# Contents

1 Third Order Mock Theta Functions: Elementary Identities  
   1.1 Introduction .................................................... 5  
   1.2 Basic Theorems .................................................. 5  
   1.3 The Third Order Identities ..................................... 7  

2 Fifth Order Mock Theta Functions. Elementary Identities . 17  
   2.1 Introduction .................................................... 17  
   2.2 Basic Theorems .................................................. 19  
   2.3 Watson's Fifth Order Identities ............................... 24  
   2.4 Ramanujan's Fifth Order Identities ........................... 26  
   2.5 Related Identities and Partitions ............................. 29  

3 Third Order Mock Theta Functions: Partial Fraction Expansions ........................................... 35  
   3.1 Introduction .................................................... 35  
   3.2 The Partial Fraction Identities ............................... 36  
   3.3 Specializations .................................................. 40  

4 The Mock Theta Conjectures: Equivalence .................... 43  
   4.1 Introduction .................................................... 43  
   4.2 Fourteen Lemmas .................................................. 45  
   4.3 The Relations Among $M_i(q)$, $1 \leq i \leq 5$ .......... 50  
   4.4 Relations to Partitions ......................................... 57  

5 Fifth Order Mock Theta Functions: Proof of the Mock Theta Conjectures .................................. 61  
   5.1 Introduction .................................................... 61  
   5.2 Hecke-type Series for $f_0(q)$ and $f_1(q)$ ................. 62  
   5.3 Theta Function Identities ..................................... 69  
   5.4 Partial Fractions and Appell–Lerch Series ................. 73  
   5.5 Proof of the Mock Theta Conjectures ........................ 78
6 Sixth Order Mock Theta Functions ........................................... 93
  6.1 Introduction ............................................................. 93
  6.2 Theta Function Identities ............................................ 94
  6.3 Hecke-type Series for the Sixth Order Mock Theta Functions ... 99
  6.4 Entries for $\phi_6(q)$ and $\psi_6(q)$ .............................. 109
  6.5 Entries for the Remaining Functions .............................. 120
  6.6 Two Further Identities .............................................. 127
  6.7 Further Work ......................................................... 130

7 Transformation Formulas for Third Order Mock Theta Functions ........................................ 131
  7.1 Introduction ............................................................. 131
  7.2 A Transformation Formula Involving $f_3(q)$ and $\omega_3(q)$ ... 131
  7.3 A Transformation Formula Involving $\phi_3(q)$ and $\psi_3(q)$ ... 141
  7.4 A Brief Survey of Further Transformation Formulas for Mock Theta Functions ....................... 147

8 Transformation Formulas: 10th Order Mock Theta Functions ................................................. 153
  8.1 Introduction ............................................................. 153
  8.2 Some Theta Function Identities ................................. 155
  8.3 Proof of Entry 8.1.1 .................................................. 170
  8.4 Proof of Entry 8.1.2 .................................................. 181

9 Two Identities Involving a Mordell Integral and Appell–Lerch Sums ........................................ 183
  9.1 Introduction ............................................................. 183
  9.2 Two Lemmas ............................................................ 187
  9.3 Proof of Theorem 9.1.1 .............................................. 192
  9.4 Proof of Theorem 9.1.2 .............................................. 199

10 Ramanujan’s Last Letter to Hardy ........................................ 201
  10.1 Introduction ........................................................... 201
  10.2 Radial Limits ......................................................... 201
  10.3 Formulas for the Taylor Series Coefficients of $f(q)$ ........ 204

11 Euler Products in Ramanujan’s Lost Notebook ...................... 207
  11.1 Introduction ........................................................... 207
  11.2 Scattered Entries on Euler Products ........................... 209
  11.3 Approach of Sun and Williams ................................. 222
  11.4 A Partial Manuscript on Euler Products ....................... 227

12 Continued Fractions ..................................................... 243
  12.1 Introduction ........................................................... 243
  12.2 Finite and Infinite Rogers-Ramanujan Continued Fractions ... 243
1 Third Order Mock Theta Functions: Elementary Identities

1.1 Introduction
To be written.

1.2 Basic Theorems
For the convenience of readers, we reproduce two theorems from [20, p. 6, Theorem 1.2.1; p. 7, Theorem 1.2.2].

**Theorem 1.2.1.** If $h$ is a positive integer, then, for $|t|, |b| < 1$,

$$
\sum_{m=0}^{\infty} \frac{(a; q^h)_m(b; q)_{km}t^m}{(q^h; q^h)_m(c; q)_{hm}} = \frac{(b; q)_{\infty}(at; q^h)_{\infty}}{(c; q)_{\infty}(t; q^h)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_m(t; q^h)_m b^m}{(q; q)_{\infty}(at; q^h)_m}.
$$

(1.2.1)

**Theorem 1.2.2.** For $|t|, |b| < 1$,

$$
\sum_{n=0}^{\infty} \frac{(a; q^2)_n(b; q)_n t^n}{(q^2; q^2)_n(c; q)_n} = \frac{(b; q)_{\infty}(at; q^2)_{\infty}}{(c; q)_{\infty}(t; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(c/b; q)_{2n}(t; q^2)_n b^{2n}}{(q; q)_{\infty}(at; q^2)_n} + \frac{(b; q)_{\infty}(atq; q^2)_{\infty}}{(c; q)_{\infty}(tq; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(c/b; q)_{2n+1}(tq; q^2)_n b^{2n+1}}{(q; q)_{\infty}(atq; q^2)_n}.
$$

(1.2.2)

In [6], purely elementary identities were derived which implied many of the results in Ramanujan’s last letter to Hardy. We shall prove limiting versions of Theorems 1 and 2 of [6].

**Theorem 1.2.3.** For $b, c \in \mathbb{C}, b \neq 0$,
Third Order Mock Theta Functions: Elementary Identities

Theorem 1.2.4.

\[ \sum_{n=0}^{\infty} \frac{(b; q)_n (-1)^n (c/b)^n q^{n^2}}{(q^2; q^2)_n (c; q)_n} = \frac{(cq/b; q^2)_\infty (cb; q^2)_\infty}{(c; q)_\infty (q; q^2)_\infty (-b; q)_\infty} \sum_{n=0}^{\infty} \frac{(b^2; q^2)_n (-1)^n q^{n^2}}{(q^2; q^2)_n (cb; q^2)_n} + \frac{b(c/b; q^2)_\infty (cbq; q^2)_\infty}{(c; q)_\infty (q; q^2)_\infty (-b; q)_\infty} \sum_{n=0}^{\infty} \frac{(b^2; q^2)_n (-1)^n q^{n^2+2n}}{(q^2; q^2)_n (cbq; q^2)_n}. \]

Proof. In Theorem 1.2.2, replace \( a \) by \( cq/(bt) \) and let \( t \to 0 \). This yields

\[ \sum_{n=0}^{\infty} \frac{(b; q)_n (-1)^n (c/b)^n q^{n^2}}{(q^2; q^2)_n (c; q)_n} = \frac{(b; q)_\infty (cq/b; q^2)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b; q)_{2n+1} b^{2n+1}}{(q; q)_{2n+1} (cq/b; q^2)_n} + \frac{(b; q)_\infty (cq/b; q^2)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b; q)_{2n} b^{2n}}{(q^2; q^2)_n (q; q^2)_n} + \frac{b(c/b; q^2)_\infty (cbq; q^2)_\infty}{(1-q)(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(cq/b; q^2)_n b^{2n}}{(q^2; q^2)_n (q^2; q^2)_n} \]

where the last equality follows by two applications of Theorem 1.2.1 with \( h = 1, b = 0, t = b^2 \), and \( q \) replaced by \( q^2 \). In the first application, \( a \) is replaced by \( c/b \) and \( c = q \); in the second, \( a \) is replaced by \( cq/b \) and \( c = q^3 \). Simplifying the last equality now yields the desired result. \( \square \)

Theorem 1.2.4. For \( b, c \in \mathbb{C}, b \neq 0, \)

\[ \sum_{n=0}^{\infty} \frac{(b; q)_n q^{n(n+1)/2}}{(q; q)_n (c; q)_n} = \frac{(bq; q^2)_\infty (-q; q)_\infty (c^2/b; q^2)_\infty}{(c; q)_\infty (-c/b; q)_\infty} \sum_{n=0}^{\infty} \frac{(b; q^2)_n (c/b)^{2n+1} q^{2n^2-n}}{(q; q)_{2n+1} (c/b; q^2)_n} + \frac{(b; q^2)_\infty (-q; q)_\infty (c^2/bq; q^2)_\infty}{(c; q)_\infty (-c/b; q)_\infty} \sum_{n=0}^{\infty} \frac{(b; q^2)_n (c/b)^{2n+1} q^{2n^2+n}}{(q; q)_{2n+1} (c^2 bq; q^2)_n}. \]

Proof. In Theorem 1.2.1, set \( h = 1 \), replace \( a \) by \( -q/t \), and let \( t \to 0 \). Hence,
1.3 The Third Order Identities

First recall the definition and product representation for Ramanujan’s theta function $\varphi(q)$, namely [20, p. 17, equation (1.4.3)], [28, pp. 36, 37; Entry 22(i), equation (22.4)]

$$\varphi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^n = \frac{(q:q)_\infty}{(-q:q)_\infty}. \quad (1.3.1)$$

In his last letter to Hardy [126, p. 354], [34, p. 222], Ramanujan offered the following identity relating three third order mock theta functions, defined by

$$f_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q:q)_n^2}, \quad (1.3.2)$$

$$\phi_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \quad (1.3.3)$$

$$\psi_3(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \quad (1.3.4)$$

**Entry 1.3.1 (p. 31, 2nd and 3rd equations).** With $\varphi(q)$, $f_3(q)$, $\phi_3(q)$, and $\psi_3(q)$ as defined above,

$$2\phi_3(-q) - f_3(q) = f_3(q) + 4\psi_3(-q) = \frac{\varphi^2(-q)}{(q:q)_\infty}. \quad (1.3.5)$$

**Proof.** In Theorem 1.2.3, set $b = q$ and $c = -q$. Using Euler’s theorem and replacing $n$ by $n - 1$ in the second series on the right-hand side in Theorem 1.2.3, we find that

$$\sum_{n=0}^{\infty} \frac{(b; q)_n q^{n(n+1)/2}}{(q; q)_n (c; q)_n} \cdot \frac{(c/b; q)_\infty (c/b; q)_n}{(q; q)_n (-q; q)_n} = \frac{(b; q)_\infty (-q; q)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b; q)_n b^n}{(q^2; q^2)_n (-c/b; q)_n}$$

$$= \frac{(b; q)_\infty (-q; q)_\infty}{(c; q)_\infty} \left\{ \frac{(c^2/b; q^2)_\infty (c^2/b; q^2)_\infty}{(-c/b; q)_\infty (b; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^{2n} q^{n(2n-1)}}{(q; q^2)_n} \right\}$$

$$+ \frac{(c^2/q; b^2)_\infty}{(-c/b; q)_\infty (b; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(bq; q^2)_n (c/b)^{2n+1} q^{n(2n+1)}}{(q; q^2)_{2n+1} (c^2/c/b; q^2)_n} \right\},$$

where we applied Theorem 1.2.2 with $t = b$, $a = (c/b)^2$, $c$ replaced by $-c/b$, and $b \to 0$. Simplification completes the proof of Theorem 1.2.4. \qed
\[ f_3(q) = \phi_3(-q) - 2\psi_3(-q). \] (1.3.6)

Next, in Theorem 1.2.3, set \( b = -q \) and \( c = q \) to deduce that

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n^2} = \frac{1}{\varphi^2(-q)} \{ \phi_3(-q) + 2\psi_3(-q) \}. \] (1.3.7)

Once we recall that [8, p. 21, equation (2.2.9)]

\[ \sum_{n=0}^{\infty} q^n \frac{(q^n; q)_n}{2^n} = \frac{1}{\psi_2(-q)}, \] (1.3.8)

we see that (1.3.5) follows directly from (1.3.6) and (1.3.7).

