

INTEGRALS OF EISENSTEIN SERIES AND DERIVATIVES OF L -FUNCTIONS

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Abstract. In his lost notebook, Ramanujan recorded a formula relating a “character analogue” of the Dedekind eta-function, the integral of a quotient of eta-functions, and the value of a Dirichlet L -function at $s = 2$. Here we derive an infinite family of formulas which includes Ramanujan’s original formula as a special case. Our results depend on a representation of values of the derivatives of Dirichlet L -functions as the limiting values of certain q -series.

1. INTRODUCTION

On page 207 in his lost notebook [13], Ramanujan recorded a formula involving an integral of Dedekind eta-functions, a certain infinite product, and a particular constant C , which Ramanujan claimed (with two question marks appended) was a simple multiple of the value of a certain Dirichlet L -function $L(s, \chi)$ evaluated at $s = 2$. (See Theorem 1.1 below for a precise statement of this formula.) S. H. Son [15] established most of Ramanujan’s claim, but he was unable to evaluate the constant C . More recently, using different techniques, Berndt and Zaharescu [4] succeeded in establishing Ramanujan’s full claim.

Ramanujan’s example is surprising since it connects three distinct, apparently disparate mathematical objects, and so it is natural to ask if there are other occurrences of such phenomena. Here, inspired by the form of this result, we show that this is indeed the case by establishing a doubly infinite family of formulas which includes Ramanujan’s original formula as a very special case.

The distinguishing feature of the integrand in Ramanujan’s formula is that it is (essentially) an Eisenstein series of weight 3 on the congruence subgroup $\Gamma_0(3)$. In this paper we obtain (Theorem 1.3) an infinite family of similar formulas associated to Eisenstein series of arbitrary weight and level. The main ingredient in our proof, which is of independent interest, is a representation (Theorem 1.4) for $L'(2 - k, \chi)$ (where $k \geq 2$ is an integer and χ is a non-trivial Dirichlet character with $\chi(-1) = (-1)^k$) as the limiting value of the antiderivative of such an Eisenstein series.

Before we state the results, we recall that the Dedekind eta-function $\eta(z)$ is given by

$$\eta(z) := e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = q^{1/24} f(-q), \quad q = e^{2\pi iz}, \quad \text{Im } z > 0, \quad (1.1)$$

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where we employ Ramanujan's notation

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n).$$

If D is a fundamental discriminant, then define the character

$$\chi_D(n) := \left(\frac{D}{n} \right),$$

where the symbol on the right side is the Kronecker symbol. If χ is a Dirichlet character, then denote by $L(s, \chi)$ the usual Dirichlet L -function, defined for $\operatorname{Re} s > 1$ by $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$.

We now state Ramanujan's original formula, whose proof was given in [4].

Theorem 1.1. *Suppose that $0 < q < 1$. Then*

$$q^{1/9} \prod_{n=1}^{\infty} (1 - q^n)^{\chi_{-3}(n)n} = \exp \left(-C_3 - \frac{1}{9} \int_q^1 \frac{f^9(-t)}{f^3(-t^3)} \frac{dt}{t} \right), \quad (1.2)$$

where

$$C_3 = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}) = L'(-1, \chi_{-3}). \quad (1.3)$$

We are grateful to D. Masser [11], who first informed us of the last equality in (1.3).

As another example of this phenomenon we state the following result.

Theorem 1.2. *Suppose that $0 < q < 1$. Then*

$$q^{1/4} \prod_{n=1}^{\infty} (1 - q^n)^{\chi_{-4}(n)n} = \exp \left(-C_4 - \frac{1}{4} \int_q^1 \frac{f^4(-t)f^6(-t^2)}{f^4(-t^4)} \frac{dt}{t} \right), \quad (1.4)$$

where

$$C_4 = \frac{2}{\pi} L(2, \chi_{-4}) = L'(-1, \chi_{-4}).$$

As described above, we have the following general theorem, which contains Theorems 1.1 and 1.2 as special cases.

