin \left( \frac{2\pi ra}{q} \right) \\
= -\frac{\log(4\pi^2 x) + \gamma}{4} \cot \left( \frac{\pi a}{q} \right) + \pi x \left( \frac{1}{2} - \frac{a}{q} \right) + \frac{1}{4\pi} \left( \gamma \left( \frac{a}{q} \right) - \gamma (1 - \frac{a}{q}) \right) \\
- \frac{1}{4\pi} \sum_{m \geq 1} \left\{ J_0 \left( \frac{4\pi \sqrt{m(n + a/q)x}}{m(n + a/q)} \right) - J_0 \left( \frac{4\pi \sqrt{m(n + 1 - a/q)x}}{m(n + 1 - a/q)} \right) \right\} \\
= -\frac{\log(4\pi^2 x) + \gamma}{4} \cot \left( \frac{\pi a}{q} \right) + \pi x \left( \frac{1}{2} - \frac{a}{q} \right) + \frac{1}{4\pi} \left( \gamma \left( \frac{a}{q} \right) - \gamma (1 - \frac{a}{q}) \right) \\
- \frac{q}{4\pi} \sum_{m = 1}^{\infty} \sum_{r \equiv a \pmod{q}}^{\infty} \frac{J_0 \left( \frac{4\pi \sqrt{mr x/q}}{mr} \right)}{mr} - \sum_{r \equiv -a \pmod{q}}^{\infty} \frac{J_0 \left( \frac{4\pi \sqrt{mr x/q}}{mr} \right)}{mr}. \tag{3.5}
\]

We demonstrate (3.5) by showing that it is equivalent to (2.9). We first prove that (2.9) implies (3.5).

**Proof of (2.9) ⇒ (3.5).** Let \(\phi(q)\) denote Euler’s totient function. We multiply both sides of (2.9) by \(\chi(a)\tau(\chi)/i\phi(q)\) and sum on \(\chi\), where \(\chi\) is an odd primitive character modulo \(q\). Then, the left-hand side of (2.9) is

\[
\frac{1}{i\phi(q)} \sum_{\chi \pmod{q}} \chi(a)\tau(\chi) \sum_{n \leq x} \sum_{r | n} \chi(n) \log \left( \frac{x}{n} \right) = \frac{1}{i\phi(q)} \sum_{n \leq x} \sum_{r | n} \chi(n) \sum_{\chi \pmod{q}} \chi(a)\chi(r)\tau(\chi) \\
= \sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r | n} \sum_{\chi \pmod{q}} \chi(n) \chi(r) \log \left( \frac{x}{n} \right) \sin \left( \frac{2\pi ra}{q} \right), \tag{3.6}
\]
where we use the identity [6, Lemma 2.5]
\[
\sin \left( \frac{2\pi na}{q} \right) = \frac{1}{i \phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \tau(\chi) \chi(n),
\]
for \((n, q) = (a, q) = 1\).

Next, using the identity [7, equation (2.8)]
\[
\frac{1}{\tau(\chi)} \sum_{1 \leq h < q} \chi(h) \cot \left( \frac{\pi h}{q} \right) = -\frac{2 \tau(\chi)}{\pi} L(1, \chi),
\]
in examining the first expression on the right side of (2.9), we find that
\[
\frac{1}{i \phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \tau(\chi) \frac{i \tau(\chi)}{2\pi} \log(2\pi x) L(1, \chi)
\]
\[
= -\log(2\pi x) \frac{\phi(q)}{4\phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \sum_{1 \leq h < q} \chi(h) \cot \left( \frac{\pi h}{q} \right)
\]
\[
= -\log(2\pi x) \frac{\phi(q)}{4} \cot \left( \frac{\pi a}{q} \right).
\]

Also, applying [7, equation (2.7)]
\[
\frac{1}{\tau(\chi)} \sum_{1 \leq h < q} \chi(h) \left( \frac{1}{2} - \frac{h}{q} \right) = -\frac{i}{\pi} L(1, \chi),
\]
we obtain, for the third expression on the right-hand side of (2.9),
\[
\frac{x}{i \phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \tau(\chi) L(1, \chi)
\]
\[
= \frac{\pi x}{\phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \sum_{1 \leq h < q} \chi(a) \tau(\chi) \left( \frac{1}{2} - \frac{h}{q} \right)
\]
\[
= \frac{\pi x}{\phi(q)} \sum_{1 \leq h < q} \left( \frac{1}{2} - \frac{h}{q} \right) \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \tau(\chi)
\]
\[
= \pi x \left( \frac{1}{2} - \frac{a}{q} \right).
\]