The next entry involves two further third order mock theta functions

\[ \rho_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{j=0}^{n}(1 + q^{2j+1} + q^{4j+2})} \] (1.3.9)

and

\[ \omega_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_{n+1}^2}. \] (1.3.10)

Also recall Ramanujan’s theta function \( \psi(q) \) and its product representation given by [20, p. 17, equation (1.4.10)], [28, p. 36, Entry 22(ii)]

\[ \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q^2; q^2)_\infty}. \] (1.3.11)

**Entry 1.3.2 (p. 15, top equation).** With \( \rho_3(q) \), \( \omega_3(q) \), and \( \psi(q) \) defined above,

\[ q^{1/2} \left\{ \frac{2}{3} \rho_3(-q) + \frac{1}{3} \omega_3(-q) \right\} = q^{1/2} \frac{\psi^2(-q^3)}{(q^2; q^2)_\infty}. \] (1.3.12)

G.N. Watson proved this result in [142, p. 63]; however, he states, “Rather strangely (particularly in view of his having discovered both sets of functions of order 5) he [Ramanujan] seems to have overlooked the existence of the set of functions which I call \( \omega(q) \), \( \nu(q) \), \( \rho(q) \).” This strongly indicates that Watson either did not possess or had totally ignored the Lost Notebook [128] in 1935 when he wrote [142].

**Proof.** We follow Watson [142]. Employing [19, p. 263, equation (12.2.5)], namely,

\[ \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - cq^{n+1/2}} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(cq^{1/2})_{n+1}(q^{1/2}/c)_{n+1}}, \] (1.3.13)
with \( q \) replaced by \( q^2 \) and \( c = 1 \), we find that

\[
\omega_3(q) = \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1 - q^{2n+1}} = \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1 + q^{2n+1})}{1 - q^{2n+1}}. \tag{1.3.14}
\]

Next, apply (1.3.13) with \( q \) replaced by \( q^2 \) and \( c = e^{2\pi i/3} \) to deduce that

\[
\rho_3(q) = \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1 - q^{4n+2})}{(1 - e^{2\pi i/3}q^{2n+1})(1 - e^{-2\pi i/3}q^{2n+1})}. \tag{1.3.15}
\]

Hence, combining (1.3.14) and (1.3.15) term by term, we find that

\[
2\rho_3(q) + \omega_3(q) = \frac{3}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1 + q^{6n+3})}{1 - q^{6n+3}} = \frac{3}{(q^2; q^2)_\infty} \frac{(q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2} = 3 \varphi^2(q^3) \tag{1.3.16}
\]

where we have used [19, p. 264, equation (12.2.9)], i.e.,

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - cq^n} = \frac{(q)_\infty^2}{(c)_\infty (q/c)_\infty}, \tag{1.3.17}
\]

with \( q \) replaced by \( q^6 \) and \( c = q^3 \), and where we also invoked (1.3.11). This last identity is equivalent to (1.3.12), and so the proof is complete.

The next entry involves another third order mock theta function

\[
\chi_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}(1 - q^j + q^{2j})}. \tag{1.3.18}
\]

**Entry 1.3.3 (p. 15, 2nd equation).** If \( f_3(q) \) is defined by (1.3.2), \( \varphi(-q) \) is given by (1.3.1), and \( \chi_3(q) \) is given by (1.3.18), then

\[
\chi_3(q) = \frac{1}{4} f_3(q) + \frac{3}{4} \varphi^2(-q^3). \tag{1.3.19}
\]

**Proof.** We begin by employing [19, p. 263, equation (12.2.3)], i.e.,

\[
\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - cq^n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(c)_{n+1}(q/c)_n}. \tag{1.3.20}
\]

twice, first with \( c = e^{\pi i/3} \) to find that
Third Order Mock Theta Functions: Elementary Identities

\[ \chi_3(q) = \frac{1}{(q; q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n) q^{3n+1/2}}{1 - q^n + q^{2n}} \right), \quad (1.3.21) \]

and second with \( c = -1 \) to find that

\[ f_3(q) = \frac{1}{(q; q)_\infty} \left( 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n+1/2}}{1 + q^{3n}} \right). \quad (1.3.22) \]

Therefore, by (1.3.21) and (1.3.22),

\[ 4\chi_3(q) - f_3(q) = \frac{1}{(q; q)_\infty} \left( 3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n+1/2}}{1 + q^{3n}} \right) \]

\[ = \frac{3}{(q; q)_\infty} \left( 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n+1/2}}{1 + q^{3n}} \right) \]

\[ = \frac{3}{(q; q)_\infty} \psi^2(-q^3), \]

by an appeal to (1.3.17) with \( c = -1 \) and \( q \) replaced by \( q^3 \), and to (1.3.1). This last identity is equivalent to (1.3.19), and so the proof is complete. \( \square \)

**Entry 1.3.4 (p. 17, 3rd equation).** If \( f_3(q) \) is defined by (1.3.2) and \( \varphi(-q) \) by (1.3.1), then

\[ \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(-q; q^3)_{n+1} (-q^2; q^3)_n} = 1 - \frac{1}{4} f_3(q^3) + \frac{1}{4} \frac{\psi^2(-q)}{(q^3; q^3)_\infty}. \quad (1.3.23) \]

We have replaced \( q \) by \( q^3 \) in Ramanujan’s original formulation.

**Proof.** We begin by employing (1.3.20) with \( q \) replaced by \( q^3 \) and \( c = -q \) to deduce that

\[ \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(-q; q^3)_{n+1} (-q^2; q^3)_n} = \frac{1}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2}}{1 + q^{3n+1}} \quad (1.3.24) \]

\[ = \frac{1}{2(q^3; q^3)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2}}{1 + q^{3n+1}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2-1}}{1 + q^{3n-1}} \right), \]

where we replaced \( n \) by \(-n\) to achieve the second sum on the right-hand side of (1.3.24). Next, by (1.3.20) with \( q \) replaced by \( q^3 \) and \( c = -1 \),

\[ f_3(q^3) = \frac{2}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2}}{1 + q^{3n}}. \quad (1.3.25) \]

Hence, by (1.3.24) and (1.3.25),
Corollary 1.7

For Entry 1.3.5 (p. 17, 4th equation).

This last equality of (1.3.26) is equivalent to (1.3.23), and so the proof is by (1.3.17) with \( c = -1 \), (1.3.1), and the pentagonal number theorem [8, p. 11, Corollary 1.7]

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty.
\]

This last equality of (1.3.26) is equivalent to (1.3.23), and so the proof is complete.

**Entry 1.3.5 (p. 17, 4th equation).** For \( \omega_3(q) \) defined by (1.3.10) and \( \psi(q) \) defined by (1.3.11),

\[
\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q; q^6)_n(q^6; q^6)_n} = \frac{1}{2} \left( 1 + q^2 \omega_3(q^3) + \frac{\psi^2(q)}{(q^6; q^6)_\infty} \right).
\]

We have replaced \( q \) by \( q^6 \) in Ramanujan’s original formulation.

**Proof.** Employing (1.3.20) with \( q \) replaced by \( q^6 \) and \( c = q \), we find that

\[
\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q; q^6)_n(q^6; q^6)_n} = \frac{1}{2(q^6; q^6)_\infty} \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)} \right)
\]

\[
= \frac{1}{2(q^6; q^6)_\infty} \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)} - \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)-1} \right).
\]
where to obtain the second sum on the right-hand side above, we replaced \( n \) by \(-n\) in the first sum on the right-hand side. Hence, by (1.3.14) and (1.3.29),

\[
2 \sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q^6; q^6)_n} + 1 - q^2 \omega_3(q^3)
\]

\[
= \frac{1}{(q^6; q^6)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1 - q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)-1}}{1 - q^{6n-1}} \right) - (q^6; q^6)_\infty - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n(n+1)+2}}{1 - q^{6n+3}}
\]

\[
= 1 \frac{1}{(q^6; q^6)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1 - q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2-3n}}{1 - q^{6n-1}} (q^{6n-1} - 1 + 1) \right) - (q^6; q^6)_\infty - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n(3n+1)(3n+2)}}{1 - q^{6n+3}}
\]

\[
= 1 \frac{1}{(q^6; q^6)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - q^{2n+1}} - (q^6; q^6)_\infty + \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2-3n} \right) - (q^6; q^6)_\infty + (q^6; q^6)_\infty
\]

\[
= 1 \frac{1}{(q^6; q^6)_\infty} \psi^2(q),
\]

(1.3.30)

where we used (1.3.17) with \( q \) replaced by \( q^2 \) and \( c = q \), (1.3.27), and (1.3.11). The last equality in (1.3.30) is equivalent to (1.3.28), and this completes the proof.

\[\square\]

**Entry 1.3.6 (p. 29, 8th equation).** If \( f_3(q) \) is given by (1.3.2) and \( \varphi(-q) \) is given by (1.3.1), then

\[
\sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(q^6; q^6)_n} = \frac{3}{4} f_3(q^3) + \frac{1}{4} \frac{\varphi^2(-q)}{(q^3; q^3)_\infty}.
\]

(1.3.31)

**Proof.** We initially apply [19, p. 273, Entry 12.4.2]

\[
(-aq)_\infty (-q/a)_\infty (q) \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(-aq^2; q^2)_n (-q^2/a; q^2)_n}
\]

\[
= 1 + \sum_{n=1}^{\infty} (2(-1)^n + a^n + a^{-n}) \frac{q^{n(n+1)/2}}{1 + q^n},
\]

(1.3.32)

with \( a = -\omega := -e^{2\pi i/3} \). In the third equality below we appeal to [28, p. 114, Entry 8(v)], to wit,
\[ \varphi^2(-q) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}. \]

Thus,

\[
(q^3; q^3) \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{\omega q^2; q^2)_n (\omega^{-1} q^2; q^2)_n} \\
= 1 + \sum_{n=1}^{\infty} (-1)^n (2 + \omega^n + \omega^{-n}) q^{n(n+1)/2} \\
= 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2} q^{n+1} + \sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2} q^{n+1} \\
= 1 + \frac{1}{4} (\varphi^2(-q) - 1) + 3 \sum_{n=1}^{\infty} (-1)^n q^{3n(n+1)/2} \\
= \frac{3}{4} \varphi^2(-q) + \frac{3}{4} (f_3(q^3)(q^3; q^3)_\infty - 1) \\
= \frac{3}{4} f_3(q^3)(q^3; q^3)_\infty + \frac{1}{4} \varphi^2(-q),
\]

where in the penultimate line we employed (1.3.22). We see that this last equality is equivalent to (1.3.31), and so the proof is complete.

**Entry 1.3.7 (p. 29, 9th equation).** We have

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q^2; q^2)_n q^{2n}}{(-q^6; q^6)_n} = \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n q^{(3n+2)(3n+1)/2} \\
+ \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} q^{-1} + \frac{1}{2} (q; q^2)_\infty \sum_{n=0}^{\infty} q^{3n+2} (1 - q^{2n+1}). \tag{1.3.33}
\]

In our statement of Entry 1.3.7 we have replaced Ramanujan’s \(x\) by \(q\). Also, the three sums on the right-hand side agree with the terms listed by Ramanujan even though it appears he would have arranged the terms differently. Finally, we note that while there are only false theta functions (instead of mock theta functions) in (1.3.33), the result is sufficiently similar to the previous entry to merit inclusion in this chapter.