Theorem 1.3. *Suppose that α is real, that $k \geq 2$ is an integer, and that χ is a non-trivial Dirichlet character which satisfies the condition $\chi(-1) = (-1)^k$. Then, for $0 < q < 1$, we have*

$$q^\alpha \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n)n^{k-2}} = \exp \left(-C - \int_q^1 \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} t^n \right\} \frac{dt}{t} \right), \quad (1.5)$$

where

$$C = L'(2 - k, \chi).$$

In special cases, such as in Theorems 1.1 and 1.2, the integrand in (1.5) can be expressed in terms of eta-functions. This will be further discussed in Sections 2 and 4. The key to proving Theorem 1.3 is the following representation for $L'(2 - k, \chi)$.

Theorem 1.4. *Let $k \geq 2$ be an integer, and let χ be a non-trivial Dirichlet character such that $\chi(-1) = (-1)^k$. Then*

$$\lim_{q \rightarrow 1^-} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{q^n}{n} = L'(2-k, \chi).$$

Suppose that χ is a non-trivial primitive character modulo N and that, for $n \geq 0$, $B_{n,\chi}$ denotes the n th generalized Bernoulli number defined by [16, p. 12]

$$\sum_{n=1}^N \frac{\chi(n) t e^{nt}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad |t| < 2\pi/N.$$

If $k \geq 2$ is an integer, χ satisfies $\chi(-1) = (-1)^k$, and $q := e^{2\pi iz}$, then after the work of E. Hecke (see, for example, [9, Prop. 5.1.2]), we know that the function

$$E_{k,\chi}(z) := 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} q^n \quad (1.6)$$

is an Eisenstein series of weight k and character χ on the congruence subgroup $\Gamma_0(N)$. We note that (with the correct choice of α) the integrand in (1.5) is such an Eisenstein series.

In the next section we use Theorem 1.3 to deduce Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.3 and 1.4, and in the final section we give further examples having the same form as (1.2) and (1.4).

2. DEDUCTION OF THEOREMS 1.1 AND 1.2

In this section we deduce Theorems 1.1 and 1.2 from Theorem 1.3. We begin with a lemma which can be found in N. J. Fine's book [8, p. 22]. We use the customary notation

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1. \quad (2.1)$$

Lemma 2.1. *If $|q| < |t| < 1/|q|$, then*

$$\frac{(q; q)_{\infty}^6 (t^{-2}q; q)_{\infty} (t^2; q)_{\infty}}{(t^{-1}q; q)_{\infty}^4 (t; q)_{\infty}^4} = \frac{1+t}{(1-t)^3} + \frac{1}{t} \sum_{n,k=1}^{\infty} k^2 (t^k - t^{-k}) q^{kn}. \quad (2.2)$$

We now note that Theorems 1.1 and 1.2 follow directly from Theorem 1.3 together with the following lemmas.

Lemma 2.2. *If $|q| < 1$, then*

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-3}(d) d^2 q^n.$$

Lemma 2.3. *If $|q| < 1$, then*

$$\frac{f^4(-q)f^6(-q^2)}{f^4(-q^4)} = 1 - 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 q^n.$$

Lemma 2.2 (in a slightly different notation) was proved by L. Carlitz [5, p. 170, eq. (3.1)] using the theory of the Weierstrass \wp -function, and Son [15, Lemma 2.5] using Lemma 2.1.

Proof of Lemma 2.3. Set $t = i$ in Lemma 2.1. The left side of (2.2) then becomes

$$\frac{(q; q)_{\infty}^6 (-q; q)_{\infty} (-1; q)_{\infty}}{(-iq; q)_{\infty}^4 (iq; q)_{\infty}^4 (1-i)^4} = -\frac{(q; q)_{\infty}^6 (-q; q)_{\infty}^2}{2(-q^2; q^2)_{\infty}^4} = -\frac{(q; q)_{\infty}^4 (q^2; q^2)_{\infty}^6}{2(-q^4; q^4)_{\infty}^4}. \quad (2.3)$$

On the other hand, the right side of (2.2) is equal to

$$\begin{aligned} \frac{1+i}{(1-i)^3} - i \sum_{n,k=1}^{\infty} k^2 (e^{\pi ik/2} - e^{-\pi ik/2}) q^{kn} &= -\frac{1}{2} + 2 \sum_{k=1}^{\infty} k^2 \sin(\pi k/2) \frac{q^k}{1-q^k} \\ &= -\frac{1}{2} + 2 \sum_{k=0}^{\infty} (-1)^k (2k+1)^2 \frac{q^{2k+1}}{1-q^{2k+1}}. \end{aligned} \quad (2.4)$$