Now, we consider
\[
\frac{1}{i \phi(q)} \sum_{\chi \text{ (mod } q) \chi \text{ odd}} \chi(a) \tau(\chi) \frac{i \tau(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{d\pi(n)}{n} J_0 \left( 4\pi \sqrt{nx/q} \right)
\]
\[
= \pi \left( \frac{1}{2} - \frac{a}{q} \right).
\]
\[ = -\frac{q}{2\pi \phi(q)} \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \sum_{m, r \geq 1} J_0\left(4\pi \sqrt{mrx/q}\right) \chi(r) \]

\[ = -\frac{q}{2\pi \phi(q)} \sum_{m, r \geq 1} J_0\left(4\pi \sqrt{mrx/q}\right) \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \chi(r) \]

\[ = -\frac{q}{2\pi \phi(q)} \sum_{m, r \geq 1} J_0\left(4\pi \sqrt{mrx/q}\right) \sum_{\chi \text{ (mod q)}} \chi(r) \left(\frac{\chi(a) - \chi(-a)}{2}\right) \]

\[ = -\frac{q}{4\pi} \sum_{m=1}^{\infty} \left\{ \sum_{r=1 \atop r \equiv a \text{ (mod q)}}^{\infty} J_0\left(4\pi \sqrt{mrx/q}\right) - \sum_{r=1 \atop r \equiv -a \text{ (mod q)}}^{\infty} J_0\left(4\pi \sqrt{mrx/q}\right) \right\}. \]

So, by (3.6), (3.9), (3.11) and (3.12), we only need to show that

\[-\frac{1}{2i\phi(q)} \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \tau(\chi)L'(0, \chi)\]

\[= -\frac{\log(2\pi) + \gamma}{4} \cot \left(\frac{\pi a}{q}\right) + \frac{1}{4\pi} \left(\gamma_1\left(\frac{a}{q}\right) - \gamma_1\left(1 - \frac{a}{q}\right)\right). \quad (3.13)\]

We shall employ the identity [1, p. 186, equation (7.2.26)]

\[L'(0, \chi) = \frac{M_1(\chi)}{q} \log q + \sum_{n=1}^{q-1} \chi(n) \log \left(\Gamma\left(\frac{n}{q}\right)\right), \quad (3.14)\]

where

\[M_1(\chi) = \sum_{n=1}^{q-1} \chi(n)n,\]

and also the elementary identity

\[\cot \left(\frac{\pi h}{q}\right) = -2 \sum_{j=1}^{q-1} j \sin \left(\frac{2\pi jh}{q}\right). \quad (3.15)\]

Applying (3.14), (3.7), and (3.15), we find that

\[\frac{1}{i\phi(q)} \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \tau(\chi)L'(0, \chi)\]

\[= \frac{1}{i\phi(q)} \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \tau(\chi) \left\{ \frac{\log q}{q} \sum_{1 \leq h < q} \chi(h)h + \sum_{1 \leq h < q} \chi(h) \log \left(\Gamma\left(\frac{h}{q}\right)\right) \right\}\]

\[= \frac{\log q}{iq\phi(q)} \sum_{1 \leq h < q} h \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \chi(h) \tau(\chi) + \frac{1}{i\phi(q)} \sum_{1 \leq h < q} \log \left(\Gamma\left(\frac{h}{q}\right)\right) \sum_{\chi \text{ (mod q)} \chi \text{ odd}} \chi(a) \chi(h) \tau(\chi)\]
We examine the latter sum on the right-hand side of (3.16). To do so, we appeal to a theorem of C. Deninger [14, (2.8) Theorem (a)]

$$\frac{1}{2} \{ \log (\Gamma(a)) - \log (\Gamma(1 - a)) \} = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\log k}{k} \sin(2\pi ka) - (\gamma + \log(2\pi)) \left( a - \frac{1}{2} \right).$$

Hence, by (3.17) and (3.15),

$$\sum_{1 \leq h < q} \log \left( \frac{\Gamma\left( \frac{h}{q} \right)}{\Gamma\left( \frac{1 - h}{q} \right)} \right) \sin \left( \frac{2\pi ah}{q} \right)$$

$$= \frac{1}{2} \sum_{1 \leq h < q} \left\{ \log \left( \Gamma\left( \frac{h}{q} \right) \right) - \log \left( \Gamma\left( \frac{1 - h}{q} \right) \right) \right\} \sin \left( \frac{2\pi ah}{q} \right)$$

$$= \sum_{1 \leq h < q} \sin \left( \frac{2\pi ah}{q} \right) \left\{ \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\log k}{k} \sin \left( \frac{2\pi kh}{q} \right) - (\gamma + \log(2\pi)) \left( \frac{h}{q} - \frac{1}{2} \right) \right\}$$

