TWO DIRICHLET SERIES FOUND ON PAGE 196 OF RAMANUJAN’S LOST NOTEBOOK

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Abstract. On page 196 in his Lost Notebook, S. Ramanujan offers evaluations of two particular Dirichlet series. In this article, we establish Ramanujan’s evaluations and more general results by various approaches. The different evaluations arising from different methods yield intriguing, unsuspecting identities.

1. Introduction

On page 196 in his Lost Notebook [6, p. 196, (i), (ii)], Ramanujan recorded the identities

\[ \sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi n^2}{a} \right)}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a} \left( 1 - \frac{r}{a} \right) \sin \left( \frac{\pi}{4} + \frac{\pi r^2}{a} \right) \]

and

\[ \sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi n^2}{a} \right)}{n^2} = -\frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a} \left( 1 - \frac{r}{a} \right) \cos \left( \frac{\pi}{4} + \frac{\pi r^2}{a} \right), \]

where \( a \) is an even positive integer. Note that when \( a = 2 \), (1.1) is equivalent to Euler’s evaluation

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

We also note that (1.1) and (1.2) are equivalent to the identity

\[ \sum_{n=1}^{\infty} \frac{e^{\pi in^2/a}}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \left( \frac{r}{a} \right) \left( 1 - \frac{r}{a} \right) e^{i(\pi/4 - \pi r^2/a)}. \]

Motivated by the left-hand side of (1.3), we let

\[ R_k(s) = \sum_{n=1}^{\infty} \frac{e^{2\pi in^2/k}}{n^s}, \quad \text{Re } s > 1, \]

where \( k \) is a positive integer. In this article, we derive several identities, or evaluations, for \( R_k(2m) \), when \( m \) is a positive integer.

To illustrate our work, we provide here two of our evaluations. The first identity expresses \( R_k(2m) \), for \( k \equiv 0 \text{ (mod } 4) \), in terms of the Bernoulli
numbers $B_n$, $n \geq 0$, defined by
\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi,
\]
and the Bernoulli polynomials $B_n(t)$, $n \geq 0$, which are defined by
\[
\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n, \quad (|x| < 2\pi).
\]

**Theorem 1.1.** Let $a$ be an even positive integer. Then
\[
R_{2a}(2m) = \frac{(-1)^{m+1} \pi^2 2^{2m-1}}{(2m)!} \times \left( B_{2m} + \frac{1}{\sqrt{a}} \sum_{\nu=1}^{a} \left( B_{2m} \left( \frac{\nu}{a} \right) - B_{2m} \right) e^{\pi i/4 - \pi i \nu^2/a} \right).
\]

When $m = 1$, (1.5) reduces to (1.3).

To describe the second identity associated with $R_k(s)$, we recall the definitions of the Stirling numbers of the second kind $S(n,h)$ and the entries $c_{n,h}$ in the Catalan triangle. For nonnegative integers $n$ and $h$, they are defined by
\[
S(n,h) = \frac{1}{h!} \sum_{j=0}^{h} (-1)^j \binom{h}{j} (h-j)^n
\]
and
\[
c_{n,h} = \frac{n-h+1}{n+1} \binom{n+h}{n},
\]
respectively, where in the last definition we also require that $h \leq n$. For a fixed positive integer $k$ and for nonnegative integers $u$ and $v$, let
\[
T_{u,v} = \sum_{r=1}^{k-1} e^{2\pi i vr/k} \frac{\omega_r^v}{(\omega_r - 1)^u},
\]
where $\omega_r = e^{2\pi i r/k} \neq 1$.

**Theorem 1.2.** Let $k$ be a positive integer. Then
\[
R_k(2m) = \frac{(-1)^{m+1} \pi^2 2^{2m-1}}{2 \cdot (2m)!} \left( \frac{2\pi}{k} \right)^{2m} \left( B_{2m} - 2m \sum_{j=1}^{m} \alpha_{2m,j} T_{2j,j} \right),
\]
where
\[
\alpha_{2m,1} = 1
\]
and, for $j \geq 2$,
\[
\alpha_{2m,j} = \sum_{s=0}^{j-2} (-1)^s c_{j-2,s} (j-s)! S(2m-1, j-s).
\]
We should note that the coefficients $\alpha_{2m,j}$ in the left-hand side of (1.8) are independent of $k$.

2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. This proof is motivated by the observation that (1.5) is similar to Dirichlet’s class number formula, which expresses a special value of a certain Dirichlet $L$-series as a finite sum of terms involving the Legendre symbol [4, p. 51]. In our considerations, values of the Hurwitz zeta function

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^s}, \quad \text{Re } s > 1,$$

take the place of values of the Legendre symbol.

Proof of (1.5). We begin by writing $R_k(s)$, for $\text{Re } s > 1$, as

$$R_k(s) = \sum_{r=1}^{k} \sum_{n=0}^{\infty} e^{2\pi i(r + kn)^2/k} \left( \frac{1}{(r + kn)^s} \right) = \frac{1}{k^s} \sum_{r=1}^{k} e^{2\pi i r^2 / k} \zeta\left( s, \frac{r}{k} \right).$$

Since $\zeta(s, x)$ has an analytic continuation into the entire complex $s$-plane, the right-hand side of (2.1) gives the analytic continuation of $R_k(s)$ to the whole complex $s$-plane.