**Proof.** We begin by recording [20, p. 122, Entry 6.3.9], namely,

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n}}{(-aq^2; q^2)_n (-q^2/a; q^2)_n} = (1 + a) \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} \\
- a(q; q^2)_\infty (-aq^2; q^2)_\infty \sum_{n=0}^{\infty} a^{3n} q^{3n+2} (1 - aq^{2n+1}).
\]
Setting \( a = \omega := e^{2\pi i/3} \), we deduce that, upon some algebraic simplification,

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q^2; q^2)_n q^{2n}}{(-q^6; q^6)_n} = (1 + \omega) \sum_{n=0}^{\infty} (-\omega)^n q^{n(1+1)/2} - \omega(q; q^2)_{\infty}(-q^2; q^2)_{\infty} \sum_{n=0}^{\infty} q^{3n^2+2n}(1 - \omega q^{2n+1}). \tag{1.3.34}
\]

If we add the complex conjugate of (1.3.34) to itself (assuming that \( q \) is real for the time being), we find that

\[
2 \sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q^2; q^2)_n q^{2n}}{(-q^6; q^6)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n(1+1)/2} ( (1 + \omega) \omega^n + (1 + \omega^{-1}) \omega^{-n} ) + \frac{(q; q^2)_{\infty}(-q^2; q^2)_{\infty}}{(-q^6; q^6)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2+2n}(1 - q^{2n+1}). \tag{1.3.35}
\]

Now,

\[
(1 + \omega) \omega^n + (1 + \omega^{-1}) \omega^{-n} = \begin{cases} -2, & \text{if } n \equiv 1 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases}
\]

Hence, using the calculation above in (1.3.35), we find that

\[
2 \sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q^2; q^2)_n q^{2n}}{(-q^6; q^6)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n(1+1)/2} + 3 \sum_{n=0}^{\infty} (-1)^n q^{(3n+2)(3n+1)/2} + \frac{(q; q)_{\infty}}{(-q^6; q^6)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2+2n}(1 - q^{2n+1}),
\]

and this is the desired result multiplied by 2. \( \square \)

For the next entry, we need to define another third order mock theta function

\[
\nu_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q^2)^{n+1}}. \tag{1.3.36}
\]

**Entry 1.3.8 (p. 31, last equation).** If \( \nu_3(q) \) is given by (1.3.36), \( \omega_3(q) \) is given by (1.3.10), and \( \psi(q) \) is given by (1.3.11), then

\[
\nu_3(-q) = q \omega_3(q^2) + \frac{\psi(q^2)}{(q^2; q^4)_{\infty}}. \tag{1.3.37}
\]
This formula was given by Watson [142, p. 63], who clearly believed that Ramanujan did not have this result. See the remark following equation (1.3.12).

Proof. In Theorem 1.2.4, replace \( q \) by \( q^2 \), then set \( b = q^2 \) and \( c = q^3 \), and lastly multiply both sides by \( 1/(1 - q) \). Thus,

\[
\psi_d(-q) = \frac{(q^4; q^4)_{\infty}}{(q; q^2)_{\infty}}(-q^2; q^2)_{\infty}(q^4; q^4)_{\infty} \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4; q^4)_{n}} \\
+ \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}}(-q^2; q^2)_{\infty}(q^4; q^4)_{\infty} \sum_{n=0}^{\infty} \frac{q^{4n^2+4n+1}}{(q^2; q^4)_{n+1}} \\
= \frac{\psi(q^2)}{(q^4; q^4)_{\infty}} + q\psi_3(q^2),
\]

by Euler’s identity, (1.3.11), (1.3.8), and (1.3.10).

**Entry 1.3.9 (p. 31, 2nd and 3rd equations).** With \( \varphi(q) \) defined by (1.3.1),

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-aq^2; q^2)_{n}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(-q; q)_{n}(-aq; q)_{n}} + \frac{1}{2} \frac{\varphi(-q)}{(-aq; q)_{\infty}} \quad (1.3.38)
\]

and

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-aq^2; q^2)_{n}} = (1 + a) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(-aq; q^2)_{n}} + \frac{\varphi(-q)}{(-aq; q)_{\infty}}. \quad (1.3.39)
\]

If we use (1.3.1), the identities (1.3.38) and (1.3.39) reduce to the assertions in (1.3.5) when \( a = 1 \). They are also equivalent to equations (3a) and (3b) in [6]. Note also that we have replaced Ramanujan’s \( x \) with \( aq \).

Proof. If we put \( b = q \) and \( c = -aq \) in Theorem 1.2.3 and use Euler’s theorem, we find that

\[
\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(-q; q)_{n}(-aq; q)_{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-aq^2; q^2)_{n}} + (1 + a) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(-aq; q^2)_{n}}.
\]

Next in Theorem 1.2.3, set \( b = -q \) and \( c = aq \) to deduce, with the help of Euler’s theorem, that

\[
\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q; q)_{n}(aq; q)_{n}} = \frac{(-aq; q)_{\infty}}{(aq; q)_{\infty}} \varphi(-q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-aq^2; q^2)_{n}} \\
- (1 + a) \frac{(-aq; q)_{\infty}}{(aq; q)_{\infty}} \varphi(-q) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(-aq; q^2)_{n}}.
\]

Recalling that [8, p. 20, Corollary 2.6]
\[
\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} = \frac{1}{(aq;q)_\infty},
\]
we see that the two identities just proved are equivalent to (1.3.38) and (1.3.39), respectively. \qed
Fifth Order Mock Theta Functions.
Elementary Identities

2.1 Introduction

In the introductory chapter, we presented Ramanujan’s last letter to Hardy. In it, Ramanujan’s ten fifth order mock theta functions were given in their original “three or four terms of the series” format. We repeat them here in standard notation. The subscript 0 will denote members of the first family, and the subscript 1 will denote members of the second family. First,

\[ f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \]  
\[ \phi_0(q) := \sum_{n=0}^{\infty} (-q; q^2)_n q^{n^2}, \]  
\[ \psi_0(q) := \sum_{n=1}^{\infty} (-q; q)_{n-1} q^{n(n+1)/2}, \]  
\[ F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}, \]  
\[ \chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_n}, \]  
\[ \tilde{\chi}_0(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1}; q)_{n+1}}. \]

We note here, as did G.N. Watson [143], upon making two applications of the following corollary of the \( q \)-binomial theorem,

\[ \frac{1}{(z; q)_N} = \sum_{j=0}^{\infty} \binom{N + j - 1}{j} z^j \]
Fifth Order Mock Theta Functions. Elementary Identities

[19, p. 200, Eq. (8.2.5)], that

\[
\chi_0(q) = 1 + \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+2}; q)_{n+1}} \quad \text{(2.1.7)}
\]

\[
= 1 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q^{n+1+m(n+2)} \left[ \binom{n+m}{m} \right]
\]

\[
= 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q^{m+1}; q)_{m+1}}
\]

\[
= \tilde{\chi}_0(q). \quad \text{(2.1.8)}
\]

Thus there are really only five different 5th order mock theta functions with the subscript 0. Second,

\[
f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n}, \quad \text{(2.1.9)}
\]

\[
\phi_1(q) := \sum_{n=1}^{\infty} (-q; q^2)_{n-1} q^{n^2}, \quad \text{(2.1.10)}
\]

\[
\psi_1(q) := \sum_{n=0}^{\infty} (-q; q)_n q^{n(n+1)/2}, \quad \text{(2.1.11)}
\]

\[
F_1(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_{n+1}}, \quad \text{(2.1.12)}
\]

\[
\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}}. \quad \text{(2.1.13)}
\]

In addition, we need several other functions familiar to Ramanujan and appearing throughout these volumes. In particular [28, p. 36, Entries 22(i), (ii)], [20, p. 150],

\[
\varphi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_\infty}{(-q; q)_\infty}, \quad \text{(2.1.14)}
\]

\[
\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad \text{(2.1.15)}
\]

\[
G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty(q^4; q^5)_\infty}, \quad \text{(2.1.16)}
\]

\[
H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q^6)_\infty(q^4; q^5)_\infty}. \quad \text{(2.1.17)}
\]
The remainder of this chapter is based on the results in [6], which generalize the original work of Watson [143] on the fifth order mock theta functions. Our treatment differs from [6] in that it is in standard notation and is slightly less general, thus making it more easily read.

2.2 Basic Theorems

**Theorem 2.2.1.** Let \( s = 0 \) or \( 1 \). Then

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{(a; q)_{2n+s}(b; q)_n t^{2n+s}}{(q; q)_{2n+s}(c; q)_n} &= \frac{1}{2} \frac{(b; q)_{\infty}(at; q)_{\infty}}{(c; q)_{\infty}(t; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_{2m}(t; q)_m (b^2 q^{-s})^m}{(q; q)_{2m}(at; q)_m} \\
+ \frac{1}{2} (-1)^s \frac{(b; q)_{\infty}(-at; q)_{\infty}}{(c; q)_{\infty}(-t; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_{2m}(-t; q)_m (b^2 q^{-s})^m}{(q; q)_{2m}(-at; q)_m} \\
+ \frac{1}{2} \frac{(b; q)_{\infty}(atq^{1/2}; q)_{\infty}}{(c; q)_{\infty}(tq^{1/2}; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_{2m+1}(tq^{1/2}; q)_m (b^2 q^{-s})^{m+1/2}}{(q; q)_{2m+1}(atq^{1/2}; q)_m} \\
+ \frac{1}{2} (-1)^s \frac{(b; q)_{\infty}(-atq^{1/2}; q)_{\infty}}{(c; q)_{\infty}(-tq^{1/2}; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_{2m+1}(-tq^{1/2}; q)_m (b^2 q^{-s})^{m+1/2}}{(q; q)_{2m+1}(-atq^{1/2}; q)_m}.
\end{align*}
\]

**Proof.** By two applications of the \( q \)-binomial theorem [20, p. 6, equation (1.2.2)],

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{(a; q)_{2n+s}(b; q)_n t^{2n+s}}{(q; q)_{2n+s}(c; q)_n} &= \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; q)_{2n+s} t^{2n+s}}{(q; q)_{2n+s}} \frac{(cq^n; q)_{\infty}}{(bq^n; q)_{\infty}} \\
&= \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; q)_{2n+s} t^{2n+s}}{(q; q)_{2n+s}} \sum_{m=0}^{\infty} \frac{(c/b; q)_m (bq^n)^m}{(q; q)_m} \\
&= \frac{1}{2} \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_m b^m q^{-ms/2}}{(q; q)_m} \sum_{n=0}^{\infty} \frac{(a; q)_n t^n q^{m(n-s)/2}}{(q; q)_n} (1 + (-1)^{n+s}) \\
&= \frac{1}{2} \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_m b^m q^{-ms/2}}{(q; q)_m} \\
&\times \left\{ \frac{(atq^{m/2}; q)_{\infty}}{(tq^{m/2}; q)_{\infty}} + (-1)^s \frac{(-atq^{m/2}; q)_{\infty}}{(-tq^{m/2}; q)_{\infty}} \right\} \\
&= \frac{1}{2} \frac{(b; q)_{\infty}}{(c; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(c/b; q)_{2m} t^{2m} q^{-ms}}{(q; q)_{2m}}.
\end{align*}
\]
Next, in the last equality, we apply Theorem 1.2.2 with

\[ \frac{1}{2} \left( \frac{c/b \cdot q_{2m+1}}{(c; q)_{2m+1}} \right) \]

Now if we separate these last sums into four separate sums and simplify, we obtain the desired result.