$$= \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\log k}{k} \sum_{1 \leq h < q} \sin \left( \frac{2\pi ah}{q} \right) \sin \left( \frac{2\pi kh}{q} \right) - (\gamma + \log(2\pi)) \sum_{1 \leq h < q} \left( \frac{h}{q} - \frac{1}{2} \right) \sin \left( \frac{2\pi ah}{q} \right)$$

$$= \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\log k}{k} \sum_{1 \leq h < q} \left\{ \cos \left( \frac{2\pi(a - k)h}{q} \right) - \cos \left( \frac{2\pi(a + k)h}{q} \right) \right\} + \frac{\gamma + \log(2\pi)}{2} \cot \left( \frac{\pi a}{q} \right)$$

$$= \frac{q}{2\pi} \lim_{N \to \infty} \left\{ \sum_{k=1}^{N} \frac{\log k}{k} - \sum_{k=1}^{N} \frac{\log k}{k} \right\} + \frac{\gamma + \log(2\pi)}{2} \cot \left( \frac{\pi a}{q} \right)$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{\log(qn + a)}{n + a/q} - \frac{\log(qn + a - 1)}{n + a/q} \right\} + \frac{\gamma + \log(2\pi)}{2} \cot \left( \frac{\pi a}{q} \right)$$

$$= \frac{\log q}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{1}{n + a/q} - \frac{1}{n + 1 - a/q} \right\} + \frac{1}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{\log(n + a/q)}{n + a/q} - \frac{\log(n + 1 - a/q)}{n + 1 - a/q} \right\}$$

$$+ \frac{\gamma + \log(2\pi)}{2} \cot \left( \frac{\pi a}{q} \right)$$

$$= \frac{\log q}{2} \cot \left( \frac{\pi a}{q} \right) + \frac{1}{2\pi} \left( - \gamma \left( \frac{a}{q} \right) + \gamma \left( 1 - \frac{a}{q} \right) \right) + \frac{\gamma + \log(2\pi)}{2} \cot \left( \frac{\pi a}{q} \right),$$

where we utilized (3.3) in the last equality. Inserting (3.18) into (3.16), we complete the proof of (3.13), and hence also of (3.5). □
Proof of (3.5) \(\Rightarrow\) (2.9). Let \(\chi\) be an odd primitive character modulo \(q\). We multiply both sides of (3.5) by \(\overline{\chi}(a)/\tau(\overline{\chi})\), and sum on \(a, 1 \leq a < q\). Observe that

\[
\frac{1}{\tau(\overline{\chi})} \sum_{n=x}^{q-1} \chi(a) \sum_{n<x} \log \left( \frac{x}{n} \right) \sum_{r|n} \sin \left( \frac{2\pi ra}{q} \right)
\]

\[
= \frac{1}{\tau(\overline{\chi})} \sum_{n<x} \log \left( \frac{x}{n} \right) \sum_{r|n} \sum_{a=1}^{q-1} \chi(a) \sin \left( \frac{2\pi ra}{q} \right)
\]

\[
= \frac{1}{2i\tau(\overline{\chi})} \sum_{n<x} \log \left( \frac{x}{n} \right) \sum_{r|n} \sum_{a=1}^{q-1} \chi(a) \left( e^{2\pi ira/q} - e^{-2\pi ira/q} \right)
\]

\[
= -i \sum_{n<x} \log \left( \frac{x}{n} \right) \sum_{r|n} \left( \chi(r) - \chi(-r) \right) / 2
\]

\[
= -i \sum_{n<x} \log \left( \frac{x}{n} \right) d_{\chi}(n), \tag{3.19}
\]

where we used the formula [13, p. 65]

\[
\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a) e^{2\pi ina/q}, \tag{3.20}
\]

for any character \(\chi\) modulo \(q\).

Now we examine the right-hand side of (3.5). Using (3.8), we have

\[
-\log(4\pi^2 x) + \gamma \sum_{a=1}^{q-1} \chi(a) \cot \left( \frac{\pi a}{q} \right) = \tau(\chi) / 2\pi L(1, \overline{\chi}) \left( \log(4\pi^2 x) + \gamma \right). \tag{3.21}
\]

Also, by (3.10),

\[
\frac{\pi x}{\tau(\overline{\chi})} \sum_{a=1}^{q-1} \chi(a) \left( \frac{1}{2} - \frac{a}{q} \right) = -i x L(1, \chi). \tag{3.22}
\]