Using the functional equation of the Hurwitz zeta function [1, p. 261, Theorem 12.8], we deduce that

$$\sum_{r=1}^{k} e^{2\pi i r^2 / k} \zeta\left( 1 - s, \frac{r}{k} \right)$$

$$= \frac{\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} e^{2\pi i r^2 / k} \sum_{\ell=1}^{k} \left( e^{-\pi i s / 2} e^{2\pi i \ell / k} + e^{\pi i s / 2} e^{-2\pi i \ell / k} \right) \zeta\left( s, \frac{\ell}{k} \right).$$

Interchanging the summations on the right-hand side of (2.2), we deduce that

$$\sum_{r=1}^{k} e^{2\pi i r^2 / k} \zeta\left( 1 - s, \frac{r}{k} \right)$$

$$= \frac{\Gamma(s)}{(2\pi k)^s} \sum_{\ell=1}^{k} \zeta\left( s, \frac{\ell}{k} \right) \left\{ e^{-\pi i s / 2} \sum_{r=1}^{k} e^{2\pi i (r^2 + r \ell) / k} + e^{\pi i s / 2} \sum_{r=1}^{k} e^{2\pi i (r^2 - r \ell) / k} \right\}$$

$$= \frac{2\Gamma(s)}{(2\pi k)^s} \cos \left( \frac{\pi s}{2} \right) \sum_{\ell=1}^{k} \zeta\left( s, \frac{\ell}{k} \right) \left( \sum_{r=1}^{k} e^{2\pi i (r^2 + r \ell) / k} \right).$$
Letting $s$ tend to $1 - 2m$, where $m$ is a positive integer, we find, using the residue of $\Gamma(s)$ at $s = 1 - 2m$ [1, p. 250], that

$$
\lim_{s \to 1 - 2m} \frac{\Gamma(s) \cos \left( \frac{\pi s}{2} \right)}{s - (1 - 2m)} = \frac{(-1)^m \pi}{2(2m - 1)!}.
$$

From [1, p. 264, Theorem 12.13], we find that

$$
\zeta \left( 1 - 2m, \frac{\ell}{k} \right) = -\frac{1}{2m} B_{2m} \left( \frac{\ell}{k} \right).
$$

Using (2.4) and (2.5) in (2.3), and then (2.3) in (2.1), with $s$ replaced by $1 - 2m$, we deduce that

$$
R_k(2m) = \frac{\pi (-1)^{m+1}}{(2\pi)^{1-2m} k \cdot (2m)!} S_{2m},
$$

where

$$
S_n = \sum_{\ell=1}^k B_n \left( \frac{\ell}{k} \right) \sum_{r=1}^k e^{2\pi i (r^2 + r\ell)/k}.
$$

Next, let $k = 2a$ where $a$ is even. Then

$$
\sum_{r=1}^{2a} e^{2\pi i (r^2 + r\ell)/(2a)} = \sum_{r=1}^a e^{2\pi i (r^2 + r\ell)/(2a)} + \sum_{r=1}^a e^{2\pi i ((a+r)^2 + (a+r)\ell)/(2a)}
$$

$$
= (1 + (-1)\ell) \sum_{r=1}^a e^{\pi i (r^2 + r\ell)/a}.
$$

Hence, the left-hand side of (2.8) vanishes when $\ell$ is odd. Let $\ell = 2\nu$. Then, we may write (2.8) as

$$
\sum_{r=1}^{2a} e^{2\pi i (r^2 + r\ell)/(2a)} = 2 \sum_{r=1}^a e^{\pi i ((r+\nu)^2 - \nu^2)/a}
$$

$$
= 2e^{-\pi i \nu^2/a} \sum_{s=1}^a e^{\pi i s^2/a} = \sqrt{2a(1 + i)} e^{-\pi i \nu^2/a} = 2\sqrt{ae^{\pi i/4 - \pi i \nu^2/a}},
$$

where we have used the fact (see [1, p. 195, (30)] or [2, p. 15, Corollary 1.2.3])

$$
\sum_{r=1}^{4c} e^{2\pi i r^2/(4c)} = (1 + i)\sqrt{4c},
$$
where $c$ is any positive integer. By (2.7)–(2.9), we deduce that

$$S_{2m} = B_{2m} \sum_{r=1}^{2a} \sum_{\ell=1}^{2a} e^{2\pi i (r^2 + r\ell)/(2a)}$$

$$+ \sum_{\ell=1}^{2a} \left( B_{2m} \left( \frac{\ell}{2a} \right) - B_{2m} \right) \sum_{r=1}^{2a} e^{2\pi i (r^2 + r\ell)/(2a)}$$

$$= 2a \cdot B_{2m} + 2\sqrt{a} \sum_{\nu=1}^{a} \left( B_{2m} \left( \nu/a \right) - B_{2m} \right) e^{\pi i/4 - \pi \nu^2/a}.$$

Substituting the last equality into (2.6), we conclude the proof of (1.5). □

3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. We first establish a few lemmas.