**Theorem 2.2.2.** If \( s = 0 \) or 1, then

\[
\sum_{n=0}^{\infty} \frac{(b; q^2)_n q^{n(n+1)/2+sn}}{(q; q)_n (c; q^2)_n}
\]

\[
= \frac{(bq^2; q^4)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q^{1+2s}b/c; q^2)_m (b^4 q^4)_m (c/b)^{2m}}{(q^2; q^2)_m}
\]

\[
+ \frac{(b; q^4)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q^{1+2s}b/c; q^2)_m (bq^2 q^4)_m (c/b)^{2m+1}}{(q^2; q^2)_m}
\]

Proof. In Theorem 1.2.1, set \( h = 2 \), interchange \( b \) and \( t \), then replace \( c \) by \(-q^{1+s}\), and finally replace \( a \) by \( c/b \), and let \( t \to 0 \). This yields

\[
\sum_{n=0}^{\infty} \frac{(b; q^2)_n q^{n(n+1)/2+sn}}{(q; q)_n (c; q^2)_n}
\]

\[
= \frac{(b; q^2)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(c/b; q^2)_m b^m}{(q^2; q^2)_m (-q^{1+s}; q^2)_m}
\]

\[
= \frac{(b; q^2)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(c/b; q^2)_m b^m}{(q^4; q^4)_m (-q^{1+2s}; q^2)_m}
\]

Next, in the last equality, we apply Theorem 1.2.2 with \( q \) replaced by \( q^2 \), \( a = 0 \), \( b \) replaced by \( c/b \), \( t = b \), and \( c = -q^{1+2s} \). Hence,

\[
\sum_{n=0}^{\infty} \frac{(b; q^2)_n q^{n(n+1)/2+sn}}{(q; q)_n (c; q^2)_n}
\]

\[
= \frac{(b; q^2)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q^{1+2s}b/c; q^2)_m (b^4 q^4)_m (c/b)^{2m}}{(q^2; q^2)_m}
\]

\[
+ \frac{(b; q^2)_\infty}{(c; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q^{1+2s}b/c; q^2)_m (bq^2 q^4)_m (c/b)^{2m+1}}{(q^2; q^2)_m}
\]

\[
\times \sum_{m=0}^{\infty} \frac{(-q^{1+2s}b/c; q^2)_m (bq^2 q^4)_m (c/b)^{2m+1}}{(q^2; q^2)_m}
\]
2.2 Basic Theorems

**Theorem 2.2.3.** If \( s = 0 \) or 1, then

\[
\sum_{n=0}^{\infty} \frac{(b; q)_n q^{n^2+n}}{(q^2; q^2)_n} = \frac{b^s}{(q; q^2)_{\infty}} (\sum_{m=0}^{\infty} \frac{(-1)^m (b^2; q^2)_m q^{2m^2+4sm}}{(q^4; q^4)_m}) \\
+ \frac{b^{1-s}(-q^2; q^2)_{\infty}(b; q)_{\infty}}{(q; q^2)_{\infty}(b^2; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2; q^4)_m q^{4m^2+4m(1-s)}}{(q^2; q^2)_{2m}} \\
- \frac{b^{1-s}(-q^2; q^2)_{\infty}(b; q)_{\infty}}{(q; q^2)_{\infty}(b^2; q^2; q^4)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^2; q^4)_m q^{2m+1} (2m+3-2s)}{(q^2; q^2)_{2m+1}}.
\]

*Proof.* We note that \( \{s, 1-s\} = \{0, 1\} \). We set \( c = 0 \), \( a = -q^{1+s}/t \), and let \( t \to 0 \) in Theorem 1.2.2. Hence,

\[
\sum_{n=0}^{\infty} \frac{(b; q)_n q^{n^2+n}}{(q^2; q^2)_n} = (b; q)_{\infty} (-q^{1+s}; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{b^{2n}}{(q; q)_2n (-q^{1+s}; q^4)_n} \\
+ (b; q)_{\infty} (-q^{2+s}; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{b^{2n+1}}{(q; q)_{2n+1} (-q^{2+s}; q^2)_n} \\
= (b; q)_{\infty} (-q; q^2)_{\infty} \sum_{m=0}^{\infty} \frac{b^{2m+s}}{(q^2; q^2)_m (q^2; q^2)_{m+s}} \\
+ (b; q)_{\infty} (-q^2; q^2)_{\infty} \sum_{m=0}^{\infty} \frac{b^{2m+1-s}}{(q^4; q^4)_m (q; q^2)_{m+1-s}} \\
= \frac{b^s}{(-b; q)_{\infty}(q; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m+4ms} (b^2; q^2)_m}{(q^4; q^4)_m} \\
+ \frac{b^{1-s}(-q^2; q^2)_{\infty}(b; q)_{\infty}}{(q; q^2)_{\infty}(b^2; q^2; q^4)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2; q^4)_m q^{4m^2+4m(1-s)}}{(q^2; q^2)_{2m}} \\
- \frac{b^{1-s}(-q^2; q^2)_{\infty}(b; q)_{\infty}}{(q; q^2)_{\infty}(b^2; q^2; q^4)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^2; q^4)_m q^{2m+1} (2m+3-2s)}{(q^2; q^2)_{2m+1}}.
\]

The first sum on the far right side above arises by applying Theorem 1.2.1 to the first sum in the previous equality with \( h = 2 \), \( q \) replaced by \( q^2 \), \( b \) replaced by \( b^2 \), \( c = 0 \), \( t = q^2/a \) if \( s = 0 \), \( t = q^6/a \) if \( s = 1 \), and \( a \to \infty \). Furthermore, when \( s = 1 \), we multiply both sides by \( 1/(1-q^2) \) in our
application of Theorem 1.2.1. The second and third sums on the far right side above come from applying Theorem 1.2.2 to the second sum in the previous equality with $q$ replaced by $q^2$, $a = b = 0$, $t = b^2$, $c = q^4$ if $s = 0$, and $c = q$ if $s = 1$. In the case of $s = 0$, we also multiply both sides by $1/(1 - q)$ in our application of Theorem 1.2.2. Thus, we obtain the desired right-hand side to complete the proof.

**Theorem 2.2.4.** If $s = 0$ or 1, then

$$
\sum_{n=0}^{\infty} \frac{(b;q)_n q^n}{(q;q)_{2n+s}}
= q^{-s/2}(1 + (-1)^s)(b;q)_{\infty} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s};q^2)_m q^{2m^2}}{(q;q)_{2m}}
+ q^{(1-s)/2}(1 - (-1)^s)(b;q)_{\infty} \sum_{m=0}^{\infty} \frac{(b^2 q^{1-s};q^2)_m q^{2m^2+2m}}{(q;q)_{2m+1}}
+ \frac{b q^{-s} (b;q)_\infty}{2(q;q)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s})^m}{(q^{m+1};q)_{m+1}}
+ \frac{(-1)^s b q^{-s} (b;q)_\infty}{2(b^2 q^{-s};q)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m (b^2 q^{-s};q)_{2m} q^{m^2+2m}}{(q^2;q^2)_m}.
$$

**Proof.** By Theorem 2.2.1 with $a = c = 0$ and $t = \sqrt{q}$,

$$
\sum_{n=0}^{\infty} \frac{(b;q)_n q^n}{(q;q)_{2n+s}}
= q^{-s/2}(b;q)_{\infty} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s})^m}{(q^2;q^2)_m (-q^{1/2};q)_m}
+ \frac{(-1)^s q^{-s/2} (b;q)_{\infty}}{2(-q^{1/2};q)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s})^m}{(q^2;q^2)_m (q^{1/2};q)_m}
+ q^{-s/2} (b;q)_{\infty} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s})^{m+1/2}}{(q^{m+1};q)_{m+1}}
+ \frac{(-1)^s q^{-s/2} (b;q)_{\infty}}{2(1-q)(-q;q)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s})^{m+1/2}}{(q;q)_m (q^2;q^2)_m}.
$$

We now apply Theorem 1.2.2 to the first and second sums on the right-hand side above with $a = b = 0$, $c = -\sqrt{q}$, and $t = b^2 q^{-s}$ in the first case and $a = b = 0$, $c = \sqrt{q}$, and $t = b^2 q^{-s}$ in the second. We also invoke Theorem 1.2.1 to the fourth sum on the right-hand side above with $h = 2$, $c = t = 0$, $a = q^3/t$, and $b$ replaced by $b^2 q^{-s}$. These applications thus give us...
\[
\sum_{n=0}^{\infty} \frac{(b; q)_n q^n}{(q; q)_{2n+s}} = \frac{q^{-s/2}(b; q)_{\infty}}{2(q^{1/2}; q)_{\infty}(-q^{1/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s}; q^2)_m q^{2m^2}}{(q; q)_{2m}} + \frac{q^{-s/2}(b; q)_{\infty}}{2(q^{1/2}; q)_{\infty}(-q^{1/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s}; q^2)_m q^{2m^2}}{(q; q)_{2m+1}} + \frac{(-1)^s q^{-s/2}(b; q)_{\infty}}{2(-q^{1/2}; q)_{\infty}(q^{1/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s}; q^2)_m q^{2m^2}}{(q; q)_{2m+1}} - \frac{q^{-s/2}(b; q)_{\infty}}{2(q; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(b^2 q^{-s}; q^2)_{m+1/2}}{(q^{m+1}; q)_{m+1}} + \frac{(-1)^s b q^{-s}(b; q)_{\infty}}{2(-q; q)_{\infty}(q; q^2)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m (b^2 q^{-s}; q)_{2m} q^{2m^2+2m}}{(q^2; q^2)_{m}}.
\]

To obtain the desired result we then combine the first sum with the third and the second with the fourth above and use Euler's theorem.

**Theorem 2.2.5.** If \( s = 0 \) or \( 1 \), then

\[
\sum_{n=0}^{\infty} \frac{(b; q)_n q^{2n}}{(q; q)_{2n+s}} = \frac{q^{-s}(b; q)_{\infty}}{2(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s})^n}{(q^{n+1}; q)_n} + \frac{(-1)^s q^{-s}(b; q)_{\infty}}{2(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n (b^2 q^{-s}; q)_{2n} q^{2n^2}}{(q^2; q^2)_n} + \frac{b q^{-3s/2}(1 + (-1)^s)(b; q)_{\infty}}{2(q; q^2)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s}; q^2)_n q^{2n^2+2n}}{(q; q)_{2n+1}} + \frac{b q^{3(1-s)/2}(1 - (-1)^s)(b; q)_{\infty}}{2(q; q^2)_{\infty}(b^2 q^{-1-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{1-s}; q^2)_n q^{2n^2+4n}}{(q; q)_{2n+1}}.
\]

**Proof.** By Theorem 2.2.1 with \( a = c = 0 \) and \( t = q \),

\[
\sum_{n=0}^{\infty} \frac{(b; q)_n q^{2n}}{(q; q)_{2n+s}} = \frac{q^{-s}(b; q)_{\infty}}{2(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s})^n}{(q^{n+1}; q)_n} + \frac{(-1)^s q^{-s}(b; q)_{\infty}}{2(-q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s})^n}{(q; q)_n(q^2)_n} + \frac{q^{-s}(b; q)_{\infty}}{2(1-q)(q^{3/2}; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s})^{m+1/2}}{(q^2; q^2)_n(-q^{3/2}; q)_n}.
\]
2 Fifth Order Mock Theta Functions. Elementary Identities

We next apply Theorem 1.2.1 to the second sum on the right side above with $h = 2$, $q$ replaced by $q^2$, $t = q/a$, $b$ replaced by $b^2 q^{-s}$, $c = 0$, and $a \to \infty$. In the third and fourth sums above, we utilize Theorem 1.2.2 with $a = b = 0$, $t = b^2 q^{-s}$, and $c = \mp q^{3/2}$, respectively. Accordingly, we find that

$$
\sum_{n=0}^{\infty} \frac{(b; q)_n q^{2n}}{(q; q)_{2n+2}} = \frac{q^{-s}(b; q)_{\infty}}{2(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b^2 q^{-s})^n}{(q^{n+1}; q)_n}
+ \frac{(-1)^s q^{-s}(b; q)_{\infty}}{2(-q; q)_{\infty}(q; q^2)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} (-1)^n (b^2 q^{-s}; q)_{2n} q^{n^2}
+ \frac{q^{-s}(b^2 q^{-s})^{1/2}(b; q)_{\infty}}{2(1-q)(q^{3/2}; q)_{\infty}(-q^{3/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} (b^2 q^{-s}; q^2)_{n} q^{2n^2 + 2n + 2n+1}
+ \frac{(-1)^s q^{-s}(b^2 q^{-s})^{1/2}(b; q)_{\infty}}{2(1-q)(q^{3/2}; q)_{\infty}(q^{3/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} (b^2 q^{-s}; q^2)_{n} q^{2n^2 + 2n + 2n+1}
- \frac{(-1)^s q^{-s}(b^2 q^{-s})^{1/2}(b; q)_{\infty}}{2(1-q)(-q^{3/2}; q)_{\infty}(q^{3/2}; q)_{\infty}(b^2 q^{-s}; q^2)_{\infty}} \sum_{n=0}^{\infty} (b^2 q^{-s}; q^2)_{n} q^{2n^2 + 2n + 2n+1}.
$$

Simplifying above with the use of Euler’s theorem and combining the third sum with the fifth, and the fourth with the sixth, we obtain Theorem 2.2.5. 

\[\square\]

2.3 Watson’s Fifth Order Identities

In [143], Watson proved nine identities among the fifth order mock theta functions. He used these to establish all of the assertions about fifth order mock theta functions in Ramanujan’s last letter to Hardy. We shall use the theorems of Section 2.2 to prove identities equivalent to Watson’s results. In Section 2.4, we shall deduce all of Ramanujan’s identities for the fifth order mock theta functions.

