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A reciprocity theorem for certain q-series found in Ramanujan's lost notebook

Bruce C. Berndt · Song Heng Chan · Boon Pin Yeap · Ae Ja Yee

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Abstract Three proofs are given for a reciprocity theorem for a certain q-series found in Ramanujan's lost notebook. The first proof uses Ramanujan's ${}_1\psi_1$ summation theorem, the second employs an identity of N. J. Fine, and the third is combinatorial. Next, we show that the reciprocity theorem leads to a two variable generalization of the quintuple product identity. The paper concludes with an application to sums of three squares.

Keywords q-series · Reciprocity theorem · Ramanujan's ${}_1\psi_1$ summation theorem · Quintuple product identity · Sums of three squares · Combinatorial bijection

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B. C. Berndt (✉) · S. H. Chan · B. P. Yeap · A. J. Yee
Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, United States
e-mail: berndt@math.uiuc.edu

S. H. Chan
Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Republic of Singapore
e-mail: matcshs@nus.edu.sg

B. P. Yeap
Department of Mathematics, University of California at San Diego, LaJolla, CA 92093, USA
e-mail: byeap@ucsd.edu

A. J. Yee
Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
e-mail: yee@math.psu.edu

1 Introduction

In his lost notebook [15, p. 40]. Ramanujan offers a beautiful reciprocity theorem for

$$\rho(a,b) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}, \tag{1.1}$$

where a and b are any complex numbers, except that $a \neq -q^{-n}$ for any positive integer n.

Theorem 1.1. *If $a,\ b \neq -q^{-n}$, then*

$$\rho(a,b)-\rho(b,a) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty(-bq)_\infty}. \tag{1.2}$$

In (1.1) and (1.2) and in the sequel, we use the customary notation

$$(a)_0 := (a;q)_0 := 1, \quad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1,$$

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a)_n := (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad -\infty < n < \infty.$$

The first proof of Theorem 1.1 was given by Andrews [2], who used considerably heavy machinery. He then employed Theorem 1.1 in a later paper [3] to prove two beautiful entries from Ramanujan's lost notebook related to Euler's famous theorem asserting that partitions of a positive integer n into odd parts are equinumerous with partitions of n into distinct parts. The purpose of this short note is to provide three new proofs of Theorem 1.1 and to show that it leads to a generalization, involving one additional variable, of the quintuple product identity. In the final section of our paper we give an application of Theorem 1.1 to sums of three squares.

In closing our introduction, we remark that there is a slightly simpler representation of $\rho(a,b)$. Write

$$\rho(a,b) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n-1}}{(-aq)_n}. \tag{1.3}$$

Now replace n by $n+1$ in the first sum on the right side of (1.3) and then recombine the sums. After elementary simplification, we find that

$$\rho(a,b) = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n-1}}{(-aq)_{n+1}}. \tag{1.4}$$

2 Proofs

In this section, we give three proofs of Theorem 1.1. The first proof rests upon Ramanujan's ${}_1\psi_1$ summation theorem and the second iterate of Heine's transformation. The second depends upon a transformation formula of N. J. Fine and the Jacobi triple product identity. Lastly, our third proof is partition theoretic and purely combinatorial.

First Proof of Theorem 1.1: Recall Ramanujan's ${}_1\psi_1$ summation formula [5, p. 34, Eq. (17.6)]. If $|b/a| < |z| < 1$, then

$$ {}_1\psi_1(a;b;z) := \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(b/a)_\infty (az)_\infty (q/(az))_\infty (q)_\infty}{(q/a)_\infty (b/(az))_\infty (b)_\infty (z)_\infty}. \tag{2.1} $$

Letting $b = 0$, replacing a by $-1/a$, setting $z = -b$, and lastly multiplying both sides by $(1 + 1/b)$, we find that

$$ \begin{aligned} &\left(1 + \frac{1}{b}\right) \sum_{n=-\infty}^{\infty} (-1/a)_n (-b)^n \\ &= \left(1 + \frac{1}{b}\right) \sum_{n=1}^{\infty} (-1/a)_n (-b)^n + \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} (-1/a)_{-n} (-b)^{-n} \\ &= \left(1 + \frac{1}{b}\right) \frac{(b/a)_\infty (aq/b)_\infty (q)_\infty}{(-b)_\infty (-aq)_\infty} \\ &= \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(bq/a)_\infty (aq/b)_\infty (q)_\infty}{(-bq)_\infty (-aq)_\infty}. \end{aligned} \tag{2.2} $$