Lemma 3.1. Suppose $\alpha \neq 1$ and

$$\frac{xe^x}{e^x - \alpha} =: \sum_{n=1}^{\infty} \frac{U_n}{n!} x^n.$$ \hfill (3.1)

Then

$$U_n = n(1 - u) \sum_{h=1}^{n-1} \sigma_{n-1,h} u^h (-1)^{h-1},$$

where

$$u = \frac{1}{1 - \alpha} \quad \text{and} \quad \sigma_{n,h} = h! S(n, h).$$ \hfill (3.2)

Proof. From (3.1), we find that

$$xe^x = (e^x - \alpha) \left( \sum_{n=1}^{\infty} \frac{U_n}{n!} x^n \right)$$

$$= \left( 1 - \alpha + \sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \left( \sum_{n=1}^{\infty} \frac{U_n}{n!} x^n \right).$$

Comparing the coefficients of $x^n$, $n \geq 1$, we find that

$$n = (1 - \alpha) U_n + \sum_{j=1}^{n-1} \binom{n}{j} U_{n-j}.$$ 

Using the value $U_1 = 1/(1 - \alpha)$, we see that $U_n$ must satisfy the recurrence relation

$$U_n = nu(1 - u) - u \sum_{j=2}^{n-1} \binom{n}{j} U_j.$$ \hfill (3.3)
Now, let
\[ U_j = j u (1 - u) V_j. \]
Then, by (3.3), \( V_j \) satisfies the recurrence relation
\[ V_n = 1 - u \sum_{j=2}^{n-1} \binom{n-1}{j-1} V_j. \]
Thus, in order to prove Lemma 3.1, it suffices to show that if
\[ W_n = \sum_{h=1}^{n-1} \sigma_{n-1,h} u^{h-1} (-1)^{h-1}, \]
then \( W_1 = V_1 \) and
\[ W_n = 1 - u \sum_{j=2}^{n-1} \binom{n-1}{j-1} W_j, \quad n \geq 2, \]
Since \( S(n-1,1) = 1 \), it is easily checked that \( W_1 = V_1 \).
Now, we observe that
\[
1 - u \sum_{j=2}^{n-1} \binom{n-1}{j-1} W_j = 1 + \sum_{j=2}^{n-1} \binom{n-1}{j-1} \sum_{h=1}^{j-1} \sigma_{j-1,h} u^{h} (-1)^{h}
= 1 + \sum_{h=1}^{n-2} u^{h} (-1)^{h} \sum_{j=h+1}^{n-1} \binom{n-1}{j-1} \sigma_{j-1,h}.
\]
In order to show that \( W_n \) satisfies (3.4), it suffices to show that
\[ \sigma_{n-1,h+1} = \sum_{j=h+1}^{n-1} \binom{n-1}{j-1} \sigma_{j-1,h}, \]
because \( S(n-1,1) = 1 \).
Now, using (3.2), we may rewrite (3.5) as
\[ (h + 1) S(n-1, h+1) = \sum_{j=h}^{n-2} \binom{n-1}{j} S(j,h), \]
or, with \( n \) replaced by \( n + 1 \),
\[ (h + 1) S(n, h+1) = \sum_{j=h}^{n-1} \binom{n}{j} S(j,h), \]
where \( h \leq n - 1 \). Adding the term \( S(n, h) \) to both sides of (3.6), we conclude that
\[ S(n, h) + (h + 1) S(n, h+1) = \sum_{j=h}^{n} \binom{n}{j} S(j,h). \]
It is known that [7, p. 43, Eq. 14(b)] the right-hand side of (3.7) equals \( S(n+1, h+1) \). Hence, (3.7) is equivalent to

\[
S(n+1, h+1) = S(n, h) + (h+1)S(n, h+1).
\]

But (3.8) is a well known property of \( S(n, h) \) [7, p. 33, (37)] and this completes the proof of Lemma 3.1. \( \square \)

Recall from (2.6) that we can express \( R_k(2m) \) in terms of \( S_{2m} \), where \( S_n \) is given by (2.7). (Note that (2.6) and (2.7) hold for any positive integer \( k \).

We now examine \( S_n \).

Separating the term with \( r = k \) in (2.8) and using the multiplication formula for Bernoulli polynomials [5, p. 590, Eq. (24.4.18)]

\[
B_n(kx) = k^{n-1} \sum_{\ell=1}^{k-1} B_n \left( x + \frac{\ell}{k} \right)
\]

with \( x = 0 \), we deduce that

\[
S_n = k^{1-n}B_n + \sum_{r=1}^{k-1} e^{2\pi i r/k} A_n(r),
\]

where

\[
A_n(r) = \sum_{\ell=1}^{k} B_n \left( \frac{\ell}{k} \right) \omega_r^\ell, \quad \omega_r = e^{2\pi i r/k}. \tag{3.10}
\]

Substituting \( t = \ell/k \) in (1.4), multiplying by \( \omega_r^\ell \) and summing over \( \ell \), \( 1 \leq \ell \leq k \), we deduce that

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{\ell=1}^{k} B_n \left( \frac{\ell}{k} \right) \omega_r^\ell \right) x^n = \frac{x}{e^x-1} \sum_{\ell=1}^{k} e^{x\ell/k} \omega_r^\ell. \tag{3.11}
\]

Since

\[
\sum_{\ell=1}^{k} \left( e^{x/k} \omega_r \right)^\ell = \frac{1-e^{x/k} \omega_r}{1-e^{x/k} \omega_r} e^{x/k} \omega_r,
\]

we find, from (3.10) and (3.11), that

\[
\sum_{n=0}^{\infty} \frac{1}{n!} A_n(r)x^n = \frac{xe^{x/k}}{e^{x/k} - \omega_r^{-1}}. \tag{3.12}
\]

Replacing \( x \) by \( kx \) (with sufficiently small \(|x|\)), we have

\[
\sum_{n=0}^{\infty} \frac{1}{n!} k^{n-1} A_n(r)x^n = \frac{xe^x}{e^x - \alpha}, \tag{3.13}
\]

where \( \alpha = \omega_r^{-1} \neq 1 \).