We now offer Watson’s nine identities. First, from Theorem 2.2.3 with $s = 0$ and $b = q$, with the help of (2.1.1), (2.1.2), (2.1.15), (2.1.17), and (2.1.4),

$$f_0(q) = \phi_0(-q^2) + q\psi(q^2)H(q^4) - F_0(q^2) + 1. \quad (2.3.1)$$

Second, employ Theorem 2.2.3 with $s = 0$ and $b = -q$. We then multiply both sides of the identity by
\[
\frac{(q; q)_\infty}{(-q; q)_\infty} = \varphi(-q).
\]

Using (2.1.16), (2.1.14), (2.1.15), (2.1.17), and (2.1.4), we find that
\[
\varphi(-q)G(q) = \phi_0(-q^2) - q\psi(q^2)H(q^4) + F_0(q^2) - 1. \tag{2.3.2}
\]

Third, utilize Theorem 2.2.2 with \(s = 1, b = q^2,\) and \(c \to 0\). Next, employ (2.1.3), (2.1.15), (2.1.17), and (2.1.4). Lastly, multiply both sides by \(q\). We thus find that
\[
\psi_0(q) = q\psi(q^2)H(q^4) + F_0(q^2) - 1. \tag{2.3.3}
\]

Fourth, from Theorem 2.2.4 with \(s = 0\) and \(b = q\), with the help of (2.1.5), (2.1.4), (2.1.6), and (2.1.2),
\[
\chi_0(q) = F_0(q) + \frac{1}{2} \tilde{\chi}_0(q) - \frac{1}{2} \phi_0(-q). \tag{2.3.4}
\]

Fifth, invoke Theorem 2.2.5 with \(s = 1\) and \(b = q\), and then multiply both sides by \(q\). We deduce that
\[
\tilde{\chi}_0(q) = F_0(q) - \frac{1}{2} \phi_0(-q) + \frac{1}{2} \chi_0(q). \tag{2.3.5}
\]

We note that the fact, \(\chi_0(q) = \tilde{\chi}_0(q)\), which we noted in (2.1.7), follows immediately from (2.3.4) and (2.3.5).

Sixth, by Theorem 2.2.3 with \(s = 1\) and \(b = q\), with the help of (2.1.9), (2.1.10), (2.1.15), (2.1.16), and (2.1.12),
\[
f_1(q) = \psi(q^2)G(q^4) - qF_1(q^2) - q^{-1}\phi_1(-q^2). \tag{2.3.6}
\]

Seventh, employ Theorem 2.2.3 with \(s = 1\) and \(b = -q\), and multiply both sides of the identity by
\[
(q; q^2)_\infty(q; q)_\infty = \frac{(q; q)_\infty}{(-q; q)_\infty} = \varphi(-q).
\]

Appealing to (2.1.14), (2.1.17), (2.1.15), (2.1.12), and (2.1.10), we arrive at
\[
\varphi(-q)H(q) = \psi(q^2)G(q^4) - qF_1(q^2) + q^{-1}\phi_1(-q^2). \tag{2.3.7}
\]

Eighth, by Theorem 2.2.2 with \(s = 0, b = q^2,\) and \(c \to 0\), with the help of (2.1.11), (2.1.15), (2.1.16), and (2.1.12),
\[
\psi_1(q) = \psi(q^2)G(q^4) + qF_1(q^2). \tag{2.3.8}
\]

Ninth, by Theorem 2.2.4 with \(s = 1\) and \(b = q\), with the aid of (2.1.13), (2.1.12), and (2.1.10),
\[
\chi_1(q) = 2F_1(q) + q^{-1}\phi_1(-q). \tag{2.3.9}
\]
2.4 Ramanujan’s Fifth Order Identities

The identities in question are found on pages 22 and 25 of the Lost Notebook and contain all of the fifth order identities found in Ramanujan’s last letter. For succinctness, we shall express the identities in Watson’s notation given in Section 2.1.

Entry 2.4.1 (p. 22). We have

\[ \psi_1(q) - q^{-1} \phi_1(-q^2) = \varphi(q) H(-q). \]

Proof. By (2.3.8) and (2.3.7), with \( q \) replaced by \( -q \),

\[ \psi_1(q) - q^{-1} \phi_1(-q^2) = (\psi(q^2) G(q^4) + q F_1(q^2)) - (-\varphi(q) H(-q) + \psi(q^2) G(q^4) + q F_1(q^2)) = \varphi(q) H(-q). \]

\[ \square \]

Entry 2.4.2 (p. 22). We have

\[ f_1(q) + 2 q F_1(q^2) = \varphi(q) H(-q). \]

Proof. By (2.3.6) and by (2.3.7), with \( q \) replaced by \( -q \),

\[ f_1(q) + 2 q F_1(q^2) = (\psi(q^2) G(q^4) - q F_1(q^2) - q^{-1} \phi_1(-q^2)) + 2 q F_1(q^2) = (\varphi(q) H(-q) - q F_1(q^2) + q^{-1} \phi_1(-q^2)) + q F_1(q^2) - q^{-1} \phi_1(-q^2) = \varphi(q) H(-q). \]

\[ \square \]

Entry 2.4.3 (p. 22). We have

\[ f_1(q) + 2 q^{-1} \phi_1(-q^2) = \varphi(-q) H(q). \]

Proof. Subtract (2.3.7) from (2.3.6).

\[ \square \]

Entry 2.4.4 (p. 22). We have

\[ \psi_1(q) - q F_1(q^2) = \psi(q^2) G(q^4). \]

Proof. Entry 2.4.4 is precisely (2.3.8).

\[ \square \]

Entry 2.4.5 (p. 22). We have

\[ \psi_0(q) + \phi_0(-q^2) = \varphi(q) G(-q). \]
2.4 Ramanujan’s Fifth Order Identities

Entry 2.4.6 (p. 22). We have
\[ f_0(q) + 2F_0(q^2) - 2 = \varphi(q)G(-q). \]

**Proof.** Replace \( q \) by \(-q\) in (2.3.2) and subtract the resulting identity from (2.3.1). □

Entry 2.4.7 (p. 22). We have
\[ f_0(q) - 2\varphi_0(-q^2) = -\varphi(-q)G(q). \]

**Proof.** Add (2.3.2) to (2.3.1). □

Entry 2.4.8 (p. 25). We have
\[ \psi_0(q) - F_0(q^2) + 1 = q\psi(q^2)H(q). \]

**Proof.** This assertion is equivalent to (2.3.3). □

Entry 2.4.9 (p. 25). We have
\[ \chi_0(q) + \phi_0(-q) = 2F_0(q). \]

The proofs of Entries 2.4.9 and 2.4.10 will be given together.

Entry 2.4.10 (p. 25). We have
\[ \chi_0(q) = \tilde{\chi}_0(q). \]

Although Entry 2.4.10 was proved in (2.1.7), we include a second proof here.

**Proof.** Subtracting (2.3.5) from (2.3.4), we deduce Entry 2.4.10. Once we have Entry 2.4.10, then Entry 2.4.9 follows immediately from (2.3.4). □

Entry 2.4.11 (p. 25). We have
\[ \chi_1(q) - q^{-1}\phi_1(-q) = 2F_1(q). \]

**Proof.** This assertion is equivalent to (2.3.9). □

We come now to three assertions that involve the fifth order mock theta functions but are not in Ramanujan’s last letter to Hardy. They exist buried in the work of Watson [143] in the middle of the proofs that Watson originally gave for these identities. This suggests that Ramanujan’s proofs of his identities, which are found on page 25 in his Lost Notebook, were along lines similar to those of Watson.

We require three new functions:
\[ W_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n(q^2; q^2)_n}, \quad (2.4.1) \]
\[ W_2(q) := \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q; q)_n(q^2; q^2)_{n+1}}, \quad (2.4.2) \]
\[ W_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q; q)_n(q^2; q^2)_{n+1}}, \quad (2.4.3) \]

**Entry 2.4.12 (p. 25).** We have

\[ W_1(q) = \frac{\phi_0(-q)}{\varphi(-q)}. \]

**Proof.** Using first Theorem 2.2.4 with \( b = -q \) and \( s = 0 \), second using Entry 2.4.10, and third using Entry 2.4.9, we deduce that

\[
W_1(q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^n}{(q; q)_{2n}} = \frac{\eta_0(q)}{\varphi(-q)} - \frac{\chi_0(q) - 1}{2\varphi(-q)} + \frac{\phi_0(-q) - 1}{2\varphi(-q)} = \frac{1}{2\varphi(-q)} (2\eta_0(q) - \chi_0(q) + \phi_0(-q)) = \frac{\phi_0(-q)}{\varphi(-q)}.
\]

\[ \square \]

**Entry 2.4.13 (p. 25).** We have

\[ W_2(q) = -\frac{\phi_1(-q)}{\varphi(-q)}. \]

**Proof.** We employ (2.4.2) and Theorem 2.2.4 with \( b = -q \) and \( s = 1 \). Using (2.1.12), (2.1.13), (2.1.10), and (2.1.14), we find that

\[
W_2(q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n+1}}{(q; q)_{2n+1}} = \frac{q \eta_1(q)}{\varphi(-q)} - \frac{q \chi_1(q) - \phi_1(-q)}{2\varphi(-q)} = \frac{q}{2\varphi(-q)} (2\eta_1(q) - \chi_1(q) - q^{-1}\phi_1(-q)) = -\frac{\phi_1(-q)}{\varphi(-q)},
\]

by Entry 2.4.11. \[ \square \]
Entry 2.4.14 (p. 25). We have

\[ W_3(q) = \frac{1 - \phi_0(-q)}{\varphi(-q)}. \]

Proof. We appeal to (2.4.3) and Theorem 2.2.5 with \( b = -q \) and \( s = 1 \). Using also (2.1.5), (2.1.2), (2.1.4), and (2.1.14), we find that

\[
W_3(q) = q \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{2n}}{(q;q)_{2n+1}}
\]

\[
= \frac{\chi_0(q)}{2\varphi(-q)} - \frac{\phi_0(-q)}{2\varphi(-q)} - \frac{(F_0(q) - 1)}{\varphi(-q)}
\]

\[
= \frac{1}{2\varphi(-q)} (-2F_0(q) + \chi_0(q) - \phi_0(-q) + 2)
\]

\[
= \frac{1 - \phi_0(-q)}{\varphi(-q)},
\]

by Entry 2.4.9. \( \square \)

It is natural to ask why there is not an identity for another function, say \( W_4(q) \), from the case \( b = -q, s = 0 \) of Theorem 2.2.5. In this case, some of the expressions that arise are not accounted by the fifth order mock theta functions, and one of the series diverges.

2.5 Related Identities and Partitions

The ten fifth order mock theta functions are easily seen to be the generating functions for various classes of partitions. The techniques for proving such assertions are standard (cf. [9, Chapter 1], [66, Chapter 19]).

Theorem 2.5.1. The mock theta function \( f_0(q) \) is the generating function for the excess of the number of partitions with differences at least two between parts and largest part odd over the number of such partitions with largest part even.

The mock theta function \( \phi_0(q) \) is the generating function for partitions into odd parts without gaps, where every part appears either once or twice.

The mock theta function \( \psi_0(q) \) is the generating function for partitions without gaps, with unique largest part and with all other parts appearing once or twice.

The mock theta function \( F_0(q) \) is the generating function for partitions into odd parts without gaps in which each part appears at least twice.

The mock theta function \( \chi_0(q) \) is the generating function for partitions with unique smallest part and no parts larger than twice the smallest part.

