ON THE BROCARD–RAMANUJAN DIOPHANTINE EQUATION $n! + 1 = m^2$

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In 1876, and then again in 1885, H. Brocard [1], [2] posed the problem of finding all integral solutions to

$$n! + 1 = m^2. \tag{1}$$

In 1913, unaware of Brocard's query, S. Ramanujan [8], [9, p. 327] formulated the problem in the form, "The number $1 + n!$ is a perfect square for the values 4, 5, 7 of n. Find other values." In 1906, A. Géradin [4] presented arguments claiming that, if $m > 71$, then m must have at least 20 digits, and, in 1935, H. Gupta [5] stated that calculations of $n!$ up to $n = 63$ gave no further solutions. Despite the fact that the problem also appears in R. K. Guy's popular book [6, pp. 193–194], we do not know of any further calculations. The purpose of this paper is to report on recent calculations up to $n = 10^9$ and to briefly discuss a related problem.

If $\left(\frac{a}{p}\right)$ denotes the Legendre symbol, we looked for solutions to

$$\left(\frac{n! + 1}{p}\right) = 1 \text{ or } 0. \tag{2}$$

Let us say that we have a "solution" if (2) holds for each of the first 40 primes p after 10^9. Computations were performed modulo p. Except for the known cases $n = 4, 5, 7$, we found no further "solutions" of (2). It follows that (1) also has no further solutions up to 10^9.

The computations were effected by writing a program in C. The search for solutions of (2) up to $n = 10^9$ took about 32 hours on a SUN SPARCstation 5 workstation.

In 1993, M. Overholt [7] proved that (1) has only finitely many solutions if the weak form of Szpiro's conjecture is true, but this remains unproved. To state the weak form of Szpiro's conjecture, which is a special case of the ABC conjecture, first set

$$N_0(n) = \prod_{p|n} p,$$

where p denotes a prime. Let a, b, and c denote positive integers, relatively prime in pairs and satisfying the equality $a + b = c$. Then the weak form of Szpiro's conjecture asserts that there exists a constant s such that

$$|abc| \leq N_0^s(abc).$$

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It is natural to consider the more general diophantine equation

$$n! + k = m^2. \tag{3}$$

A. Dabrowski [3] easily showed that, for each fixed k that is not a square, there is only a finite number of solutions. More precisely, if n yields a solution, n is less than the least prime p for which $\left(\frac{k}{p}\right) = -1$. Thus, (3) is only interesting when k is a square. For $k = s^2, 2 \le s \le 50$, we searched for solutions of (3) up to $n = 10^5$ and found either zero or one solution in each case. In this range, the solution giving the largest n is $11! + 18^2 = 6318^2$.

Dabrowski [3] also proved that, when k is a square, (3) has only finitely many solutions if the weak form of Szpiro's conjecture is true.

References

1. H. Brocard, *Question 166*, Nouv. Corresp. Math. **2** (1876), 287.
2. H. Brocard, *Question 1532*, Nouv. Ann. Math. **4** (1885), 391.
3. A. Dabrowski, *On the diophantine equation* $n! + A = y^2$, Nieuw Arch. Wisk. **14** (1996), 321–324.
4. A. Gérardin, *Contribution a l'étude de l'équation* $1 \cdot 2 \cdot 3 \cdot 4 \cdots z + 1 = y^2$, Nouv. Ann. Math. (4) **6** (1906), 222–226.
5. H. Gupta, *On a Brocard–Ramanujan problem*, Math. Student **3** (1935), 71.
6. R. Guy, *Unsolved Problems in Number Theory*, Springer–Verlag, New York, 1994.
7. M. Overholt, *The diophantine equation* $n! + 1 = m^2$, Bull. London Math. Soc. **25** (1993), 104.
8. S. Ramanujan, *Question 469*, J. Indian Math. Soc. **5** (1913), 59.
9. S. Ramanujan, *Collected Papers*, Chelsea, New York, 1962.

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