Next, we use (3.13) and (3.8) to obtain

\[
\frac{1}{4\pi\tau(\overline{\chi})} \sum_{a=1}^{q-1} \chi(a) \left( \gamma_1 \left( \frac{a}{q} \right) - \gamma_1 \left( 1 - \frac{a}{q} \right) \right)
\]

\[
= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q-1} \chi(a) \left\{ \log(2\pi) + \gamma \cot \left( \frac{\pi a}{q} \right) - \frac{1}{2i\phi(q)} \sum_{\chi_1 \equiv 1 \pmod{q}, \chi_1 \text{ odd}} \chi_1(a) \tau(\overline{\chi_1}) L'(0, \chi_1) \right\}
\]

\[
= -\frac{\log(2\pi) + \gamma}{2\pi} \tau(\chi) L(1, \chi) - L'(0, \chi) \sum_{a=1}^{q-1} \chi(a) \chi(a)
\]
Lastly, observe that
\[
\sum_{a=1}^{q-1} \chi(a) = 2 \sum_{r=1}^{\infty} \frac{J_0(4\pi \sqrt{mrx/q})}{mr} - \sum_{r=1}^{\infty} \frac{J_0(4\pi \sqrt{mrx/q})}{mr}
\]
\[
\sum_{a=1}^{q-1} \chi(a) = \sum_{r=1}^{\infty} \frac{\chi(a) J_0(4\pi \sqrt{mrx/q})}{mr} + \sum_{r=1}^{\infty} \frac{\chi(-a) J_0(4\pi \sqrt{mrx/q})}{mr}
\]
\[
= 2 \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{\chi(r) J_0(4\pi \sqrt{mrx/q})}{mr}
\]
\[
= 2 \sum_{n=1}^{\infty} \frac{d(n)}{n} J_0 \left( \frac{4\pi \sqrt{nx/q}}{mr} \right).
\]
(3.24)

If we insert our calculations from (3.19), (3.21), (3.22), (3.23), and (3.24) into (3.5) and multiply both sides by \(i\), we then complete the proof of (2.9).

4. LOGARITHMIC MEAN IDENTITY FOR EVEN PRIMITIVE CHARACTERS

Let \(\chi\) be an even primitive non-principal character modulo \(q\). As before, the generating function for \(d_\chi(n)\) is the Dirichlet series \(\zeta(s)L(s, \chi)\). Set
\[
F(s, \chi) := \zeta(2s)L(2s, \chi).
\]
Then \(F(s, \chi)\) satisfies the functional equation [13, pp. 59, 71], [5, p. 2065]
\[
\left( \frac{\pi}{\sqrt{q}} \right)^{-2s} \Gamma^2(s) F(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \left( \frac{\pi}{\sqrt{q}} \right)^{2s-1} \Gamma^2 \left( \frac{1}{2} - s \right) F \left( \frac{1}{2} - s, \chi \right).
\]

We now apply Theorem 1.3 (i) when \(m = 2, r = \frac{1}{2}\), and \(a = 1\). First, we calculate
\[
R(x; 1) = \frac{1}{2\pi i} \int_{C} \frac{(\pi/\sqrt{q})^{-2s} L(2s, \chi)}{s^2} \frac{\zeta(2s)x^s}{ds},
\]
(4.1)
where \(C\) is a simple closed, positively oriented contour containing the simple poles \(s = \frac{1}{2}\) and \(s = 0\) (since \(L(0, \chi) = 0\)) on the interior of \(C\).

First, with the help of (2.4), we record the Taylor expansions about \(s = 0\),
\[
\left( \frac{q^x}{\pi^2} \right)^s = 1 + \log \left( \frac{q^x}{\pi^2} \right) s + \cdots,
\]
\[
\zeta(2s) = -\frac{1}{2} - \log(2\pi) s + \cdots.
\]
and
\[ L(2s, \chi) = 0 + 2L'(0, \chi)s + \cdots. \]

Hence,
\[ R_0 = -L'(0, \chi). \tag{4.2} \]

Next,
\[ R_{1/2} = \frac{\sqrt{q}}{\pi} \frac{1}{2} L(1, \chi) \sqrt{x} = \frac{2\sqrt{qx}}{\pi} L(1, \chi). \tag{4.3} \]

Employing (4.2) and (4.3) in (4.1), we deduce that
\[ R(x; 1) = \frac{2\sqrt{qx}}{\pi} L(1, \chi) - L'(0, \chi). \tag{4.4} \]

Second, first recall that [18, p. 54, equation (3)]
\[ J_{1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \sin z. \]

Thus,
\[
K(x; -\frac{1}{2}, \frac{1}{2}; -; 2) = \int_0^\infty u^{-1} J_{1/2}(u) J_{1/2}(x/u) du \\
= \int_0^\infty u^{-1} \sqrt{u} \sin u \sqrt{\frac{2u}{x}} \sin(x/u) du \\
= \frac{2}{\pi \sqrt{x}} \int_0^\infty u^{-1} \sin u \sin(x/u) du \\
= \frac{2}{\pi \sqrt{x}} \left( \frac{\pi}{2} Y_0(2\sqrt{x}) + K_0(2\sqrt{x}) \right) \\
= \frac{1}{\sqrt{x}} \left( Y_0(2\sqrt{x}) + \frac{2}{\pi} K_0(2\sqrt{x}) \right) : = \frac{1}{\sqrt{x}} P_0(2\sqrt{x}), \tag{4.5} \]

where, in the penultimate line, we use an evaluation from [18, p. 184], which also can be found in [2, p. 371, Example 2].