Combining (2.3) and (2.4) and recalling the notation (1.1) and (2.1), we see that

$$\frac{f^4(-q)f^6(-q^2)}{f^4(-q^4)} = 1 - 4 \sum_{k=0}^{\infty} (-1)^k (2k+1)^2 \frac{q^{2k+1}}{1-q^{2k+1}} = 1 - 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 q^n,$$

as desired. \square

We remark that the latter two lemmas could also be proved by using the fact that the functions on either side of each desired equality lie in the same finite dimensional vector space of modular forms. Therefore the equalities can be established by comparing an appropriate number of terms in each Fourier expansion.

3. PROOFS OF THEOREMS 1.3 AND 1.4

We begin the proof of Theorem 1.3 by taking the logarithm of both sides of (1.5) to find that (1.5) is equivalent to

$$\sum_{n=1}^{\infty} \chi(n) n^{k-2} \log(1-q^n) = -C + \int_q^1 \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} t^n \frac{dt}{t}. \quad (3.1)$$

Expanding $\log(1-q^n)$ in a Taylor series about the origin and rearranging, we find that (3.1) is in turn equivalent to

$$-\sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{q^n}{n} = -C + \lim_{Q \rightarrow 1^-} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{t^n}{n} \Bigg|_q^Q. \quad (3.2)$$

Finally, we see that (3.2) is equivalent to the statement

$$C = \lim_{Q \rightarrow 1^-} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{Q^n}{n}.$$

Therefore, Theorem 1.3 is a direct consequence of Theorem 1.4. The remainder of this section is devoted to a proof of the latter theorem.

We begin by showing that the general result follows from the case when χ is primitive. To this end, suppose that Theorem 1.4 holds for primitive characters. Let $\tilde{\chi}$ be an imprimitive character modulo \tilde{N} with conductor N , and let χ be the corresponding primitive character defined modulo N . Let S be the set of primes which divide \tilde{N} but not N . By [6, p. 37] we know that

$$L(s, \tilde{\chi}) = \prod_{p \in S} (1 - \chi(p)p^{-s}) \cdot L(s, \chi).$$

Since $L(2 - k, \chi) = 0$ [6, pp. 69, 71], we find upon differentiation that

$$L'(2 - k, \tilde{\chi}) = \prod_{p \in S} (1 - \chi(p)p^{k-2}) \cdot L'(2 - k, \chi). \quad (3.3)$$

On the other hand, define

$$H(q) := \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{q^n}{n} = \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi(d) d^{k-2} \frac{q^{nd}}{n}. \quad (3.4)$$

Then, using an inclusion-exclusion argument, we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{d|n} \tilde{\chi}(d) d^{k-1} \frac{q^n}{n} &= \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \tilde{\chi}(d) d^{k-2} \frac{q^{nd}}{n} \\ &= \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi(d) d^{k-2} \frac{q^{nd}}{n} - \sum_{p \in S} \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi(pd) (pd)^{k-2} \frac{q^{pnd}}{n} \\ &\quad + \sum_{p_1, p_2 \in S} \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi(p_1 p_2 d) (p_1 p_2 d)^{k-2} \frac{q^{p_1 p_2 nd}}{n} \\ &\quad - \sum_{p_1, p_2, p_3 \in S} \chi(p_1 p_2 p_3 d) (p_1 p_2 p_3 d)^{k-2} \frac{q^{p_1 p_2 p_3 nd}}{n} + \dots \end{aligned}$$

In other words,

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{d|n} \tilde{\chi}(d) d^{k-1} \frac{q^n}{n} &= H(q) - \sum_{p \in S} \chi(p) p^{k-2} H(q^p) + \sum_{p_1, p_2 \in S} \chi(p_1 p_2) (p_1 p_2)^{k-2} H(q^{p_1 p_2}) \\ &\quad - \sum_{p_1, p_2, p_3 \in S} \chi(p_1 p_2 p_3) (p_1 p_2 p_3)^{k-2} H(q^{p_1 p_2 p_3}) + \dots \end{aligned} \quad (3.5)$$