We are now ready to prove the next lemma, which because of its importance, we elevate to the status of a theorem.
Theorem 3.2. Let \( T_{u,v} \) be given by (1.7). Then
\[
\sum_{n=1}^{\infty} \frac{e^{2\pi i n^2/k}}{n^{2m}} = \frac{(-1)^{m+1}(2\pi)^{2m-1}}{k(2m)!} \left( 2m \sum_{s=1}^{2m-1} \sigma_{2m-1,s} T_{s+1,1} \right).
\]

Proof. Lemma 3.1 and (3.13) give the representation of \( A_n(r) \) in terms of Stirling numbers of the second kind, namely,
\[
k^{n-1} A_n(r) = n(1-u) \sum_{h=1}^{n-1} \sigma_{n-1,h} u^h (-1)^{h-1},
\]
where \( u = 1/(1 - \omega_r^{-1}) \). Set \( n = 2m \) in (3.15) and then substitute (3.15) in (3.9). Using also (2.6), we then find that
\[
R_k(2m) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2/k}}{n^{2m}} = \frac{(-1)^{m+1}(2\pi)^{2m-1}}{k(2m)!} S_{2m}
\]
\[
= \frac{(-1)^{m+1}(2\pi)^{2m-1}}{k(2m)!} \left\{ \frac{B_{2m}}{k^{2m-1}} + \frac{1}{k^{2m-1}} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \right\}
\]
\[
\times 2m \sum_{h=1}^{2m-1} (-1)^h \sigma_{2m-1,h} \frac{\omega_r^h}{(\omega_r - 1)^{h+1}}
\]
\[
= \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} \left\{ \frac{B_{2m}}{k^{2m-1}} + 2m \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \right\}
\]
\[
\times \sum_{h=1}^{2m-1} (-1)^h \sigma_{2m-1,h} \frac{\omega_r^h}{(\omega_r - 1)^{h+1}}
\].

Now replace \( r \) by \( k - r \) in the summation over \( r \) on the far right-hand side above. Employing (1.7), we then obtain the assertion (3.14). \( \square \)

In order to prove Theorem 1.2, we need to replace the terms
\[
T_{s+1,1}, \quad 1 \leq s \leq 2m - 1,
\]
in Theorem 3.2 by
\[
T_{2j,j}, \quad 1 \leq j \leq m.
\]
To effect such a change, we need two lemmas.

Lemma 3.3. Let \( j \) be a positive integer. Then
\[
T_{2j,j} = \sum_{\nu=0}^{j-1} \binom{j-1}{\nu} T_{j+\nu+1,1}.
\]
Proof. From the definition (1.7),
\[ T_{2j,j} = \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \frac{\omega_r^j}{(\omega_r - 1)^{2j}} \]
\[ = \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \omega_r (\omega_r)^{j-1} \frac{1}{(\omega_r - 1)^{2j}} \]
\[ = \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \omega_r \left( \sum_{\nu=0}^{j-1} \left( j-1 \nu \right) (\omega_r - 1)^\nu \right) \]
\[ = \sum_{\nu=0}^{j-1} \left( j-1 \nu \right) T_{j+\nu+1,1}, \]
where in the penultimate line we replaced \( \nu \) by \( j-1-\nu \). \( \square \)

Lemma 3.4. Let \( j \) be a positive integer. Then
\[ (3.17) \quad T_{2j+1,1} = -\frac{1}{2} \sum_{h=0}^{2j-2} \binom{2j-1}{h} T_{h+2,1}. \]

Proof. Replacing \( r \) by \( k-r \) and then introducing the notation \( v = \omega_r - 1 \), we find that
\[ T_{2j+1,1} = \frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \left( \frac{\omega_r}{(\omega_r - 1)^{2j+1}} + \frac{\omega_r^{-1}}{(\omega_r^{-1} - 1)^{2j+1}} \right) \]
\[ = -\frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \omega_r (\omega_r^{2j-1} - 1) \frac{1}{v^{2j+1}} \]
\[ = -\frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi ir^2/k} \omega_r \left( \sum_{h=0}^{2j-2} \binom{2j-1}{h} v^{2j-1-h} \right) \]
\[ = -\frac{1}{2} \sum_{h=0}^{2j-2} \binom{2j-1}{h} T_{h+2,1}. \] \( \square \)

Proof of Theorem 1.2. Let
\[ \mathcal{M}_s = \{T_{j,1} | 1 \leq j \leq s\} \]
and
\[ \mathcal{N}_s = \{T_{2j,j} | 1 \leq j \leq s\}, \]
except that if \( s = 1 \), then \( \mathcal{M}_2 \) contains only the element \( T_{2,1} \). We claim that for fixed \( t \), every term in \( \mathcal{M}_{2t} \) is a linear combination of elements in \( \mathcal{N}_t \).
We prove this by induction on $t \geq 1$. Keeping in mind that $M_2 = \{T_{2,1}\}$, we can easily see that the case $t = 1$ is trivial. By induction, it suffices to show that both $T_{2t-1,1}$ and $T_{2t,1}$ can be expressed as linear combinations of elements in $N_t$. Now, by (3.17), $T_{2t-1,1}$ is a linear combination of elements in $M_{2(t-1)}$ and by induction, each term in $M_{2(t-1)}$ is a linear combination of elements in $N_{t-1}$. Next, by (3.16), $T_{2t,1}$ is a linear combination of elements in $M_{2(t-1)} \cup \{T_{2t,1}\}$. By induction again, we conclude that $T_{2t,1}$ is a linear combination of elements in $N_t$.