We now examine the two sums on the right side of the first equality in (2.2). For the first sum, we use Rogers's transformation, or the second iterate of Heine's transformation [14], [5, p. 15, fourth line from the bottom of the page]. For $|z|, |c/b| < 1$,

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n = \frac{(c/b)_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n (b)_n}{(bz)_n (q)_n} \left(\frac{c}{b}\right)^n. \tag{2.3} $$

Now let $b = q$ and let $c \to 0$ to deduce from (2.3) that

$$ \sum_{n=0}^{\infty} (a)_n z^n = \frac{1}{1 - z} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} a^n z^n}{(zq)_n}. \tag{2.4} $$

Next, in (2.4), let $z = -b$ and replace a by $-q/a$ to deduce that

$$\left(1+\frac{1}{b}\right)\sum_{n=1}^{\infty}(-1/a)_n(-b)^n = \left(1+\frac{1}{b}\right)\left(1+\frac{1}{a}\right)(-b)\sum_{n=0}^{\infty}(-q/a)_n(-b)^n$$
$$= -\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^{-n}b^n}{(-bq)_n}$$
$$= -\rho(b,a). \tag{2.5}$$

The second sum on the right side of (2.2) is easier to examine. Observe that

$$(-1/a)_{-n} = \frac{1}{(-q^{-n}/a)_n} = \frac{a^n q^{n(n+1)/2}}{(-aq)_n},$$

after elementary algebra, so that

$$\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}(-1/a)_{-n}(-b)^{-n} = \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$
$$= \rho(a,b). \tag{2.6}$$

Using (2.5) and (2.6) in (2.2), we complete our first proof. □

Second Proof of Theorem 1.1: We begin with a transformation from Fine's text [10, p. 7, Eqs. (8.2) and the equality $F + G = HS$ above]. Replacing u by b and b by $-a$ in this formula and correcting a misprint, we find that

$$(1+a)\sum_{n=0}^{\infty}(-q/b)_n(-a)^n - \frac{b}{1+b}\sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n(-bq)_n}$$
$$= \frac{1}{(-aq)_\infty(-b)_\infty}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n q^{n(n+1)/2}. \tag{2.7}$$

(In the formula for H in (8.2) of [10, p. 7], replace $(u)_\infty$ by $(-u)_\infty$. There are three similar misprints in (8.1).) Return to (2.4) and replace a by $-q/b$ and set $z = -a$ to deduce that

$$\sum_{n=0}^{\infty}(-q/b)_n(-a)^n = \frac{1}{1+a}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n} = \frac{b}{(1+a)(1+b)}\rho(a,b). \tag{2.8}$$

Using (2.8) in (2.7) and multiplying both sides by $(1+b)/b$, we find that

$$\rho(a,b) - \sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n(-bq)_n} = \frac{1}{b(-aq)_\infty(-bq)_\infty}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n q^{n(n+1)/2}. \tag{2.9}$$

Now rewrite (2.9) with a and b interchanged to find that

$$\rho(b,a)-\sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n(-bq)_n}=\frac{1}{a(-aq)_\infty(-bq)_\infty}\sum_{n=0}^{\infty}\left(-\frac{b}{a}\right)^n q^{n(n+1)/2}. \tag{2.10}$$

Subtracting (2.10) from (2.9), we deduce that

$$\begin{aligned}\rho(a,b)-\rho(b,a)\\ &=\frac{1}{(-aq)_\infty(-bq)_\infty}\left\{\frac{1}{b}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n q^{n(n+1)/2}-\frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{b}{a}\right)^n q^{n(n+1)/2}\right\}\\ &=\frac{1}{(-aq)_\infty(-bq)_\infty}\left\{\frac{1}{b}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n q^{n(n+1)/2}-\frac{1}{a}\sum_{n=-\infty}^{-1}\left(-\frac{a}{b}\right)^{n+1} q^{n(n+1)/2}\right\}.\end{aligned} \tag{2.11}$$