Third, we observe that
\[
\lambda_n = \mu_n = \frac{\pi^2 n^2}{q}, \quad \xi_n = 4\pi n \sqrt{\frac{x}{q}}, \quad a(n) = d_\chi(n), \quad b(n) = \frac{\tau(\chi)}{\sqrt{q}} d_\chi(n). \tag{4.6} \]

We now apply Theorem 1.3 (i), while invoking (4.4)–(4.6), to conclude that
\[
\sum_{\lambda_n \leq x} d_\chi(n) \log \left( \frac{x}{\lambda_n} \right) = \frac{2\sqrt{qx}}{\pi} L(1, \chi) - L'(0, \chi) \\
+ \frac{2\tau(\chi)}{\sqrt{q}} \sum_{n=1}^{\infty} d_\chi(n) \left( \frac{xq}{\pi^2 n^2} \right)^{1/4} \frac{1}{(4\pi n \sqrt{x/q})^{1/2}} P_0 \left( 2 \left( 4\pi n \sqrt{x/q} \right)^{1/2} \right). \tag{4.7} \]

We now replace \( x \) by \( \pi^2 x^2/q \) in (4.7) to deduce the following more simplified identity.
Theorem 4.1. If $\chi$ is an even, non-principal, primitive character of modulus $q$, then
\begin{equation}
\sum_{n \leq x} d_\chi(n) \log \left( \frac{x}{n} \right) = xL(1, \chi) - \frac{1}{2} L'(0, \chi) + \frac{\tau(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{d_\chi(n)}{n} P_0(4\pi \sqrt{nx/q}). \tag{4.8}
\end{equation}

5. Logarithmic Mean Analogue for Ramanujan’s Identity, Entry 1.2

Theorem 5.1. Let $x > 0$ and $0 < \theta < 1$. Then
\begin{equation}
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r \mid n} \cos(2\pi r \theta) = \frac{\log(4\pi^2 x) + \gamma}{4} - x \log(2 \sin(\pi \theta)) - \frac{1}{8} (\gamma_0(\theta) + \gamma_0(1 - \theta)) \\
+ \frac{1}{4\pi} \sum_{\substack{m \geq 1 \\mod q \atop n \geq 0}} \left\{ P_0 \left( 4\pi \sqrt{m(n + \theta)x} \right) + P_0 \left( 4\pi \sqrt{m(n + 1 - \theta)x} \right) \right\}, \tag{5.1}
\end{equation}

where $\gamma_0(\theta)$ and $\gamma_0(1 - \theta)$ are generalized Stieltjes constants defined in (3.2).

As we argued prior to the proof of Theorem 3.1, it suffices to establish (5.1) for $\theta = a/q$, with $q$ prime and $1 \leq a < q$, and consequently our task is to prove the identity
\begin{equation}
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r \mid n} \cos \left( \frac{2\pi ra}{q} \right) = \frac{\log(4\pi^2 x) + \gamma}{4} - x \log(2 \sin(\pi \theta)) - \frac{1}{8} \left( \gamma_0 \left( \frac{a}{q} \right) + \gamma_0 \left( 1 - \frac{a}{q} \right) \right) \\
+ \frac{1}{4\pi} \sum_{\substack{m \geq 1 \\mod q \atop n \geq 0}} \left\{ P_0 \left( 4\pi \sqrt{m(n + a/q)x} \right) + P_0 \left( 4\pi \sqrt{m(n + 1 - a/q)x} \right) \right\}. \tag{5.2}
\end{equation}

Similarly, we demonstrate that (5.2) is equivalent to (4.8).