Let $\tilde{S}_{2m}$ denote the sum on the right-hand side of (3.14), namely, with $s$ replaced by $j + 1$,

\begin{equation}
\tilde{S}_{2m} = \sum_{j=0}^{2m-2} \sigma_{2m-1,j+1} T_{j+2,1}.
\end{equation}

From the argument above, we can conclude that

\begin{equation}
\tilde{S}_{2m} = \sum_{j=1}^{m} \alpha_{2m,j} T_{2j,j} = \sum_{j=0}^{2m-1} \alpha_{2m,j+1} T_{2j+2,j+1},
\end{equation}

for certain rational numbers $\alpha_{2m,j+1}$.

We substitute (3.16) into (3.19) and obtain

\begin{equation}
\tilde{S}_{2m} = \sum_{j=0}^{2m-2} \sum_{h=0}^{m-1} \alpha_{2m,j+1} \binom{j}{h} T_{j+h+2,1}
\end{equation}

Comparing (3.20) and (3.18), we conclude that if $\alpha_{2m,j+1}$ can be chosen so as to satisfy the relations

\begin{equation}
\sum_{j=0}^{m-1} \alpha_{2m,j+1} \binom{j}{\ell - j} = \sigma_{2m-1,\ell+1}, \quad 0 \leq \ell \leq 2m - 2,
\end{equation}

then (3.19) holds with these $\alpha_{2m,j+1}$. Our next task is to invert the relations (3.21). Note that in (3.18) the binomial coefficient

\[ \binom{j}{\ell - j} \]

vanishes when $\ell - j < 0$ and $j < \ell - j$. Replacing $j$ by $r = \ell - j$ in (3.21), we find that

\begin{equation}
\sum_{0 \leq r \leq \ell/2} \alpha_{2m,\ell+1-r} \binom{\ell - r}{r} = \sigma_{2m-1,\ell+1}.
\end{equation}
If we set $b_n = \alpha_{2m,n+1}$ and $a_n = \sigma_{2m-1,n+1}$, then (3.22) becomes

$$\sum_{0 \leq r \leq \ell/2} \binom{\ell-r}{r} b_{\ell-r} = a_n.$$ 

Hence, by the inversion formula [8, p. 62, Formula 5, Table 2.3], we find that (3.23)

$$b_{\ell} = \sum_{k \geq 0} (-1)^k \left\{ \binom{\ell+k-1}{k} - \binom{\ell+k-1}{k-1} \right\} a_{\ell-k}.$$ 

Note that, by (1.6),

$$\left( \binom{\ell+k-1}{k} - \binom{\ell+k-1}{k-1} \right) = \frac{\ell-k}{\ell} \left( \binom{\ell+k-1}{k} \right) = c_{\ell-1,k}.$$ 

From (3.23), the formula above, and our auxiliary notation, we obtain the explicit formula

$$\alpha_{2m,\ell+1} = \sum_{k \geq 0} (-1)^k c_{\ell-1,k} \sigma_{2m-1,\ell-k+1}.$$ 

If we now return to Theorem 3.2 and then use (3.18), (3.19), (3.21), and (3.2), we see that (3.24) enables us to complete the proof of Theorem 1.2. □

When $k = 2a$, with $a$ an even positive integer, Theorem 3.2 (or Theorem 1.2) provides an alternative way to evaluate the left-hand side of (1.5). For example, when $m = 1$ and $k = 2a$, where $a$ is an even positive integer, we find that

$$\sum_{n=1}^{\infty} \frac{e^{i n^2/a}}{n^2} = \frac{\pi^2}{24a^2} + \frac{\pi^2}{8a^2} \sum_{r=1}^{2a-1} e^{i r^2/a} \csc^2 \left( \frac{\pi r}{2a} \right).$$ 

We now demonstrate that (3.25) can be recast in the equivalent formulotions

$$\sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi n^2}{a} \right)}{n^2} = \frac{\pi^2}{6a^2} + \frac{\pi^2 \cos(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \cos \left( \frac{\pi j^2}{a} \right) \csc^2 \left( \frac{\pi j}{a} \right)$$ 

and

$$\sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi n^2}{a} \right)}{n^2} = \frac{\pi^2 \sin(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \sin \left( \frac{\pi j^2}{a} \right) \csc^2 \left( \frac{\pi j}{a} \right).$$ 

In (3.25), replace $r$ by $2a - r$, $a + 1 \leq r \leq 2a - 1$, and isolate the term when $r = a$. Then replace $r$ by $a - r$, $\frac{a}{2} + 1 \leq r \leq a - 1$, and separate the term
when \( r = a/2 \). Hence,
\[
\sum_{n=1}^{\infty} \frac{e^{\pi in^2/a}}{n^2} = \frac{\pi^2}{24a^2} + \frac{\pi^2}{8a^2} \left( 2 \sum_{r=1}^{a-1} e^{\pi ir^2/a} \csc^2 \left( \frac{\pi r}{2a} \right) + 1 \right)
\]
\[
= \frac{\pi^2}{6a^2} + \frac{\pi^2}{4a^2} \sum_{r=1}^{a-1} e^{\pi ir^2/a} \csc^2 \left( \frac{\pi r}{2a} \right)
\]
\[
= \frac{\pi^2}{6a^2} + \frac{\pi^2}{4a^2} \left( \sum_{r=1}^{\frac{a-1}{2}} e^{\pi ir^2/a} \left( \csc^2 \left( \frac{\pi r}{2a} \right) + \csc^2 \left( \frac{\pi}{2} - \frac{\pi r}{2a} \right) \right) + 2e^{\pi ia/4} \right)
\]
\[
= \frac{\pi^2}{6a^2} + \frac{\pi^2}{2a^2} e^{\pi ia/4} + \frac{\pi^2}{a^2} \sum_{r=1}^{\frac{a-1}{2}} e^{\pi ir^2/a} \csc^2 \left( \frac{\pi r}{a} \right).
\]
Equating real and imaginary parts on the extremal sides above, we deduce (3.26) and (3.27), respectively.