Since for any positive integer P , $\lim_{q \rightarrow 1^-} H(q^P) = \lim_{q \rightarrow 1^-} H(q)$, we find, upon taking limits in (3.5), that

$$\lim_{q \rightarrow 1^-} \sum_{n=1}^{\infty} \sum_{d|n} \tilde{\chi}(d) d^{k-1} \frac{q^n}{n} = \prod_{p \in S} (1 - \chi(p) p^{k-2}) \lim_{q \rightarrow 1^-} H(q).$$

Combining this with (3.3), we conclude that the general case indeed follows from the case when χ is primitive.

Suppose for the duration that χ is primitive. We must show that if $H(q)$ is defined as in (3.4), and if

$$C := C(k, \chi) = \lim_{q \rightarrow 1^-} H(q),$$

then

$$C = L'(2 - k, \chi).$$

For non-negative integers j , define

$$f_j(q) := \sum_{d=1}^{\infty} \chi(d) d^j q^d. \quad (3.6)$$

Let N denote the conductor of χ . Then it is clear that

$$f_0(q) = \frac{1}{1 - q^N} \sum_{a=1}^{N-1} \chi(a) q^a, \quad (3.7)$$

and that, for each nonnegative integer j , we have

$$f_j(q) = \left(q \frac{d}{dq} \right)^j f_0(q). \quad (3.8)$$

Moreover, from the definitions (3.4) and (3.6), we find that

$$H(q) = \sum_{n=1}^{\infty} \frac{1}{n} f_{k-2}(q^n).$$

Therefore, making the substitution $q = e^{-1/M}$, we find that

$$C = \lim_{q \rightarrow 1^-} H(q) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{n=1}^{\infty} \frac{1}{n/M} f_{k-2}(e^{-n/M}) = \int_0^{\infty} \frac{f_{k-2}(e^{-z})}{z} dz. \quad (3.9)$$

We now define

$$F_j(z) := f_j(e^z). \quad (3.10)$$

By (3.8), we see that $f_j(q) = q f'_{j-1}(q)$ for $j \geq 1$. In other words,

$$F_j(z) = f_j(e^z) = \frac{d}{dz} f_{j-1}(e^z) = F'_{j-1}(z). \quad (3.11)$$

Moreover, from (3.7), we see that

$$F_0(-z) = \frac{1}{1 - e^{-Nz}} \sum_{a=1}^{N-1} \chi(a) e^{-az} = \frac{-1}{1 - e^{Nz}} \sum_{a=1}^{N-1} \chi(a) e^{(N-a)z} = -\chi(-1) F_0(z).$$

Therefore, (3.11) gives, for $j \geq 0$,

$$F_j(-z) = (-1)^{j+1} \chi(-1) F_j(z).$$

Since $\chi(-1) = (-1)^k$, we see that $F_{k-2}(z)$ is odd. Thus (3.9) yields

$$-2C = 2 \int_{-\infty}^0 \frac{F_{k-2}(z)}{z} dz = \int_{-\infty}^{\infty} \frac{F_{k-2}(z)}{z} dz. \quad (3.12)$$

To evaluate the integral in (3.12), we start by shifting the line of integration on the right side of (3.12) by a small quantity $y_0 > 0$, where y_0 is assumed to be small enough so that $F_{k-2}(z)$ has no poles in the horizontal strip $0 \leq \text{Im } z \leq y_0$. Note that $F_{k-2}(z)/z$ vanishes at both ends of any given horizontal strip. Thus, by Cauchy's theorem,

$$-2C = \int_{-\infty}^{\infty} \frac{F_{k-2}(x + iy_0)}{x + iy_0} dx. \quad (3.13)$$

Next, we use (3.11) to integrate (3.13) by parts. We find that

$$\int_{-\infty}^{\infty} \frac{F_{k-2}(x + iy_0)}{x + iy_0} dx = \int_{-\infty}^{\infty} \frac{F_{k-3}(x + iy_0)}{(x + iy_0)^2} dx.$$