The mock theta function \( \tilde{\chi}_0(q) \) is the generating function for partitions with largest part odd and no parts smaller than half the largest part.
Proof. As mentioned prior to the statement of Theorem 2.5.1, each assertion is easily deduced by standard methods. For example,

\[ F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{1+1} q^{3+3} \cdots q^{(2n-1)+(2n-1)}}{(1-q)(1-q^3) \cdots (1-q^{2n-1})} = \sum_{n=0}^{\infty} \sum_{r_1=2}^{\infty} \sum_{r_3=2}^{\infty} \cdots \sum_{r_{2n-1}=2}^{\infty} q^{r_1-1+r_3-3+\cdots+r_{2n-1}-(2n-1)} \cdot \]

The other five assertions follow in a similar manner. \(\square\)

Corollary 2.5.1. We have \(\chi_0(q) = \tilde{\chi}_0(q)\).

The proof that follows, taken from [7], will be the third proof of this result in this chapter.

Proof. We begin with the Ferrers graph of a partition enumerated by \(\tilde{\chi}_0(q)\):

\[
\begin{array}{c|c}
2m+1 \\
\hline
m+1 & m \\
\hline
\end{array}
\]

We now translate the nodes to the right of the vertical bar down below the nodes on the left:
We now read the columns of the resulting graph, and we see that we have a partition of the type enumerated by $\chi_0(q)$. The mapping is clearly reversible thus establishing a bijection between the two classes of partitions. Hence $\chi_0(q) = \tilde{\chi}_0(q)$.

**Theorem 2.5.2.** The mock theta function $f_1(q)$ is the generating function for the excess of the number of partitions with no one’s, with differences at least two between parts, and with largest part even over the number of such partitions with largest part odd.

The mock theta function $\phi_1(q)$ is the generating function for partitions into odd parts without gaps with unique largest part and with all other parts appearing once or twice.

The mock theta function $\psi_1(q)$ is the generating function for partitions without gaps in which each part appears once or twice.

The mock theta function $F_1(q)$ is the generating function for partitions in which each even integer not exceeding the largest part appears exactly twice.

The mock theta function $q\chi_1(q)$ is the generating function for partitions in which the largest part is less than twice the smallest part.

**Proof.** Our situation is exactly like that for Theorem 2.5.1. We content ourselves with one example:

$$
\psi_1(q) = 1 + \sum_{n=1}^{\infty} (-q; q)_{n}q^{n(n+1)/2} \\
= 1 + \sum_{n=1}^{\infty} (q^1 + q^{1+1})(q^2 + q^{2+2}) \cdots (q^n + q^{n+n}).
$$

It is possible to provide combinatorial interpretations of $W_1(q)$, $W_2(q)$, and $W_3(q)$. However, it is not possible to interpret them as generating functions for some classes of ordinary partitions, because their $n^{th}$ coefficients are eventually larger than $p(n)$.

Nonetheless, there are functions related to the fifth order mock theta functions that have easily described partition-theoretic interpretations, namely,

$$
V_1(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{2n+1}}{(q^2; q^2)_{2n}},
$$

$$
V_2(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_{n+1} q^{2n+1}}{(q^2; q^2)_{2n+1}},
$$

$$
V_3(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{4n+2}}{(q^2; q^2)_{2n}}.
$$

The following identities do not appear in the Lost Notebook. However, they are natural companions to Ramanujan’s identities for $W_i(q)$, $i = 1, 2, 3$. The proof is omitted for brevity.
Hence, it seems appropriate to include them, especially as they have nice partition-theoretic interpretations.

**Theorem 2.5.3.** We have

\[ V_1(q) = \frac{q\psi_1(-q)}{\psi(-q)}, \quad (2.5.4) \]
\[ V_2(q) = -\frac{\psi_0(-q)}{\psi(-q)}, \quad (2.5.5) \]
\[ V_3(q) = \frac{q^2(1 + \psi_0(-q))}{\psi(-q)}. \quad (2.5.6) \]

**Proof.** In Theorem 2.2.4 with \( s = 0 \), we replace \( q \) by \( q^2 \), then set \( b = -q \), multiply both sides by \( q \), and use (2.1.16), (2.1.15), (2.1.13), and (2.1.10) to deduce that

\[ V_1(q) = \frac{q(-q;q^2)\infty G(q^4)}{(q^2;q^4)\infty} - \frac{q^2(-q;q^2)\infty}{2(q^2;q^4)\infty} \chi_1(q^2) + \frac{(-q;q^2)\infty}{2(q^2;q^4)\infty} \phi_1(-q^2) \]
\[ = \frac{q^2}{\psi(-q)} \left( q^{-1}\psi(q^2)G(q^4) - \frac{1}{2} \left( \chi_1(q^2) - q^{-2}\phi_1(-q^2) \right) \right) \]
\[ = \frac{q^2}{\psi(-q)} (q^{-1}\psi(q^2)G(q^4) - F_1(q^2)) \]
\[ = \frac{q\psi_1(-q)}{\psi(-q)}, \]

where in the penultimate line we employed Entry 2.4.11 with \( q \) replaced by \( q^2 \), and in the last line invoked (2.3.8) with \( q \) replaced by \( -q \).

Next in Theorem 2.2.4, we set \( s = 1 \), replace \( q \) by \( q^2 \), then set \( b = -q^3 \), and multiply both sides by \( q(1 + q) \). Thus, using (2.1.17), Entry 2.4.9, and (2.1.2), we find that

\[ V_2(q) = \frac{q(-q;q^2)\infty}{(q^2;q^4)\infty} H(q^4) - \frac{(-q;q^2)\infty}{2(q^2;q^4)\infty} (\chi_0(q^2) - 1) \]
\[ - \frac{(-q;q^2)\infty}{2(q^2;q^4)\infty} (\phi_0(-q^2) - 1) \]
\[ = \frac{(-q;q^2)\infty}{(q^2;q^4)\infty} \left( q\psi(q^2)H(q^4) - \frac{1}{2} \left( \chi_0(q^2) + \phi_0(-q^2) \right) + 1 \right) \]
\[ = \frac{1}{\psi(-q)} (q\psi(q^2)H(q^4) - F_0(q^2) + 1) \]
\[ = -\frac{\psi_0(-q)}{\psi(-q)}, \]

where in the penultimate line we used Entry 2.4.9 with \( q \) replaced by \( q^2 \), and in the last employed (2.3.3) with \( q \) replaced by \( -q \).
Finally, in Theorem 2.2.5, we set \( s = 0 \), replace \( q \) by \( q^2 \), then set \( b = -q \), and multiply both sides by \( q^2 \). Thus,

\[
V_3(q) = \frac{q^2(-q; q^2)_\infty}{2(q^2; q^2)_\infty} \chi_0(q^2) + \frac{q^2(-q; q^2)_\infty}{2(q^2; q^2)_\infty} \phi_0(-q^2) - \frac{q^3(-q; q^2)_\infty}{(q^2; q^4)_\infty^2} H(q^4)
\]

\[
= \frac{q^2}{\psi(-q)} \left( \frac{1}{2} \chi_0(q^2) + \frac{1}{2} \phi_0(-q^2) - q \psi(q^2) H(q^4) \right)
\]

\[
= \frac{q^2}{\psi(-q)} (F_0(q^2) - q \psi(q^2) H(q^4))
\]

\[
= \frac{q^2(1 + \psi_0(-q))}{\psi(-q)},
\]

where Entry 2.4.9 with \( q \) replaced by \( q^2 \) was employed in the penultimate line and (2.3.3) with \( q \) replaced by \( -q \) was used in the last line.

\( \square \)

**Theorem 2.5.4.** The quotient \( V_1(q) \) is the generating function for partitions with no repeated odd parts and no parts as large as twice the largest odd part.

The quotient \( V_2(q) \) is the generating function for partitions in which the largest odd part may appear at most twice, no other odd part is repeated, and no parts are larger than twice the largest odd part.

The function \( V_3(q) \) is the generating function for partitions in which the largest odd part appears twice, no other odd parts are repeated, and no parts are as large as twice the largest odd part.

**Proof.** Again we require only standard arguments to establish these three assertions. We treat \( V_3(q) \) as typical. Thus,

\[
V_3(q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{4n+2}}{(q^2; q^2)_{2n}}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(1 + q)(1 + q^3) \cdots (1 + q^{2n-1})} q^{(2n+1)(2n+1) + (2n+1)}
\]

\[
\times \frac{1}{1 - q^2} \frac{1}{1 - q^4} \cdots \frac{1}{1 - q^{4n}},
\]

and the geometric series expansion of each of the factors \( 1/(1 - q^{2i}) \) yields the partitions as exponents of \( q \) when all the expressions are multiplied out. \( \square \)

Since \( V_1(q) \), \( V_2(q) \), and \( V_3(q) \) have partition-theoretic interpretations, it is natural to ask if (2.5.4)–(2.5.6), upon possibly multiplying both sides by \( \psi(-q) \) in each case, have meaningful partition-theoretic interpretations. If so, then one should try to find bijective proofs.
Entry 2.4.1, Entry 2.4.2, Entry 2.4.3, Entry 2.4.4, Entry 2.4.5, Entry 2.4.6,
Entry 2.4.7.

Page 25

Entry 2.4.8, Entry 2.4.9, Entry 2.4.10, Entry 2.4.11, Entry 2.4.12, Entry
2.4.13, Entry 2.4.14.
3.1 Introduction

Partial fractions arise again and again in the Lost Notebook. Indeed, we have already seen instances of partial fractions (e.g. [19, p. 271]) that specialize to mock theta functions. We have reserved for this chapter those results on partial fractions that seem especially close to the third order mock theta functions.

Our starting point is a corollary of Lemma 2.3.2 from page 19 of Part III [21].

Theorem 3.1.1. For \( z, \zeta \in \mathbb{C} \),

\[
R(z, \zeta, q) := \zeta^3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^3 q^{n(3n+1)/2}}{1 - z\zeta q^n} + \zeta \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-3} q^{3n(n+1)/2}}{1 - zq^n / \zeta} \\
- \zeta \frac{(\zeta^2 q^2 / \zeta^2; q)}{(\zeta, q / \zeta; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - zq^n} \\
= \frac{z(\zeta, q \zeta / z, z, q / z, z\zeta, q / (z\zeta); q)_\infty}{(z / \zeta, q\zeta / z, z, q / z, z\zeta, q / (z\zeta); q)_\infty}.
\]

(3.1.1)

Proof. Define

\[
S(z, \zeta, q) := \zeta^3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^3 q^{3n(n+1)/2}}{1 - z\zeta q^n} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-3} q^{3n(n+1)/2}}{1 - zq^n / \zeta} \\
- \zeta \frac{(\zeta^2 q^2 / \zeta^2; q)}{(\zeta, q / \zeta; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - zq^n}.
\]

Then Lemma 2.3.2 of Part III [21, p. 19] asserts that

\[
S(z, \zeta, q) = \frac{(\zeta, q / \zeta, \zeta^2 q / \zeta^2, q, q; q)_{\infty}}{(z / \zeta, q\zeta / z, z, q / z, z\zeta, q / (z\zeta); q)_{\infty}}.
\]
So to conclude the proof of this theorem we need only show that

\[ S(z, \zeta, q) = \frac{1}{z} R(z, \zeta, q). \]

Now, using the pentagonal number theorem (1.3.27) in the second equality below, we find that

\[
S(z, \zeta, q) = \zeta^3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z \zeta q^n} \left( \frac{1}{z \zeta} - \frac{1}{z \zeta} \left( 1 - \frac{z}{\zeta} q^n \right) \right)
+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z q^n / \zeta} \left( \frac{\zeta}{z} - \zeta \left( 1 - \frac{z}{\zeta} q^n \right) \right)
- \frac{\zeta (\zeta^2 q / \zeta^2 : q)_{\infty}}{(\zeta, q/\zeta : q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - z q^n} \left( \frac{1}{z} - \frac{1}{z} (1 - z q^n) \right)
\]

\[
= \frac{1}{z} R(z, \zeta, q) - \frac{\zeta}{z} \left( \zeta \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z q^n} \left( \frac{1}{z} - \frac{1}{z} (1 - z q^n) \right) \right)
+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z q^n} - \frac{(\zeta^2 q / \zeta^2 : q)_{\infty}}{(\zeta, q/\zeta : q)_{\infty}} (q : q)_{\infty}
= \frac{1}{z} R(z, \zeta, q),
\]

because the expression inside the large parentheses equals 0, in that it is a restatement of the quintuple product identity [21, p. 221, equation (8.2.18)], [31, p. 18]. In particular, if we take the formulation from [31]

\[
\sum_{n=-\infty}^{\infty} q^{3n^2 + n} (z q^{3n} - z^{3n+1} q^{3n+1})
= (q^2 : q^2)_{\infty} (q z : q^2)_{\infty} (q^2 : q^4)_{\infty} (q^2 : q^4)_{\infty} (q : q)_{\infty}, \quad (3.1.2)
\]

replace \( q \) by \( \sqrt{q} \), and then set \( z = -\zeta \sqrt{q} \), we find that the sum of the expressions within large parentheses on the far right side above equals 0. \( \square \)

### 3.2 The Partial Fraction Identities

Each of the entries in this section involves

\[
f_n(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n} (1 + a q^j + q^{2j})}. \quad (3.2.1)
\]

We shall use again and again the observation that if

\[ a = -x - x^{-1}, \]

then

\[
 f_a(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq; q)_{x-1} q^n (q; q)_{x+1}} = \frac{(1-x)}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^n (3n+1)/2}{1-x q^n}, \tag{3.2.2}
\]

by (1.3.20).