In general, we see from Theorem 1.2 that we can represent \( R_{2a}(2m) \), when \( a \) is an even positive integer, in terms of the trigonometric sums
\[
\sum_{r=1}^{2a-1} e^{\pi ir^2/a} \csc^2 \left( \frac{\pi r}{2a} \right), \quad 1 \leq j \leq a.
\]

We conclude this section with one further observation. Let \( s \) tend to \(-2m \) (\( m \geq 1 \)) in (2.3). The limit of the left-hand side is \( R_k(2m + 1) \). Since \( \lim_{s \to -2m} \cos(\pi s/2) = (-1)^m \), it follows, from the obvious analogue of (2.6) and (2.7), that
\[
S_{2m+1}(k) = \sum_{\ell=1}^{k} B_{2m+1} \left( \frac{\ell}{k} \right) \sum_{r=1}^{k} e^{2\pi i(r^2+r\ell)/k} = 0, \quad m \geq 1.
\]
However, we can show (3.29) directly. First, recalling that \( B_{2m+1} = 0 \), \( m \geq 1 \), and secondly replacing \( r \) by \( k - r \), we find that
\[
S_{2m+1} = \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \sum_{\ell=1}^{k-1} B_{2m+1} \left( \frac{\ell}{k} \right) e^{2\pi i r\ell/k}
\]
\[
= \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \sum_{\ell=1}^{k-1} B_{2m+1} \left( \frac{\ell}{k} \right) e^{-2\pi i r\ell/k}.
\]
Next replace \( \ell \) by \( k - \ell \) in the inner sum and use the property \( B_n(1-x) = (-1)^n B_n(x) \), \( n \geq 2 \) [5, p. 589, Eq. (24.4.3)], to conclude that
\[
S_{2m+1} = \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \sum_{\ell=1}^{k-1} B_{2m+1} \left( 1 - \frac{\ell}{k} \right) e^{2\pi i r\ell/k} = -S_{2m+1}.
\]
Hence, (3.29) follows.
4. Evaluations Using the Theory of Periodic Zeta Functions

We now use the theory of periodic zeta functions as developed in the paper by the first author and L. Schoenfeld [3] to evaluate the general series

\[ S_a(r) := \sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi n^2}{2a} \right)}{n^r} \quad \text{and} \quad T_a(r) := \sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi n^2}{2a} \right)}{n^r}, \]

where \( r \) and \( a \) are even positive integers. In order to effect these evaluations, we need to introduce periodic Bernoulli numbers.

Let \( A = \{a_n\}, -\infty < n < \infty \), denote a sequence of numbers with period \( k \). Then the periodic Bernoulli numbers \( B_n(A), n \geq 0 \), can be defined by [3, p. 55, Proposition 9.1], for \( |z| < 2\pi/k \),

\[ \frac{z \sum_{n=0}^{k-1} a_n e^{nz}}{e^{kz} - 1} = \sum_{n=0}^{\infty} \frac{B_n(A)}{n!} z^n. \]

Furthermore [3, p. 56, Eq. (9.5)], for each positive integer \( n \),

\[ B_n(A) = k^{n-1} \sum_{j=0}^{k-1} a_{-j} B_n \left( \frac{j}{k} \right), \]

where \( B_n(x), n \geq 0 \), denotes the \( n \)th Bernoulli polynomial. We say that \( A \) is even if \( a_n = a_{-n} \) for every integer \( n \).

The complementary sequence \( B = \{b_n\}, -\infty < n < \infty \), is defined by [3, p. 32]

\[ b_n = \frac{1}{k} \sum_{j=0}^{k-1} a_j e^{-2\pi i jn/k}. \]

It is easily checked that if \( A \) is even, then \( B \) is even, and that (4.3) holds if and only if

\[ a_n = \sum_{j=0}^{k-1} b_j e^{2\pi i jn/k}, \quad -\infty < n < \infty. \]

Now set

\[ \zeta(s; A) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{Re} \ s > 1. \]

If \( A \) and \( r \) are even and if \( r \geq 2 \), then [3, p. 49, Eq. (6.25)]

\[ \zeta(r; B) = \frac{(-1)^{r+1} B_r(A)}{2r!} \left( \frac{2\pi i}{k} \right)^r. \]

From (4.3) and (4.4), we see that the sequences \( A \) and \( B \) are not symmetric. Thus, we note from above that, since \( A \) is even,

\[ \zeta(r; A) = \frac{(-1)^{r+1} B_r(B) k}{2r!} \left( \frac{2\pi i}{k} \right)^r. \]
We are now ready to state general evaluations in closed form for $S_a(r)$ and $T_a(r)$.

**Theorem 4.1.** If $S_a(r)$ and $T_a(r)$ are defined by (4.1) and if $r$ and $a$ are even positive integers, then

$$S_a(r) = \frac{(-1)^{1+r/2r}2^{r-1}\pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r \left( \frac{m}{a} \right) \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right)$$  \hspace{1cm} (4.6)

and

$$T_a(r) = \frac{(-1)^{1+r/2r}2^{r-1}\pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r \left( \frac{m}{a} \right) \cos \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right).$$  \hspace{1cm} (4.7)

In our work below, we need the value of the Gauss sum [2, p. 43, Exer. 5; p. 15, Corollary 1.2.3]

$$c^{-1} \sum_{n=0}^{c-1} e^{\pi n^2/c} = e^{\pi i/4} \sqrt{c},$$  \hspace{1cm} (4.8)

where $c$ is an even positive integer.