Proof of (4.8) ⇒ (5.2). Let $\chi_0$ be the principal character modulo $q$. Using the identity [6, Lemma 2.5]
\begin{equation}
\cos \left( \frac{2\pi na}{q} \right) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a) \tau(\chi) \chi(n) \tag{5.3}
\end{equation}
for $(n, q) = (a, q) = 1$, we have
\begin{equation}
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r \mid n} \cos \left( \frac{2\pi ra}{q} \right) = \sum_{n \leq x} \log \left( \frac{x}{n} \right) \left\{ \sum_{\substack{q \mid n \atop q \neq r}} 1 + \sum_{\chi \mod q} \chi(a) \chi(r) \tau(\chi) \right\}
\end{equation}
\begin{equation}
= \sum_{k \leq x/q} \log \left( \frac{x}{qk} \right) d(k) + \frac{1}{\phi(q)} \sum_{n \leq x} \log \left( \frac{x}{n} \right) \sum_{r \mid n} \sum_{\chi \mod q} \chi(a) \chi(r) \tau(\chi)
\end{equation}
where to achieve the last line, we used the observation
\[
d_{\chi_0}(n) = \sum_{d \mid n} \chi_0(d) = \sum_{d \mid n} 1 - \sum_{q \mid d \mid n} 1 = d(n) - d(n/q).
\]

We examine the last summation in (5.4). By (4.8),
\[
\frac{1}{\phi(q)} \sum_{\chi \not\equiv \chi_0 \text{ even}} \chi(a) \tau(\chi) \sum_{n \leq x} \log \left( \frac{x}{n} \right) d_\chi(n)
= \frac{1}{\phi(q)} \sum_{\chi \not\equiv \chi_0 \text{ even}} \chi(a) \tau(\chi) \left\{ xL(1, \chi) - \frac{1}{2} L'(0, \chi) + \frac{\tau(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{d_\chi(n)}{n} P_0(4\pi \sqrt{nx/q}) \right\}.
\]

We recall from [10, p. 8]
\[
L(1, \chi) = -\frac{1}{\tau(\chi)} \sum_{h=1}^{q-1} \overline{\chi}(h) \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right).
\]

Also [15, p. 41, formula 1.392, no. 1]
\[
\prod_{n=1}^{q-1} \sin \left( \frac{\pi n}{q} \right) = \frac{q}{2^{q-1}}.
\]

Using the last two identities, we find that, for the first expression on the right-hand side of (5.5),
\[
\frac{x}{\phi(q)} \sum_{\chi \not\equiv \chi_0 \text{ even}} \chi(a) \tau(\chi) L(1, \chi) = -\frac{x}{\phi(q)} \sum_{\chi \not\equiv \chi_0 \text{ even}} \chi(a) \sum_{h=1}^{q-1} \overline{\chi}(h) \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right)
\]
= - \frac{x}{\phi(q)} \sum_{h=1}^{q-1} \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right) \left( \sum_{\chi (\text{mod } q) \chi \neq \chi_0 \text{ even}} \chi(a)\tau(\chi)\chi(h) - 1 \right) \\
= -x \log \left( 2 \sin \left( \frac{\pi a}{q} \right) \right) + \frac{x \log q}{\phi(q)}.
\tag{5.8}

Next, to examine the second expression on the right-hand side of (5.5), recall first from [1, equation (7.3.18)] that

\[ L'(0, \chi) = -\frac{1}{2} \sum_{n=1}^{q-1} \chi(n) \log \left( 2 \sin \left( \frac{\pi n}{q} \right) \right). \tag{5.9} \]

Second [14, p. 176, equation (2.7.1)], [15, pp. 45, 46, equations 1.441, nos. 1, 2],

\[ \sum_{k=1}^{\infty} e^{2\pi ikx} \frac{k}{k} = - \log(2 \sin(\pi x)) - i\pi \left( x - \frac{1}{2} \right). \tag{5.10} \]

Hence, by (5.9), (5.3), (5.7), and (5.10), we obtain

\[ S = \frac{1}{\phi(q)} \sum_{\chi (\text{mod } q)} \chi(a)\tau(\chi)L'(0, \chi) \]

\[ = -\frac{1}{2\phi(q)} \sum_{\chi (\text{mod } q) \chi \neq \chi_0 \text{ even}} \chi(a)\tau(\chi) \sum_{h=1}^{q-1} \chi(h) \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right) \]

\[ = -\frac{1}{2\phi(q)} \sum_{h=1}^{q-1} \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right) \left( \sum_{\chi (\text{mod } q) \chi \neq \chi_0 \text{ even}} \chi(a)\tau(\chi)\chi(h) + 1 \right) \]

\[ = -\frac{1}{2} \sum_{h=1}^{q-1} \log \left( 2 \sin \left( \frac{\pi h}{q} \right) \right) \cos \left( \frac{2\pi a h}{q} \right) - \log q \frac{2}{2\phi(q)} \]