The following entry from Ramanujan’s lost notebook was first proved by H. Yesilyurt [146].

**Entry 3.2.1 (p. 2).** If \(a^2 + b^2 = 4\), then

\[
 \begin{align*}
 b - a + 2 \frac{f_a(-q)}{4} + b + a + 2 \frac{f_{-a}(-q) - b}{2} f_b(q) \\
 = \frac{(q^4; q^4)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^{\infty} \frac{(1 - bq^n + q^{2n})}{(1 + (a^2b^2 - 2)q^n + q^{2n})}. \tag{3.2.3}
\end{align*}
\]

**Proof.** First we note that we may parameterize the circle \(a^2 + b^2 = 4\) by

\[
 b = -2 \cos \theta, \quad a = -2 \sin \theta, \tag{3.2.4}
\]

and with \(z = e^{i\theta}\), we find that

\[
 b = -z - z^{-1} \quad \text{and} \quad a = i(z - z^{-1}). \tag{3.2.5}
\]

Hence,

\[
 a^2 b^2 - 2 = -z^4 - z^{-4}. \tag{3.2.6}
\]

Let us now set \(\zeta = i = \sqrt{-1}\) in Theorem 3.1.1. Thus, by (3.2.2), (3.2.5), and (3.2.6),

\[
 \begin{align*}
 - &\sum_{n=-\infty}^{\infty} \frac{(-1)^n i^{3n} q^n (3n+1)/2}{1 - z i q^n} + i \sum_{n=-\infty}^{\infty} \frac{(-1)^n i^{-3n} q^n (3n+1)/2}{1 + z i q^n} \\
 &- i(1, -q; q)_\infty \cdot (q; q) \prod_{n=1}^{\infty} \frac{(1 - bq^n + q^{2n})}{(1 + (a^2b^2 - 2)q^n + q^{2n})}. \tag{3.2.7}
\end{align*}
\]

Multiply both sides of (3.2.7) by

\[
 \frac{1 + z^2}{2z(1 - i)(-q; q)_\infty}. \tag{3.2.7}
\]

Upon doing so, we then see that the right-hand side of (3.2.7) becomes the right-hand side of (3.2.3). The third expression on the left-hand side of (3.2.7) then becomes
Therefore, we will complete the proof if we can show that

\[
\frac{(1 + z^2)(1 - z)(-i) (-1, -q^2 q; q)_\infty}{2z(1 - i)(-q; q)_\infty (i, -iq; q)_\infty (1 - z)} f_b(q) = -\frac{b}{2} f_b(q).
\]

Combining sums on each side of (3.2.9), we see that our task has been reduced to proving that

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n i^{3n} q^{n(3n+1)/2}}{1 + ziq^n} + i \sum_{n=-\infty}^{\infty} \frac{(-1)^n i^{-3n} q^{n(3n+1)/2}}{1 + ziq^n} = 0.
\]

Combining sums on each side of (3.2.9), we see that our task has been reduced to proving that

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 + z^2 q^{2n}} \left( -i^{3n}(1 + ziq^n) + i^{-3n}(1 - ziq^n) \right) = 0,
\]

and this follows immediately because

\[-i^{3n} + i^{-3n} = (-1)^n(3n+1)/2(i - 1)\]

and

\[-i^{1+3n} - i^{2-3n} = (-1)^n(3n+1)/2, (-1 + i).\]

The last two assertions are most easily proved by noting that each expression is periodic with period 4, and the assertions hold for \(n = 0, 1, 2, 3\). \(\square\)
3.2 The Partial Fraction Identities

Entry 3.2.2 (p. 2). If \( a^2 + ab + b^2 = 3 \), then

\[
(a + 1)f_{-a}(q) + (b + 1)f_{-b}(q) - (a + b - 1)f_{a+b}(q) = \frac{3(q^3; q^3)_{\infty}^2}{(q; q)_{\infty} \prod_{n=1}^{\infty} (1 + ab(a + b)q^{3n} + q^{6n})}. \tag{3.2.10}
\]

Proof. First we note that we may parameterize the ellipse \( a^2 + ab + b^2 = 3 \) by \( a = 2 \cos (\theta + 2\pi/3) \), \( b = 2 \cos \theta \). So with \( z = e^{i\theta} \), we find that

\[
b = z + z^{-1} \quad \text{and} \quad a = z\omega + (z\omega)^{-1}, \tag{3.2.11}
\]

where \( \omega = e^{2\pi i/3} \). Hence,

\[
a + b = -z\omega^2 - (z\omega^2)^{-1} \tag{3.2.12}
\]

and

\[
ab(a + b) = -z^3 - z^{-3}. \tag{3.2.13}
\]

Therefore, we now set \( \zeta = \omega \) in (3.1.1). Thus, the resulting right-hand side, by (3.2.12), equals

\[
\frac{z(1 - \omega)(1 - \omega^2)(q^3; q^3)^2_{\infty}}{(1 - z^3)(z^3q^3, z^{-3}q^3; q^3)_{\infty}} = \frac{z(q; q)_{\infty}}{(1 - z^3)(1 - z^3)} \frac{3(q^3; q^3)_{\infty}^2}{(q; q)_{\infty} \prod_{n=1}^{\infty} (1 + ab(a + b)q^{3n} + q^{6n})}. \tag{3.2.14}
\]

We now observe that the latter quotient on the right-hand side of (3.2.14) is the same as the right-hand side of (3.2.10). We are thus led to multiply the left-hand side of (3.1.1) with \( \zeta = \omega \) by

\[
\frac{(1 - z^3)}{z(q; q)_{\infty}}
\]

to deduce, with the help of three applications of (3.2.2), that

\[
\frac{(1 - z^3)}{z(q; q)_{\infty}} \left( \omega^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - z\omega q^n} + \omega \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - z\omega^2 q^n} \right) - \frac{\omega(1 - \omega^2)}{(1 - \omega)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - zq^n}
\]

\[
= \frac{(1 - z^3)}{z(1 - \omega z)^3} f_{-a}(q) + \frac{(1 - z^3)\omega}{z(1 - \omega^2 z)} f_{a+b}(q) - \frac{(1 - z^3)\omega(1 - \omega^2)}{z(1 - \omega)(1 - z)} f_{-b}(q)
\]

\[
= (a + 1)f_{-a}(q) - (a + b - 1)f_{a+b}(q) + (b + 1)f_{-b}(q),
\]

which is the left-hand side of (3.2.10). This completes the proof. \qed
3.3 Specializations

Referring to the notation of (3.2.1), we see that

\[ f_{-2}(q) = \frac{1}{(q; q)_\infty}, \quad (3.3.1) \]

by (1.3.8), and

\[ f_{2}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}, \]

which is the third order mock theta function \( f_3(q) \) defined in (1.3.2). Also, by (1.3.3) and (1.3.18), respectively,

\[ f_{0}(q) = \phi_{3}(q), \quad (3.3.2) \]

and

\[ f_{-1}(q) = \chi_{3}(q). \quad (3.3.3) \]

If we set \( a = 2 \) and \( b = 0 \) and replace \(-q\) by \( q \) in Entry 3.2.1, we obtain (3.3.1). If we set \( b = 2 \) and \( a = 0 \) in Entry 3.2.1, employ (3.3.2) and (1.3.1), and use the fact that \( f_{2}(q) = f_{3}(q) \), we find, in the notation of Entry 1.3.3, that

\[ 2\phi_{3}(-q) - f_{3}(q) = \frac{\psi^2(-q)}{(q; q)_\infty}, \]

which is the assertion in (1.3.5).

If we set \( a = b = 1 \) in Entry 3.2.2, use (3.3.3), and also use (1.3.1), we find that

\[ 4\chi_{3}(q) - f_{3}(q) = \frac{3\psi^2(-q^3)}{(q; q)_\infty}, \]

which is Entry 1.3.3.

**Entry 3.3.1 (p. 17).** We have

\[
\frac{1 + \sqrt{3}}{2} f_{-1}(-q) + \frac{3 + \sqrt{3}}{6} f_{1}(-q) - f_{\sqrt{3}}(q) = \frac{2}{\sqrt{3}} \psi^2(-q) \frac{(q^4; q^4)_\infty}{(q^6; q^6)_\infty} \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{3} q^n + q^{2n}}. \quad (3.3.4)
\]

**Proof.** Let \( a = 1 \) and \( b = \sqrt{3} \) in Entry 3.2.1 to deduce that

\[
\frac{1 + \sqrt{3}}{4} f_{1}(-q) + \frac{3 + \sqrt{3}}{4} f_{-1}(-q) - \frac{\sqrt{3}}{2} f_{\sqrt{3}}(q) = \frac{(q^4; q^4)_\infty}{(q^6; q^6)_\infty} \prod_{n=1}^{\infty} \frac{1 - \sqrt{3} q^n + q^{2n}}{1 + q^{4n} + q^{8n}}. \quad (3.3.5)
\]
3.3 Specializations

Now multiply both sides of (3.3.5) by \( \frac{2}{\sqrt{3}} \) to arrive at

\[
\frac{3 + \sqrt{3}}{6} f_1(-q) + \frac{1 + \sqrt{3}}{2} f_{-1}(-q) - f_{\sqrt{3}}(q)
= \frac{2}{\sqrt{3}} \frac{(q^4; q^4)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^\infty \frac{1 - \sqrt{3}q^n + q^{2n}}{1 + q^{4n} + q^{8n}}.
\]

(3.3.6)

Examining (3.3.4) and (3.3.6), we see that we are required to show that

\[
\psi(-q) = \frac{(q^6; q^6)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^\infty \frac{1 - \sqrt{3}q^n + q^{2n} + \sqrt{3}q^n + q^{2n}}{1 + q^{4n} + q^{8n}}.
\]

(3.3.7)

To that end,

\[
\frac{(q^6; q^6)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^\infty \frac{1 - \sqrt{3}q^n + q^{2n} + \sqrt{3}q^n + q^{2n}}{1 + q^{4n} + q^{8n}}
= \frac{(q^6; q^6)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^\infty \frac{1 - q^{2n} + q^{4n}}{1 + q^{4n} + q^{8n}}
= \frac{(q^6; q^6)_\infty}{(-q; q^2)_\infty} \prod_{n=1}^\infty \frac{(1 - q^{2n})}{(1 + q^{2n} + q^{4n})(1 - q^{2n})}
= \frac{(q^6; q^6)_\infty (q^2; q^2)_\infty}{(-q; q^2)_\infty (q^6; q^6)_\infty} \frac{(q^2; q^2)_\infty}{(1 - q; q^2)_\infty} = \psi(-q),
\]

by (1.3.11). Thus, we have shown (3.3.7), and so the proof of Entry 3.3.1 is finished.

