Abel's Theorem (key Step)

Let $W(X^{(1)}, \ldots, X^{(n)})$ be the Wronskian of n solutions to $X' = P(t)X$. The key first step in proving Abel's theorem is to prove that

$$W'(t) = \sum_{j=1}^{n} p_{jj}(t) \cdot W(t). \qquad (1)$$

We prove (1) for the case $n=2$. We leave the case of general n to Math 441 students. Let $X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$ and $X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix}$

be solutions of $X' = P(t)X$. Thus,

$$\begin{bmatrix} x_1^{(1)\prime} \\ x_2^{(1)\prime} \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} \end{bmatrix} \qquad (2')$$

$$\begin{bmatrix} x_1^{(2)\prime} \\ x_2^{(2)\prime} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{bmatrix}. \qquad (2'')$$

Now

$$W(t) = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}.$$

Thus

$$W'(t) = \begin{vmatrix} x_1^{(1)\prime} & x_1^{(2)\prime} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)\prime} & x_2^{(2)\prime} \end{vmatrix}$$

$$= \begin{vmatrix} p_{11}x_1^{(1)} + p_{12}x_2^{(1)} & p_{11}x_1^{(2)} + p_{12}x_2^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$+ \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ p_{21}x_1^{(1)} + p_{22}x_2^{(1)} & p_{21}x_1^{(2)} + p_{22}x_2^{(2)} \end{vmatrix} \qquad (\text{by } (2'), (2''))$$

$$= (p_{11}x_1^{(1)} + \cancel{p_{12}x_2^{(1)}})x_2^{(2)} - (p_{11}x_1^{(2)} + \cancel{p_{12}x_2^{(2)}})x_2^{(1)}$$

$$+ (\cancel{p_{21}x_1^{(2)}} + p_{22}x_2^{(2)})x_1^{(1)} - (\cancel{p_{21}x_1^{(1)}} + p_{22}x_2^{(1)})x_1^{(2)}$$

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$$= p_{11} x_1^{(1)} x_2^{(2)} - p_{11} x_2^{(1)} x_1^{(2)} + p_{22} x_1^{(1)} x_2^{(2)} - p_{22} x_1^{(2)} x_2^{(1)}$$

$$= p_{11} \left(x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)} \right) + p_{22} \left(x_1^{(1)} x_2^{(2)} - x_1^{(2)} x_2^{(1)} \right)$$

$$= p_{11} W(x^{(1)}, x^{(2)}) + p_{22} W(x^{(1)}, x^{(2)})$$

$$= (p_{11} + p_{22}) W(t).$$

The case of general n can be proved similarly. For example, when calculating the derivative of the Wronskian, there will be 18 terms for $n=3$, 32 terms for $n=4$, and in general $2 \cdot n^2$ terms for degree n.