\[ = \frac{1}{2} \sum_{h=1}^{q-1} \cos \left( \frac{2\pi a h}{q} \right) \left( \sum_{k=1}^{\infty} e^{2\pi i k h/q} \frac{k}{k} + i\pi \left( \frac{h}{q} - \frac{1}{2} \right) \right) - \log q \frac{2}{2\phi(q)} \]

\[ = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{h=1}^{q-1} \frac{1}{k} \left( \cos \left( \frac{2\pi kh}{q} \right) + i \sin \left( \frac{2\pi kh}{q} \right) \right) \cos \left( \frac{2\pi a h}{q} \right) \]

\[ + \frac{i\pi}{2q} \sum_{h=1}^{q-1} h \cos \left( \frac{2\pi a h}{q} \right) - \frac{i\pi}{4} \sum_{h=1}^{q-1} \cos \left( \frac{2\pi a h}{q} \right) - \log q \frac{2}{2\phi(q)}. \tag{5.11} \]

From [15, p. 38, equation 1.352],

\[ \sum_{h=1}^{q-1} h \cos \left( \frac{2\pi a h}{q} \right) = -\frac{q}{2}. \]
Thus, we find that from (5.11) that
\[
S = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \sum_{h=1}^{q-1} \cos \left( \frac{2\pi k h}{q} \right) \cos \left( \frac{2\pi a h}{q} \right) + i \sum_{h=1}^{q-1} \sin \left( \frac{2\pi k h}{q} \right) \cos \left( \frac{2\pi a h}{q} \right) \right\} - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k} \left( \cos \left( \frac{2\pi (k-a) h}{q} \right) + \cos \left( \frac{2\pi (k+a) h}{q} \right) \right) - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{-2}{k} + \sum_{k=1}^{N} \frac{q-2}{k} \right) - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{-2}{k} + \sum_{k=1}^{N} \frac{q}{k} \right) - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{-2}{k} + \sum_{n=0}^{\lfloor N-a \rfloor} \frac{1}{n + a/q} + \sum_{n=0}^{\lfloor N+a \rfloor - 1} \frac{1}{n + 1 - a/q} \right) - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \left( \gamma_0 \left( \frac{a}{q} \right) + \gamma_0 \left( 1 - \frac{a}{q} \right) - 2\gamma - 2 \log q \right) - \frac{\log q}{2\phi(q)}
\]
\[
= \frac{1}{4} \left( \gamma_0 \left( \frac{a}{q} \right) + \gamma_0 \left( 1 - \frac{a}{q} \right) \right) - \frac{\log q}{2\phi(q)},
\] (5.12)

where we used (3.3) for the last equality.

Now, we consider the last expression in (5.5). To that end,

\[
\frac{q}{2\pi \phi(q)} \sum_{\chi \equiv \chi_0 \text{ even}} \chi(a) \sum_{n=1}^{\infty} \frac{d\chi(n)}{n} P_0(4\pi \sqrt{nx/q})
\]
\[
= \frac{q}{2\pi \phi(q)} \sum_{m,r \geq 1} \frac{P_0(4\pi \sqrt{mrx/q})}{mr} \sum_{\chi \equiv \chi_0 \text{ even}} \chi(a)\chi(r)
\]
\[
= \frac{q}{2\pi \phi(q)} \sum_{m,r \geq 1} \frac{P_0(4\pi \sqrt{mrx/q})}{mr} \left( \sum_{\chi \equiv \chi_0 \text{ even}} \chi(a)\chi(r) - 1 \right)
\]
\[
= \frac{q}{4\pi} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{P_0(4\pi \sqrt{mrx/q})}{mr} - \frac{q}{2\pi \phi(q)} \sum_{m,r \geq 1} \frac{P_0(4\pi \sqrt{mrx/q})}{mr}
\]
\[
= \frac{1}{4\pi} \sum_{n \geq 0}^{\infty} \left\{ \frac{P_0 \left( 4\pi \sqrt{mn + a/q}x \right)}{(m(n+a/q))} + \frac{P_0 \left( 4\pi \sqrt{m(n+1-a/q)x} \right)}{(m(n+1-a/q))} \right\}
\]
- \frac{q}{2\pi\phi(q)} \sum_{m,r \geq 1} \frac{P_0(4\pi \sqrt{mrx/q})}{mr} + \frac{1}{2\pi\phi(q)} \sum_{m,n \geq 1} \frac{P_0(4\pi \sqrt{mnx})}{mn}

= \frac{1}{4\pi} \sum_{m \geq 1} \left\{ \frac{P_0\left(4\pi \sqrt{m(n + a/q)x}\right)}{(m(n + a/q))} + \frac{P_0\left(4\pi \sqrt{m(n + 1 - a/q)x}\right)}{(m(n + 1 - a/q))} \right\}