Integrating by parts $k - 2$ times, we find that

$$\int_{-\infty}^{\infty} \frac{F_{k-2}(x + iy_0)}{x + iy_0} dx = \int_{-\infty}^{\infty} \frac{F_{k-3}(x + iy_0)}{(x + iy_0)^2} dx = \cdots = (k - 2)! \int_{-\infty}^{\infty} \frac{F_0(x + iy_0)}{(x + iy_0)^{k-1}} dx,$$

which gives, by (3.12),

$$C = -\frac{(k - 2)!}{2} \int_{-\infty}^{\infty} \frac{F_0(x + iy_0)}{(x + iy_0)^{k-1}} dx. \quad (3.14)$$

We now apply contour integration. Let γ_{R_m} , $1 \leq m < \infty$, be a sequence of positively oriented rectangles with vertices $\pm\sqrt{R_m} + iy_0$ and $\pm\sqrt{R_m} + R_m^{3/2}i$, which are chosen so that the points $R_m^{3/2}i$ remain at a bounded distance from the points $2\pi in/N$ as m tends to ∞ . For brevity, let $L_1 = L_1(m)$ and $L_2 = L_2(m)$ denote, respectively, the left and right sides, and let $L_3 = L_3(m)$ denote the top side of γ_{R_m} . For $j = 1, 2$ we have

$$\left| \int_{L_j} \frac{F_0(z)}{z^{k-1}} dz \right| = O_N \left(R_m e^{-\sqrt{R_m}} \right) \quad (3.15)$$

as $R_m \rightarrow \infty$. Moreover,

$$\left| \int_{L_3} \frac{F_0(z)}{z^{k-1}} dz \right| = O_N \left(\frac{1}{R_m} \right) \quad (3.16)$$

as $R_m \rightarrow \infty$. The inequalities (3.15) and (3.16) imply that, if $\gamma'_{R_m} = L_1 \cup L_2 \cup L_3$, then

$$\int_{\gamma'_{R_m}} \frac{F_0(z)}{z^{k-1}} dz = o(1) \quad (3.17)$$

as $R_m \rightarrow \infty$. By the residue theorem we conclude that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F_0(x + iy_0)}{(x + iy_0)^{k-1}} dx = \sum_{n=1}^{\infty} \text{Res} \left(\frac{F_0(z)}{z^{k-1}}, \frac{2\pi in}{N} \right). \quad (3.18)$$

Hence, combining (3.14) with (3.18), we find that

$$C = -(k-2)! \pi i \sum_{n=1}^{\infty} \operatorname{Res} \left(\frac{F_0(z)}{z^{k-1}}, \frac{2\pi in}{N} \right). \quad (3.19)$$

If n is a fixed positive integer, then, for brevity, set $\alpha = 2\pi in/N$. Since α is a simple pole of $F_0(z)$, we see by (3.7) and (3.10) that

$$\begin{aligned} \operatorname{Res} \left(\frac{F_0(z)}{z^{k-1}}, \alpha \right) &= \lim_{z \rightarrow \alpha} \frac{(z-\alpha)F_0(z)}{z^{k-1}} \\ &= \frac{\sum_{a=1}^{N-1} \chi(a)e^{a\alpha}}{\alpha^{k-1}} \lim_{z \rightarrow \alpha} \left(\frac{z-\alpha}{1-e^{Nz}} \right) \\ &= -\frac{1}{N\alpha^{k-1}} \sum_{a=1}^{N-1} \chi(a)e^{a\alpha} = -\frac{N^{k-2}}{(2\pi in)^{k-1}} G_n(\chi), \end{aligned} \quad (3.20)$$

where $G_n(\chi)$ denotes the Gauss sum

$$G_n(\chi) = \sum_{a=1}^{N-1} \chi(a)e^{2\pi ian/N}.$$

Set $G(\chi) := G_1(\chi)$. By [6, p. 65, eq. (2)], for each integer n , G_n satisfies the factorization theorem

$$G_n(\chi) = \bar{\chi}(n)G(\chi).$$

Therefore, by (3.20),

$$\operatorname{Res} \left(\frac{F_0(z)}{z^{k-1}}, \alpha \right) = -\frac{N^{k-2}}{(2\pi in)^{k-1}} \bar{\chi}(n)G(\chi).$$

Summing over n , $1 \leq n < \infty$, recalling that $k \geq 2$, and utilizing (3.19), we find that

$$C = \frac{(k-2)!N^{k-2}G(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}} L(k-1, \bar{\chi}).$$

It is now seen that Theorem 1.4, and so also Theorem 1.3, is an immediate consequence of the following lemma.