Recall that Ramanujan's definition of his theta function $f(\alpha,\beta)$ and the Jacobi triple product identity [5, pp. 34–35] are given by

$$f(\alpha,\beta):=\sum_{n=-\infty}^{\infty}\alpha^{n(n+1)/2}\beta^{n(n-1)/2}=(-\alpha;\alpha\beta)_\infty(-\beta;\alpha\beta)_\infty(\alpha\beta;\alpha\beta)_\infty. \tag{2.12}$$

Hence, by (2.12), (2.11) can be rewritten as

$$\begin{aligned}\rho(a,b)-\rho(b,a)&=\frac{1}{(-aq)_\infty(-bq)_\infty}\frac{1}{b}f(-aq/b,-b/a)\\ &=\frac{1}{(-aq)_\infty(-bq)_\infty}\frac{1}{b}(aq/b)_\infty(b/a)_\infty(q)_\infty\\ &=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty},\end{aligned}$$

which completes the second proof. □

Third Proof of Entry 1.1: Replace b by $-b$ in (1.2). After some simplification, we find that

$$\begin{aligned}\frac{(-aq/b)_\infty(-b/a)_\infty}{(b)_\infty}\\ &=\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(a/b)^n}{(-aq)_n}+\frac{(-a)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(b/a)^{n+1}}{(b)_{n+1}} && (2.13)\\ &=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-aq^{n+1})_\infty(a/b)^n+\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1/a)_n b^n, && (2.14)\end{aligned}$$

where the last equality is obtained as follows. We put the first factor $(1+a)$ of $(-a)_\infty$ in the second expression of (2.13) into the summation, and replace n by $(n-1)$. Then we obtain

$$\sum_{n=1}^{\infty}(1+a)\frac{q^{n(n-1)/2}(b/a)^n}{(b)_n}=\sum_{n=1}^{\infty}(1+1/a)\frac{q^{n(n-1)/2}(1/a)^{n-1}}{(b)_n}b^n,$$

each term of which generates partitions into n nonnegative distinct parts. Thus we can rewrite it as the second summation on the right side of (2.14).

A generalized Frobenius partition, or F-partition, of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

where a_i and b_i are weakly decreasing sequences of nonnegative integers and $s+\sum_{i=1}^{s} a_i+\sum_{i=1}^{r} b_i=n$. The left hand side of (2.13) can be interpreted as a generating function for F-partitions, where the top rows are partitions into distinct nonnegative parts and the bottom rows are overpartitions, which are partitions where the first occurrence of a nonnegative number may be overlined. Let $s=r+k$. We consider two cases: when $k\geq 0$ and when $k<0$.

Case I. $k\geq 0$. This case explains the first expression on the right side of (2.14).

1. Rearrange the parts in the bottom row such that overlined parts follow unrestricted parts, overlined parts are in decreasing order, and unrestricted parts are in weakly increasing order. We denote the bottom row so obtained by $(\beta_1\,\beta_2\dots\beta_r)$.
2. Divide the top row $(a_1\,a_2\dots a_{r+k})$ into two ordinary partitions λ and μ, where

$$\lambda_i=a_i-(r+k-i),$$
$$\mu_i=(r+k-i)+1.$$

3. Produce a partition $\nu^{(1)}$ into distinct parts and a partition $\nu^{(2)}$ as follows: for $1\leq i\leq r$,

   – put $\beta_i+\mu_{r-i+1}$ as part of $\nu^{(1)}$ if β_{r-i+1} is unrestricted,
   – put $\beta_i+\mu_{r-i+1}$ as a part of $\nu^{(2)}$ if β_{r-i+1} is overlined.

   Note that the parts of $\nu^{(1)}$ are greater than k, since μ_{r-i+1} equals $(k+i)$. Thus $\nu^{(1)}$ generated by $(-aq^{k+1})_\infty$ in the first summation on the right side of (2.14). Moreover, the parts $\nu^{(2)}$ are greater than or equal to $(r+k)$, since β_i is greater than or equal to $(r-i)$ if β_i is overlined, and μ_{r-i+1} equals $(k+i)$. Rearrange the parts of each partition in weakly decreasing order.
4. The remaining parts $\mu_{r+1},\dots,\mu_{r+k}$ of μ form the partition ρ into parts from $1,2,\dots,k$, which is generated by $(a/b)^k q^{k(k+1)/2}$ in the first summation of the right side of (2.14).