- \frac{q}{2\pi\phi(q)} \sum_{n=1}^\infty \frac{d(n)}{n} P_0(4\pi \sqrt{nx/q}) + \frac{1}{2\pi\phi(q)} \sum_{n=1}^\infty \frac{d(n)}{n} P_0(4\pi \sqrt{nx}). \quad (5.13)

Put (5.8), (5.12), and (5.13) into (5.5), and then put this new representation for (5.5) into (5.4). Finally, also applying the identity [2, p. 371]

\sum_{n \leq x} d(n) \log \left(\frac{x}{n}\right) = x(\log x - 2 + 2\gamma) + \frac{1}{4} \log(4\pi^2 x) + \frac{1}{2\pi} \sum_{n=1}^\infty \frac{d(n)}{n} P_0(4\pi \sqrt{nx}),

in (5.4), we obtain (5.2). \qed

Proof of (5.2) ⇒ (4.8). We multiply both sides of (5.2) by \( \chi(a)/\tau(\chi) \), and sum on \( a, 1 \leq a < q \). First, observe that

\[
\frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \sum_{n \leq x} \log \left(\frac{x}{n}\right) \sum_{r|n} \cos \left(\frac{2\pi ra}{q}\right) = \frac{1}{\tau(\chi)} \sum_{n \leq x} \log \left(\frac{x}{n}\right) \sum_{r|n} \sum_{a=1}^{q-1} \chi(a) \cos \left(\frac{2\pi ra}{q}\right)
\]

\[
= \frac{1}{2\tau(\chi)} \sum_{n \leq x} \log \left(\frac{x}{n}\right) \sum_{r|n} \sum_{a=1}^{q-1} \chi(a) \left(e^{2\pi i ra/q} + e^{-2\pi i ra/q}\right)
\]

\[
= \sum_{n \leq x} \log \left(\frac{x}{n}\right) \sum_{r|n} \frac{\chi(r) + \chi(-r)}{2}
\]

\[
= \sum_{n \leq x} \log \left(\frac{x}{n}\right) d(\chi(n)), \quad (5.14)
\]

where we used the formula [13, p. 65]

\[
\chi(n)\tau(\chi) = \sum_{h=1}^{q} \overline{\chi(h)} e^{2\pi i nh/q},
\]

for any character \( \chi \) modulo \( q \).

Now, we examine the right-hand side of (5.2). Since \( \chi \) is non-principal, we find that

\[
\log(4\pi^2 x) + \gamma \sum_{a=1}^{q-1} \chi(a) = 0. \quad (5.15)
\]
Also, by (5.6),
\[
\frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \log \left( 2 \sin \left( \frac{\pi a}{q} \right) \right) = -L(1, \chi). \tag{5.16}
\]

Next, note that the last expression on the right-hand side in (5.12) is equal to $S$, which is defined in (5.11). Thus, equating that definition of $S$ with the last line of (5.12), multiplying both sides by $\chi(a)/\tau(\chi)$, and summing on $a$, we find that
\[
\frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \left( \gamma_0 \left( \frac{a}{q} \right) + \gamma_0 \left( 1 - \frac{a}{q} \right) \right) = 4 \frac{\tau(\chi)}{\phi(q)} \sum_{\chi \pmod{q}} \chi(\chi_1) L'(0, \chi_1) \\
= 4 L'(0, \chi) + 4 \frac{\tau(\chi)}{\phi(q)} \sum_{\chi \equiv \pm \chi_0 \pmod{q}, \chi \text{ even}} \tau(\chi) L'(0, \chi_1) \sum_{a=1}^{q-1} \chi(a) \chi_1(a) \\
= 4 L'(0, \chi). \tag{5.17}
\]
Lastly, observe that
\[
\frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \sum_{n \geq 1} \sum_{m \geq 1} \frac{P_0 \left( 4\pi \sqrt{mn(a/q)x} / m \right)}{(m(n + a/q))} + \frac{P_0 \left( 4\pi \sqrt{mn(1 - a/q)x} / m \right)}{(m(n + 1 - a/q))} \\
= \frac{q}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{P_0(4\pi \sqrt{mrx/q})}{mr} \\
= \tau(\chi) \sum_{m=1}^{\infty} \sum_{a=1}^{q-1} \sum_{r=1}^{\infty} \chi(r) \frac{P_0(4\pi \sqrt{mrx/q})}{mr} \\
= 2 \tau(\chi) \sum_{n=1}^{\infty} \frac{d_\chi(n)}{n} P_0(4\pi \sqrt{nx/q}). \tag{5.18}
\]
Hence, inserting (5.14) – (5.18) into (4.8), we complete the proof of (5.2). \qed

REFERENCES


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