Before proceeding further, we show that (1.1) and (1.2) are special cases of (4.6) and (4.7), respectively. Let $r = 2$ in Theorem 4.1, and recall that $B_2(x) = x^2 - x + \frac{1}{6}$. Then

$$S_a(2) = \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left( \frac{m}{a} \right)^2 - \frac{m}{a} + \frac{1}{6} \right\} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right)$$

$$= \frac{\pi^2}{6\sqrt{a}} \sum_{m=0}^{a-1} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right) + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \frac{m}{a} \right\} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right)$$

$$= \frac{\pi^2}{6} + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \frac{m}{a} \right\} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right),$$

upon the use of (4.8) twice.

The proof of (1.2) follows along the same lines, but note that in this case, by (4.8),

$$\sum_{m=0}^{a-1} \cos \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right) = 0.$$

**Proof of Theorem 4.1.** Let

$$a_n = \cos \left( \frac{\pi n^2}{a} \right), \quad -\infty < n < \infty,$$

which is an even periodic sequence with period $a$, since $a$ is even. Then, from (4.3) and (4.8),

$$b_m = \frac{1}{a} \sum_{j=0}^{a-1} \cos \left( \frac{\pi j^2}{a} \right) e^{2\pi ij/m}$$
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\[
\begin{align*}
= & \frac{1}{2a} \sum_{j=0}^{a-1} e^{-\pi im/2a} \sum_{j=0}^{a-1} e^{-\pi i(j+m)^2/2a} \\
= & \frac{1}{2a} e^{-\pi im^2/2a} \sum_{j=0}^{a-1} e^{\pi ij^2/a} + \frac{1}{2a} e^{\pi im^2/2a} \sum_{j=0}^{a-1} e^{-\pi ij^2/a} \\
= & \frac{1}{2a} e^{-\pi im^2/a+\pi i/4} \sqrt{a} + \frac{1}{2a} e^{\pi im^2/2a-\pi i/4} \sqrt{a} \\
= & \frac{1}{\sqrt{a}} \cos \left( \frac{\pi m^2}{a} - \frac{\pi}{4} \right) \\
= & \frac{1}{\sqrt{a}} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right).
\end{align*}
\]

Therefore, by (4.2), with \( B \) in place of \( A \),

\begin{equation}
(4.9) \quad B_n(B) = a^{n-3/2} \sum_{m=0}^{a-1} \sin \left( \frac{\pi m^2}{a} + \frac{\pi}{4} \right) B_n \left( \frac{m}{a} \right).
\end{equation}

If we substitute (4.9) into (4.5) and simplify, we deduce (4.6).

The proof of (4.7) is analogous to that for (4.6). Now we set

\[ a_n = \sin \left( \frac{\pi n^2}{a} \right), \quad -\infty \leq n \leq \infty, \]

which of course is even, and repeat the same kind of argument that we gave above. \( \square \)

5. FURTHER REPRESENTATIONS IN TERMS OF TRIGONOMETRIC SUMS

Toward the end of Section 3, we mentioned that \( R_{2a}(2m) \) could be represented in terms of certain csc sums (3.28). In this section, we use contour integration to make that statement slightly more explicit.

Theorem 5.1. Let \( a \) and \( r \) be even positive integers, and define \( S_a(r) \) and \( T_a(r) \) by (4.6) and (4.7), respectively. Then if \( a \geq 2 \),

\begin{equation}
(5.1) \quad S_a(r) = \left( \frac{-1)^{r/2-1} \pi^{r}}{2 r! a^r} \right) \left( 1 + \cos \left( \frac{\pi a}{4} \right) (2^r - 1) \right) B_r \\
- \frac{\pi^{r}}{a^r (r-1)!} \sum_{j=1}^{a-1} \cos \left( \frac{\pi j^2}{a} \right) \cot^{(r-1)} \left( \frac{\pi j}{a} \right)
\end{equation}

and

\begin{equation}
(5.2) \quad T_a(r) = \left( \frac{-1)^{r/2-1} \pi^{r}}{2 r! a^r} \right) \sin \left( \frac{\pi a}{4} \right) (2^r - 1)B_r \\
- \frac{\pi^{r}}{a^r (r-1)!} \sum_{j=1}^{a-1} \sin \left( \frac{\pi j^2}{a} \right) \cot^{(r-1)} \left( \frac{\pi j}{a} \right).
\end{equation}
Proof. Setting \( n = ka + j, \) \( 0 \leq k < \infty, \) \( 1 \leq j \leq a, \) remembering that \( r \) is even, and using Euler’s formula for \( \zeta(r) \), we find that

\[
S_a(r) = \sum_{j=1}^{a} \cos\left(\frac{\pi j^2}{a}\right) \sum_{k=0}^{\infty} \frac{1}{(ka+j)^r}
\]

(5.3)

\[
= \frac{(-1)^{r/2-1}\pi^r}{2 r! a^r} B_r + \frac{1}{a^r} \sum_{j=1}^{a-1} \cos\left(\frac{\pi j^2}{a}\right) \sum_{k=0}^{\infty} \frac{1}{(k+j/a)^r}.
\]

Singling out the term for \( j = a/2 \), noting that the terms in the outer sum with indices \( j \) and \( a-j \) are identical, and using Euler’s formula for \( \zeta(r) \) once again, we find from (5.3) that,

\[
S_a(r) = \frac{(-1)^{r/2-1}\pi^r}{2 r! a^r} B_r + \cos\left(\frac{\pi a}{4}\right) (2^r - 1) \zeta(r) + \frac{1}{a^r} \sum_{j=1}^{a-1} \cos\left(\frac{\pi j^2}{a}\right) U(j,a,r),
\]

(5.4)

say. There remains the evaluation of \( U(j,a,r) \).