5. Add the parts of the conjugate of λ to $\nu^{(2)}$ as parts. Note that the conjugate of λ has parts less than or equal to $(r+k)$, since λ has at most $(r+k)$ positive parts. We see that $\nu^{(2)}$ is generated by $1/(q)_\infty$ in the first expression of the right side of (2.14).

Case II. $k<0$. This case explains the second expression on the right side of (2.14).

1. Rearrange the parts in the bottom row such that the resulting array $(\beta_1\,\beta_2\ldots\beta_r)$ satisfies the condition that β_i' are weakly decreasing, where

$$\beta_i' = \begin{cases} \beta_i - (r-i), & \text{if } \beta_i \text{ is overlined,} \\ \beta_i, & \text{if } \beta_i \text{ is unrestricted.} \end{cases}$$

2. Divide the bottom row $(\beta_1\,\beta_2\cdots\beta_r)$ into two ordinary partitions λ and μ, where

$$\lambda_i = \begin{cases} \beta_i - (r-i), & \text{if } \beta_i \text{ is overlined,} \\ \beta_i, & \text{if } \beta_i \text{ is unrestricted.} \end{cases}$$

$$\mu_i = \begin{cases} (r-i), & \text{if } \beta_i \text{ is overlined,} \\ 0, & \text{if } \beta_i \text{ is unrestricted.} \end{cases}$$

3. Produce a partition $\nu^{(1)}$ into distinct parts and a partition $\nu^{(2)}$ as follows: for $1\le i\le(r+k)$,

   - put $a_i+1+\mu_{r+k-i+1}$ as a part of $\nu^{(1)}$, if $\beta_{r+k-i+1}$ is unrestricted,
   - put $a_i+1+\mu_{r+k-i+1}$ as a part of $\nu^{(2)}$, if $\beta_{r+k-i+1}$ is overlined.

   Note that the parts of $\nu^{(1)}$ are distinct since the parts a_i are distinct. Thus $\nu^{(1)}$ is generated by $(-aq)_\infty$ in the second summation of (2.14). Moreover, the parts of $\nu^{(2)}$ are greater than or equal to r, since a_i is greater than or equal to $(r+k-i)$ and $\mu_{r+k-i+1}$ equals $(-k+i-1)$ if β_i is overlined.
4. The remaining parts $\mu_{r+k+1,\ldots,}\mu_r$ of μ form an array ρ, which is generated by $(-1/a)_k b^k$ in the second summation of the right side of (2.14).
5. Add the parts of the conjugate of λ to $\nu^{(2)}$ as parts. Note that conjugate of λ has parts less than or equal to r, since λ has at most r positive parts. We see that $\nu^{(2)}$ is generated by $1/(q)_\infty$ in the second summation of the right side of (2.14).

The arguments in our third proof can be extended to give a completely combinatorial proof of Ramanujan's ${}_1\psi_1$ summation theorem [17].

3 Theorem 1.1 as a two variable generalization of the quintuple product identity

We show in this section that by utilizing a specialized version of the Rogers-Fine identity (3.1), we may express (1.2) as a two variable generalization of the quintuple product identity. We remark that the method we employ is similar to that used in [7].

Theorem 3.1 (A Two Variable Generalization of the Quintuple Product Identity). *For* $a, b \neq q^{-n}$, $1 \leq n < \infty$,

$$\left(\frac{1}{a}-\frac{1}{b}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(aq)_\infty(bq)_\infty}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n(1/a)_n a^{-n-1}b^{2n}q^{3n(n+1)/2}(1-bq^{2n+1}/a)}{(bq)_{n+1}}$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n(1/b)_n a^{2n}b^{-n-1}q^{3n(n+1)/2}(1-aq^{2n+1}/b)}{(aq)_{n+1}}.$$

Proof: First recall the Rogers-Fine identity [10, p. 15, Eq. (14.1)]