Lemma 3.1. *Let χ be a primitive character of conductor N , and let $k \geq 2$ be an integer such that $\chi(-1) = (-1)^k$. Then*

$$\frac{(k-2)!N^{k-2}G(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}} L(k-1, \bar{\chi}) = L'(2-k, \chi).$$

Proof. Assume first that χ and k are odd. Then the functional equation (see [6, p. 71, eq. (11)]) is given by

$$\pi^{-(2-s)/2} N^{(2-s)/2} \Gamma \left(\frac{2-s}{2} \right) L(1-s, \chi) = \frac{i\sqrt{N}}{G(\bar{\chi})} \pi^{-(s+1)/2} N^{(s+1)/2} \Gamma \left(\frac{s+1}{2} \right) L(s, \bar{\chi}). \quad (3.21)$$

Allowing s to approach $k - 1$ in (3.21), we obtain

$$\pi^{(k-3)/2} N^{(3-k)/2} \lim_{s \rightarrow k-1} \left(\Gamma \left(\frac{2-s}{2} \right) L(1-s, \chi) \right) = \frac{i\sqrt{N}}{G(\bar{\chi})} \pi^{-k/2} N^{k/2} \Gamma \left(\frac{k}{2} \right) L(k-1, \bar{\chi}). \quad (3.22)$$

At any negative integer $-m$ the function $\Gamma(s)$ has a simple pole with residue

$$\text{Res}(\Gamma(s), -m) = \frac{(-1)^m}{m!}.$$

On combining this with the fact that

$$\lim_{s \rightarrow k-1} \left(\Gamma \left(\frac{2-s}{2} \right) L(1-s, \chi) \right) = 2 \text{Res} \left(\Gamma(s), \frac{3-k}{2} \right) L'(2-k, \chi),$$

we find that

$$\lim_{s \rightarrow k-1} \left(\Gamma \left(\frac{2-s}{2} \right) L(1-s, \chi) \right) = \frac{2(-1)^{(k-3)/2}}{((k-3)/2)!} L'(2-k, \chi). \quad (3.23)$$

From (3.22) and (3.23) it follows that

$$L'(2-k, \chi) = \frac{(-1)^{(k-3)/2} N^{k-3/2}}{2\pi^{k-3/2}} \cdot \frac{i\sqrt{N}}{G(\bar{\chi})} \cdot \Gamma \left(\frac{k}{2} \right) \Gamma \left(\frac{k-1}{2} \right) L(k-1, \bar{\chi}). \quad (3.24)$$

Since $\Gamma \left(\frac{k}{2} \right) \Gamma \left(\frac{k-1}{2} \right) = \sqrt{\pi} \cdot 2^{-k} (k-2)!$ and $G(\chi)G(\bar{\chi}) = -N$, we find from (3.24) that

$$L'(2-k, \chi) = \frac{(-1)^{(k-3)/2} (k-2)! N^{k-1} i}{2^{k-1} \pi^{k-2} G(\bar{\chi})} L(k-1, \bar{\chi}) = \frac{(k-2)! N^{k-2} G(\chi)}{2^{k-1} \pi^{k-2} i^{k-2}} L(k-1, \bar{\chi}).$$

This completes the proof when χ and k are odd. In the case when k and χ are even, we employ the functional equation of $L(s, \chi)$ for even characters [6, p. 71, eq. (11)]. Since the details are similar, we omit them. The proof of Lemma 3.1, and with it the proofs of Theorems 1.3 and 1.4, is therefore complete. \square

4. FURTHER EXAMPLES

In this section we exhibit further examples of the form given by Theorems 1.1 and 1.2, in other words, examples for which the quantity

$$\alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} q^n$$

can be expressed in terms of the eta-function.