First observe that if for \( -\infty < k \leq -1 \), we set \( k = -\ell - 1 \), then

\[
\sum_{k=-\infty}^{\infty} \frac{1}{(k+j/a)^r} + \frac{1}{(k+(a-j)/a)^r}
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{(k+j/a)^r} + \frac{1}{(k+(a-j)/a)^r}
\]

\[
+ \sum_{\ell=0}^{\infty} \frac{1}{(-\ell-1+j/a)^r} + \frac{1}{(-\ell-j/a)^r} = 2U(j,a,r).
\]

(5.5)

It therefore suffices to evaluate the bilateral sum in (5.5).

To evaluate \( U(j,a,r) \), recall the partial fraction decomposition

\[
\pi \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{1}{z+k} + \frac{1}{z-k} \right), \quad 0 < |z| < 1.
\]

Differentiating \( r-1 \) times above, we find that

\[
\pi^r \cot^{(r-1)}(\pi z) = (-1)^{r-1}(r-1)! \sum_{k=-\infty}^{\infty} \frac{1}{(z+k)^r}.
\]

(5.6)
Recalling that $r$ is even and putting $z = j/a$ in (5.6), we deduce that

$$U(j, a, r) = -\frac{\pi^r}{(r-1)!} \cot^{(r-1)}(\pi j/a).$$

(5.7)

Putting (5.7) in (5.5), we complete the proof of (5.1).

The proof of (5.2) follows along exactly the same lines. In analogy with (5.3), we now easily deduce that

$$T_a(r) = \frac{1}{a^r} \sum_{j=1}^{a-1} \sin\left(\frac{\pi j^2}{a}\right) \sum_{k=0}^{\infty} \frac{1}{(k+j/a)^r}.$$

By the same identical argument that we used to produce (5.4), we arrive at

$$T_a(r) = \sin(\pi a/4) \left(\frac{-1}{a^r} \sum_{j=1}^{a/2-1} \pi^r (2^r - 1) B_r + \frac{1}{a^r} \sum_{j=1}^{a/2-1} \sin\left(\frac{\pi j^2}{a}\right) U(j, a, r)\right).$$

Using (5.7) above, we complete the proof of (5.2).

In conclusion, our attempts to establish Ramanujan’s original evaluations (1.1) and (1.2) and their generalizations (4.1) in Theorems 1.1, 1.2, 3.2, 4.1, and 5.1 have given us various representations for these sums in terms of Catalan triangle numbers, Stirling numbers of the second kind, Bernoulli numbers and polynomials, and trigonometric functions. Equating the different evaluations provide identities that would be surprising had not we had known of their origins. But nonetheless, these identities are intriguing. For example, let us return to the case $r = 2$ in Theorem 5.1. Since $\cot (\pi z) = -\pi \csc^2(\pi z)$, we easily conclude from Theorem 5.1 that

$$S_a(2) = \frac{\pi^2}{6a^2} + \frac{\pi^2 \cos(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \cos\left(\frac{\pi j^2}{a}\right) \csc^2\left(\frac{\pi j}{a}\right)$$

and

$$T_a(2) = \frac{\pi^2 \sin(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \sin\left(\frac{\pi j^2}{a}\right) \csc^2\left(\frac{\pi j}{a}\right).$$

(See also (3.26) and (3.27).) Hence, combining (1.1) and (1.2) with (5.8) and (5.9), respectively, we deduce the identities

$$\frac{\pi^2}{6a^2} + \frac{\pi^2 \cos(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \cos\left(\frac{\pi j^2}{a}\right) \csc^2\left(\frac{\pi j}{a}\right) = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a} \left(1 - \frac{r}{a}\right) \sin\left(\frac{\pi}{4} + \frac{\pi r^2}{a}\right).$$
and

\[ \frac{\pi^2 \sin(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{j=1}^{a/2-1} \sin(\frac{\pi j}{a}) \csc^2(\frac{\pi j}{a}) \]
\[ = -\frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a} \left( 1 - \frac{r}{a} \right) \cos(\frac{\pi}{4} + \frac{\pi r^2}{a}). \]

Note that on the left-hand sides above, the sums contain only trigonometric functions, while on the right-hand sides the sums contain both polynomials and trigonometric functions. Trigonometric identities involving polynomials in the summands appear to be rare. The sums on both sides of the identities may be regarded as new analogues of Gauss sums.

We record a few examples to illustrate our evaluations, namely,

\[ S_2(2) = \frac{\pi^2}{24}, \quad S_4(2) = -\frac{\pi^2}{48} + \frac{\pi^2 \sqrt{2}}{16}, \quad S_6(2) = -\frac{\pi^2}{72} + \frac{\pi^2 \sqrt{3}}{18}, \]
\[ T_2(2) = \frac{\pi^2}{8}, \quad T_4(2) = \frac{\pi^2 \sqrt{2}}{16}, \quad T_6(2) = \frac{\pi^2}{24} + \frac{\pi^2 \sqrt{3}}{54}. \]

References