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n\tau^n}{(\beta)_n}=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha\tau q/\beta)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta)_n(\tau)_{n+1}}. \tag{3.1}$$

Setting $\beta=0$ in (3.1), we find that

$$\sum_{n=0}^{\infty}(\alpha)_n\tau^n=\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n\alpha^n\tau^{2n}q^{n(3n-1)/2}(1-\alpha\tau q^{2n})}{(\tau)_{n+1}}. \tag{3.2}$$

Applying (2.4) and (3.2) to (1.4), we find that

$$\rho(a,b)=1+\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_{n+1}}$$
$$=1+\frac{1}{b}\sum_{n=0}^{\infty}(-1/b)_n(-aq)^n$$
$$=1+\sum_{n=0}^{\infty}\frac{(-1/b)_n a^{2n}b^{-n-1}q^{3n(n+1)/2}(1-aq^{2n+1}/b)}{(-aq)_{n+1}}. \tag{3.3}$$

Rewriting Theorem 1.1 with the representation of $\rho(a,b)$ given in (3.3) and replacing both a and b with $-a$ and $-b$, respectively, we complete the proof of Theorem 3.1. □

Corollary 3.2 (Quintuple Product Identity). ([5, p. 80, Eq. (38.2)]) *For any complex number a,*

$$\frac{(a^2)_\infty(q/a^2)_\infty(q)_\infty}{(a)_\infty(q/a)_\infty}=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}(a^{3n+1}+a^{-3n}). \tag{3.4}$$

Proof: Set $b = 1/(aq)$ in Theorem 3.1 and multiply both sides by a^2. After simplifying, we find that

$$\begin{aligned}
\frac{(a^2)_\infty(q/a^2)_\infty(q)_\infty}{(a)_\infty(q/a)_\infty} &= \sum_{n=0}^{\infty}\frac{(-1)^n(1/a)_n a^{-3n+1}q^{n(3n-1)/2}(1-q^{2n}/a^2)}{(1/a)_{n+1}}\\
&\quad -\sum_{n=0}^{\infty}\frac{(-1)^n(aq)_n a^{3n+3}q^{(n+1)(3n+2)/2}(1-a^2q^{2n+2})}{(aq)_{n+1}}\\
&= \sum_{n=0}^{\infty}(-1)^n a^{-3n+1}q^{n(3n-1)/2}\left(1+\frac{q^n}{a}\right)\\
&\quad +\sum_{n=1}^{\infty}(-1)^n a^{3n}q^{n(3n-1)/2}(1+aq^n)\\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}(a^{3n+1}+a^{-3n}),
\end{aligned}$$

and this completes the proof of (3.4). □

4 A new representation for the generating function for sums of three squares

Letting $b \to 1$ in Theorem 1.1, we find that

$$\begin{aligned}
&2\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}a^k}{(-aq)_k} - \left(1+\frac{1}{a}\right)\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}a^{-k}}{(-q)_k}\\
&= \left(1-\frac{1}{a}\right)\frac{(aq)_\infty(q/a)_\infty(q)_\infty}{(-aq)_\infty(-q)_\infty}.
\end{aligned}$$

Dividing both sides by $a-1$ and letting $a \to 1$, we find that

$$\begin{aligned}
\frac{(q)_\infty^3}{(-q)_\infty^2} &= \frac{d}{da}\left(2\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}a^k}{(-aq)_k}\right)_{a=1}\\
&\quad -\frac{d}{da}\left(\left(1+\frac{1}{a}\right)\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2}a^{-k}}{(-q)_k}\right)_{a=1} =: S_1 + S_2. \qquad (4.1)
\end{aligned}$$

To evaluate these last two expressions, we need the q-analogue of Euler's transformation, or the third iterate of Heine's transformation [5, p. 15, third line from the bottom of the page], given by

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}z^n = \frac{(abz/c)_\infty}{(z)_\infty}\sum_{n=0}^{\infty}\frac{(c/a)_n(c/b)_n}{(c)_n(q)_n}\left(\frac{abz}{c}\right)^n. \qquad (4.2)$$