We remark first that, given an example like Theorem 1.1, it is possible to construct further examples corresponding to imprimitive extensions of the relevant character. For example, let $p \neq 3$ be prime, and let $\tilde{\chi}_{-3}$ denote the imprimitive extension of χ_{-3} to modulus $3p$. Using Lemma 2.2, i.e.,

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-3}(d) d^2 q^n = 1 - 9 \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi_{-3}(d) d^2 q^{nd},$$

we find that

$$\begin{aligned}
& \frac{f^9(-q)}{f^3(-q^3)} - \chi_{-3}(p)p^2 \frac{f^9(-q^p)}{f^3(-q^{3p})} \\
&= 1 - \chi_{-3}(p)p^2 - 9 \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi_{-3}(d)d^2q^{nd} + 9 \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \chi_{-3}(pd)(pd)^2q^{pnd} \\
&= 1 - \chi_{-3}(p)p^2 - 9 \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \tilde{\chi}_{-3}(d)d^2q^{nd} = 1 - \chi_{-3}(p)p^2 - 9 \sum_{n=1}^{\infty} \sum_{d|n} \tilde{\chi}_{-3}(d)d^2q^n.
\end{aligned} \tag{4.1}$$

Therefore, by Theorem 1.3, with $\alpha := (1 - \chi_{-3}(p)p^2)/9$, we obtain

$$q^\alpha \prod_{n=1}^{\infty} (1 - q^n)^{\tilde{\chi}_{-3}(n)n} = \exp \left(-C - \frac{1}{9} \int_q^1 \left\{ \frac{f^9(-q)}{f^3(-q^3)} - \chi_{-3}(p)p^2 \frac{f^9(-q^p)}{f^3(-q^{3p})} \right\} \frac{dt}{t} \right),$$

where

$$C = L'(-1, \tilde{\chi}_{-3}).$$

Using similar arguments, it is possible to produce examples for any composite modulus $3N$.

After Theorem 1.4, we see that in order to produce more examples in the same vein as Ramanujan's, we need only to express the Eisenstein series $E_{k,\chi}(z)$ (see (1.6)) in terms of the eta-function. By way of illustration, we list a few such examples here. Our list is by no means exhaustive; further examples could certainly be obtained using the same ideas.

For another example of weight 3, we use the identity

$$E_{3,\chi_{-7}}(z) = 1 - \frac{7}{8} \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-7}(d)d^2q^n = \frac{\eta^7(z)}{\eta(7z)} + \frac{49}{8}\eta^3(z)\eta^3(7z). \tag{4.2}$$

The formula (4.2) was first stated by Ramanujan in his unpublished manuscript on the partition and tau functions; see [13, eq. (8.2), p. 145] for its appearance in his original handwritten manuscript, or see [3, p. 55] for a typed, edited version. The first proof of (4.2) known to us is by N. J. Fine [8], while O. Kolberg [10] has given a proof using modular forms. Together with Theorem 1.3, (4.2) gives, in analogy with Theorem 1.1, the formula

$$q^{8/7} \prod_{n=1}^{\infty} (1 - q^n)^{\chi_{-7}(n)n} = \exp \left(-C_7 - \frac{8}{7} \int_q^1 \left\{ \frac{f^7(-t)}{f(-t^7)} + \frac{49}{8}t f^3(-t)f^3(-t^7) \right\} \frac{dt}{t} \right),$$

where

$$C_7 = L'(-1, \chi_{-7}).$$

We discuss three examples in the case $k = 2$. The first such example can be found in Ramanujan's lost notebook [13], and was first proved by G. E. Andrews [1, pp. 188, 206–207]. To state it in the form recorded by Ramanujan, we define the Rogers–Ramanujan continued fraction $R(q)$ by

$$R(q) := \frac{1}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots = \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}, \quad |q| < 1.$$

(The latter equality was first proved by L. J. Rogers [14] in 1894.) Andrews, and undoubtedly Ramanujan as well, used Ramanujan's famous identity