Letting $a=0, b=-1$, and $c=-q$, and then replacing z by $-zq$ in (4.2), we find that

$$2\sum_{n=0}^{\infty}\frac{(-q)^n z^n}{(1+q^n)(q)_n}=\frac{1}{(-zq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}z^n}{(-q)_n}. \tag{4.3}$$

Similarly, letting $a=0, b=-c$, and $z=-q$, and replacing c by $-cq$, in (4.2), we find that

$$\sum_{n=0}^{\infty}\frac{(1+c)(-q)^n}{(1+cq^n)(q)_n}=\frac{1}{(-q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}c^n}{(-cq)_n}. \tag{4.4}$$

Applying (4.4) to S_1 in (4.1), we find that

$$\begin{aligned} S_1 &= 2(-q)_\infty \frac{d}{da}\left((1+a)\sum_{k=0}^{\infty}\frac{(-q)^k}{(1+aq^k)(q)_k}\right)_{a=1}\\ &= 2(-q)_\infty\sum_{k=0}^{\infty}\frac{(-q)^k}{(1+q^k)(q)_k}\left(1-2\frac{q^k}{1+q^k}\right). \end{aligned}$$

Applying (4.3) to S_2 in (4.1), we find that

$$\begin{aligned} S_2 &= 2\frac{d}{da}\left(\left(\frac{-q}{a}\right)_\infty\left(1+\frac{1}{a}\right)\sum_{k=0}^{\infty}\frac{(-q)^k a^{-k}}{(1+q^k)(q)_k}\right)_{a=1}\\ &= 4(-q)_\infty\sum_{k=0}^{\infty}\frac{(-q)^k}{(1+q^k)(q)_k}\left(-k-\sum_{n=0}^{\infty}\frac{q^n}{1+q^n}\right). \end{aligned}$$

Using the last two calculations in (4.1), dividing both sides by $(-q)_\infty$, and using the Jacobi triple product identity (2.12), we find that

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty}(-q)^{k^2}\right)^3 &= \frac{(q)_\infty^3}{(-q)_\infty^3}\\ &= \sum_{k=0}^{\infty}\frac{(-q)^k}{(1+q^k)(q)_k}\left(2-4\frac{q^k}{1+q^k}+4k+4\sum_{n=0}^{\infty}\frac{q^n}{1+q^n}\right). \end{aligned} \tag{4.5}$$

We close this section by offering a few further formulas for the generating function for sums of three squares.

Andrews [4] proved that

$$\frac{(q)_\infty^3}{(-q)_\infty^3}=1+4\sum_{m=1}^{\infty}\frac{(-1)^m q^m}{1+q^m}-2\sum_{m\geq 1,|j|<m}\frac{q^{m^2-j^2}(1-q^m)(-1)^j}{1+q^m}.$$

Using their ${}_2\psi_2$ summation formula, Adiga and Bhargava [6, Eq. (3.7)] obtained the representation

$$\frac{(q)_\infty^3}{(-q)_\infty^3} = 1 + 2\sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(-q)_m(1+q^m)} + 4\sum_{m=1}^{\infty} \frac{(-q)_{m-1}(-q)^m}{1+q^m}.$$

In [8, Eq. (8)], using a generalized ${}_1\psi_1$ summation formula of Andrews [4, Thm. 6], Bhargava, Adiga, and D. D. Somashekara proved that

$$\frac{(-q;-q)_\infty^3}{(q;-q)_\infty^3} = 1 + 2\sum_{m=1}^{\infty} \frac{(-q;q^2)_m q^m}{(1+q^{2m})(-q^2;q^2)_m} + 4\sum_{m=1}^{\infty} \frac{(q^2;q^2)_{m-1} q^m}{(1+q^{2m})(q;q^2)_m}.$$

Since this paper was prepared in early 2003, several further proofs as well as generalizations of Theorem 1.1 have emerged. Somashekara and S. N. Fathima [16], Bhargava, Somashekara, and Fathima [9], T. Kim, Somashekara, and Fathima [12], and Adiga and N. Anitha [1] have each given proofs of Theorem 1.1. Distinct generalizations of Theorem 1.1 have been devised by S.–Y. Kang [11], Z.–G. Liu [13], and Z. Zhang [18].

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