$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5 \sum_{n=1}^{\infty} \frac{\chi_5(n)nq^n}{1-q^n}, \quad (4.3)$$

found in Ramanujan's notebooks [12], to prove that

$$q^{1/5}R(q) = \frac{\sqrt{5}-1}{2} \exp\left(-\frac{1}{5} \int_q^1 \frac{f^5(-t)}{f(-t^5)} \frac{dt}{t}\right). \quad (4.4)$$

(For references to several proofs of (4.3), see [2, pp. 261–262].) On the other hand, Theorem 1.3 gives (4.4) in the form

$$q^{1/5} \prod_{n=1}^{\infty} (1-q^n)^{\chi_5(n)} = \exp\left(-C_5 - \frac{1}{5} \int_q^1 \frac{f^5(-t)}{f(-t^5)} \frac{dt}{t}\right), \quad (4.5)$$

where

$$C_5 = L'(0, \chi_5).$$

A comparison of (4.4) and (4.5) shows that

$$L'(0, \chi_5) = \log\left(\frac{\sqrt{5}+1}{2}\right). \quad (4.6)$$

We note that (4.6) agrees with a general formula for $L'(0, \chi)$ due to C. Deninger [7, pp. 181–182, eqs. (3.2), (3.5)].

For another example, we can use the identity

$$E_{2, \chi_8}(z) = 1 - 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_8(d) dq^n = \frac{\eta^3(4z)\eta(2z)\eta^2(z)}{\eta^2(8z)},$$

first proved by Carlitz [5, eq. (6.2), p. 172], in order to obtain

$$q^{1/2} \prod_{n=1}^{\infty} (1-q^n)^{\chi_8(n)} = \exp\left(-C_8 - \frac{1}{2} \int_q^1 \frac{f^3(-t^4)f(-t^2)f^2(-t)}{f^2(-t^8)} \frac{dt}{t}\right),$$

where

$$C_8 = L'(0, \chi_8).$$

For a third example when $k=2$, we use the identity

$$E_{2, \chi_{12}}(z) = 1 - \sum_{n=1}^{\infty} \sum_{d|n} \chi_{12}(d) dq^n = \frac{\eta^2(6z)\eta^2(4z)\eta(3z)\eta(z)}{\eta^2(12z)},$$

which can be verified using the fact that each side lies in the space of modular forms of weight 2 and character χ_{12} on $\Gamma_0(12)$. From this and Theorem 1.3 we obtain

$$q \prod_{n=1}^{\infty} (1-q^n)^{\chi_{12}(n)} = \exp\left(-C_{12} - \int_q^1 \frac{f^2(-t^6)f^2(-t^4)f(-t^3)f(-t)}{f^2(-t^{12})} \frac{dt}{t}\right),$$

where

$$C_{12} = L'(0, \chi_{12}).$$

For a more complicated example, we use the fact that the Eisenstein series

$$E_{6,\chi_{13}}(z) = 1 - \frac{13}{33463} \sum_{n=1}^{\infty} \sum_{d|n} \chi_{13}(d) d^5 q^n$$

lies in the space of modular forms of weight 6 and character χ_{13} on $\Gamma_0(13)$. Since this space has a basis consisting entirely of eta-quotients, we obtain (after a short computation) the following example:

$$\begin{aligned} q^{33463/13} \prod_{n=1}^{\infty} (1 - q^n)^{\chi_{13}(n)n^4} = \exp \left(-C_{13} - \frac{33463}{13} \int_q^1 \left\{ \frac{f^{13}(-t)}{f(-t^{13})} + \frac{435006}{33463} t f^{11}(-t) f(-t^{13}) \right. \right. \\ \left. \left. + \frac{2610374}{33463} t^2 f^9(-t) f^3(-t^{13}) + \frac{8700120}{33463} t^3 f^7(-t) f^5(-t^{13}) + \frac{16508258}{33463} t^4 f^5(-t) f^7(-t^{13}) \right. \right. \\ \left. \left. + \frac{15594306}{33463} t^5 f^3(-t) f^9(-t^{13}) + \frac{4826809}{33463} t^6 f(-t) f^{11}(-t^{13}) \right\} \frac{dt}{t} \right), \end{aligned}$$

where

$$C_{13} = L'(-4, \chi_{13}).$$

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