We note that in [142] Watson proved many of the entries in Chapter 1 using an identity equivalent to Theorem 3.1.1. We have chosen to limit this chapter to these two identities generalizing third order mock theta functions.

It is uncertain whether Ramanujan used the methods of Chapter 1 or those of partial fractions to prove his identities for third order mock theta functions. It is clear that he was the master of both methods.

The identities of this chapter were first established by H. Yesilyurt [146]. Our approaches to Entries 3.2.1 and 3.2.2 are different from his, but our proof of Entry 3.3.1 is the same as his.
4.1 Introduction

This chapter and the next are devoted to the proofs of ten identities from the Lost Notebook. We place them, five each, in the following two entries. The remainder of this chapter is devoted to proving that the assertions in each entry are equivalent, i.e., they are all true or all false. The following chapter is devoted to proving that the fifth identity in each entry is true.

In formulating these assertions, we use the formulation that appears in [23]. This will most easily facilitate the equivalence proofs that comprise the remainder of the chapter. We refer the reader to (2.1.1)–(2.1.13) for the definitions of the ten fifth order mock theta functions.

Besides $\varphi(q)$ and $\psi(q)$, several other special cases of Ramanujan’s general theta function $f(a,b)$, defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

arise in the sequel. In particular, use is often made of the Jacobi triple product identity [28, p. 35, Entry 19]

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \quad (4.1.2)$$

We also need the factorisation formula [28, p. 34, Entry 18(iv)]; for any integer $n$,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab))^n, b(ab)^{-n}. \quad (4.1.3)$$

Furthermore [28, p. 34, Entry 18(iii)],

$$f(-1, q) = 0. \quad (4.1.4)$$

**Entry 4.1.1 (pp. 18, 19).** Let
and

\[ A(q) := \frac{G^2(q)(q^5;q^5)_\infty}{H(q)} \tag{4.1.5} \]

and

\[ \Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q^5)_{n+1}(q^4; q^5)_n}. \tag{4.1.6} \]

Then

\[ M_1(q) := \chi_0(q) - 2 - 3\Phi(q) + A(q) = 0, \tag{4.1.7} \]
\[ M_2(q) := F_0(q) - 1 - \Phi(q) + q\psi(q^5)H(q^2) = 0, \tag{4.1.8} \]
\[ M_3(q) := \phi_0(-q) + \Phi(q) - \frac{(q^7; q^5)_\infty G(q^2)H(q)}{H(q^2)} = 0, \tag{4.1.9} \]
\[ M_4(q) := \psi_0(q) - \Phi(q^2) - qH(q)f(-q^5, -q) = 0, \tag{4.1.10} \]
\[ M_5(q) := F_0(q) + 2\Phi(q^2) - \varphi(-q^5)G(q) = 0. \tag{4.1.11} \]

**Entry 4.1.2 (pp. 19–20).** Let

\[ D(q) := \frac{H^2(q)(q^5; q^5)_\infty}{G(q)} \tag{4.1.12} \]

and

\[ \Psi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2; q^5)_{n+1}(q^3; q^5)_n}. \tag{4.1.13} \]

Then

\[ M_6(q) := q\chi_1(q) - 3\Phi(q) - qD(q) = 0, \tag{4.1.14} \]
\[ M_7(q) := qF_1(q) - \Psi(q) - q\psi(q^5)G(q^2) = 0, \tag{4.1.15} \]
\[ M_8(q) := -\phi_1(-q) + \Psi(q) - \frac{q(q^5; q^5)_\infty G(q)H(q^2)}{G(q^2)} = 0, \tag{4.1.16} \]
\[ M_9(q) := \psi_1(q) - \frac{1}{q}\Psi(q^2) - G(q)f(-q^7, -q^3) = 0, \tag{4.1.17} \]
\[ M_{10}(q) := F_1(q) + \frac{2}{q}\Psi(q^2) - \varphi(-q^5)H(q) = 0. \tag{4.1.18} \]

We proceed to establish Entries 4.1.1 and 4.1.2 in this chapter and the next as follows. In Section 4.2, we use Ramanujan’s \( \psi_1 \)-summation to represent several theta products as Lambert series. In Section 4.3, we prove that the functions \( M_i(q) \) are identical for \( 1 \leq i \leq 5 \), and in Section 4.4 we prove that the functions \( M_i(q) \) are identical for \( 6 \leq i \leq 10 \). In Chapter 5, we provide D. Hickerson’s proofs \([86]\) that \( M_5(q) = 0 \) and \( M_{10}(q) = 0 \). These proofs will then finally establish Entries 4.1.1 and 4.1.2. When comparing the identities in Entries 4.1.1 and 4.1.2 with those in Chapter 2, we see that the only fundamentally different feature is that Entries 4.1.1 and 4.1.2 contain, respectively, the functions \( \Phi(q) \) and \( \Psi(q) \).
4.2 Fourteen Lemmas

Every result in this section is proved in exactly the same way. Each theta product is identified with an instance of Ramanujan’s \( \psi_1 \)-summation [19, p. 118, equation (4.4.6)] given by

\[
\psi_1 \left( \frac{a}{b}; q, z \right) = \frac{(q, b/a, az, q/(az); q)_\infty}{(b, q/a, z, b/(az); q)_\infty}. \tag{4.2.1}
\]

where \(|b/a| < |z| < 1\). Indeed, the applications will all follow from the identity

\[
\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - bq^{5n}} = \frac{(bz, q^2/(bz), q^3, q^5; q^5)_\infty}{(z, q^5/z, b, q^5/b; q^5)_\infty}, \tag{4.2.2}
\]

In (4.2.1), replace \( q \) by \( q^5 \), then replace \( b \) by \( bq^5 \), then replace \( a \) by \( b \), and finally divide both sides by \( 1 - b \). We therefore arrive at (4.2.2).

**Lemma 4.2.1.** Let \( \varphi(q) \) and \( G(q) \) be defined by (1.3.1) and (2.1.16), respectively. Then

\[
\varphi(-q)G(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 + q^{5n+1}}. \tag{4.2.3}
\]

**Proof.** By (1.3.1) and (2.1.4),

\[
\varphi(-q)G(q) = \frac{(q^5, q^2, q^3; q^5)_\infty}{(-q; q)_\infty} = \frac{1}{(q^{10}; q^{10})_\infty} \frac{(q^3, q^2, q^5; q^5)_\infty}{(-q, -q^2, -q^3 - q^4; q^5)_\infty} = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 + q^{5n+1}},
\]

which follows from (4.2.2) with \( z = -q^2 \) and \( b = -q \).  

**Lemma 4.2.2.** Recall that \( \varphi(q) \) and \( H(q) \) are defined by (1.3.1) and (2.1.17), respectively. Then

\[
\varphi(-q)H(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}q^{2n+1}}{1 + q^{5n+1}}. \tag{4.2.4}
\]

**Proof.** By (1.3.1) and (2.1.5),
\[ \varphi(-q)H(q) = \frac{(q^5, q, q^4; q^5)_{\infty}}{(-q; q)_{\infty}} \]
\[ = \frac{1}{(q^{10}; q^{10})_{\infty}} \frac{(q^4, q, q^5, q^5; q^5)_{\infty}}{(-q, -q^2, -q^3, -q^4; q^5)_{\infty}} \]
\[ = \frac{(-q)}{(-q)} \frac{(q^6, q^{-1}, q^5, q^5; q^5)_{\infty}}{(-q, -q^2, -q^3, -q^4; q^5)_{\infty}} \]
\[ = \frac{1}{(q^{10}; q^{10})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{2n+1}}{1 + q^{5n+4}}, \]

which follows from (4.2.2) with \( z = -q^2 \) and \( b = -q^4 \).

**Lemma 4.2.3.** If \( \psi(q) \) and \( H(q) \) are defined by (1.3.11) and (2.1.17), respectively, then
\[
\psi(q^2)H(q^4) = \frac{1}{(q^{10}; q^{10})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1 - q^{20n+14}}. \quad (4.2.5)
\]

**Proof.** By (1.3.11) and (2.1.17),
\[
\psi(q^2)H(q^4) = \frac{(q^4, q^{16}, q^{20}; q^{20})_{\infty}}{(q^{2}; q^{4})_{\infty}} \frac{(q^4, q, q^{20}, q^{20}; q^{20})_{\infty}}{(q^{2}, q^{6}, q^{10}, q^{14}, q^{18}, q^{20}, q^{20}; q^{20})_{\infty}} \]
\[ = \frac{1}{(q^{10}; q^{10})_{\infty}} \frac{(q^4, q^{16}, q^{20}; q^{20})_{\infty}}{(q^{2}, q^{6}, q^{14}, q^{18}; q^{20})_{\infty}} \]
\[ = \frac{1}{(q^{10}; q^{10})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1 - q^{20n+14}}, \]

which follows from (4.2.2) with \( q \) replaced by \( q^4 \), and then with \( z = q^2 \) and \( b = q^{14} \).\[\]
\[ \psi(q^2)G(q^4) = \frac{(q^8, q^{12}, q^{20}; q^{20})_\infty}{(q^2; q^4)_\infty} = \frac{1}{(q^{10}, q^{10})_\infty} \frac{(q^8, q^{12}, q^{20}; q^{20})_\infty}{(q^2, q^6, q^{14}, q^{18}, q^{20}; q^{20})_\infty} = \frac{1}{(q^{10}, q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1 - q^{20n+6}}, \]

which follows from (4.2.2) with \( q \) replaced by \( q^4 \), and then with \( z = q^2 \) and \( b = q^6 \).

**Lemma 4.2.5.** Recall that \( A(q) \) is defined by (4.1.5). Then

\[ A(q) = \frac{1}{(q^6; q^6)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{6n+1}}. \]  

**Proof.** Equation (4.2.7) is identical with equation (4.4.9) in [19, p. 119]. We provide a separate proof. Set \( b = z = q^2 \) in (4.2.2) to deduce that

\[ \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{6n+1}} = \frac{(q^2, q^3, q^5, q^5; q^5)_\infty}{(q^2, q^4, q^4, q^4; q^4)_\infty} = \frac{G^2(q^5; q^5)_\infty}{H(q)}, \]

from which (4.2.7) readily follows. \( \square \)

**Lemma 4.2.6.** For \( D(q) \) defined by (4.1.12),

\[ D(q) = \frac{1}{(q^6; q^6)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1 - q^{6n+2}}. \]  

**Proof.** Identity (4.2.8) is the same as identity (4.4.10) in [19, p. 119]. As in the proof of Lemma 4.2.5 above, we can construct a similar proof of Lemma 4.2.6. Setting \( z = b = q^2 \) in (4.2.2), we easily deduce (4.2.8). \( \square \)

**Lemma 4.2.7.** If \( \psi(q) \) and \( H(q) \) are defined by (1.3.11) and (2.1.17), respectively, then

\[ \psi(q^{10})H(q^4) = -\frac{1}{(q^{10}, q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{14n+6}}{1 - q^{20n+12}}. \]  

**Proof.** Replace \( q \) by \( q^4 \), and then set \( z = q^6 \) and \( b = q^8 \) in (4.2.2). On the left side, we obtain

\[ \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1 - q^{20n+8}} = \sum_{n=-\infty}^{\infty} \frac{q^{-14n-8}}{q^{20n-8} - 1} = -\sum_{m=-\infty}^{\infty} \frac{q^{14m+6}}{1 - q^{20m+12}}, \]
where we set \( n = -m - 1 \). On the other hand, on the right-hand side of (4.2.2), with the help of (2.1.17), Euler’s theorem, and (1.3.11), we find that
\[
\left( \frac{q^{14}, q^6, q^{20}, q^{20}; q^{20}}{q^6, q^{14}, q^8, q^{12}; q^{20}} \right)_\infty = H(q^4)(q^{20}; q^{20})^2
\]
\[
= H(q^4)(q^{20}; q^{20}) \left( q^{10}; q^{10} \right)_\infty (-q^{10}; q^{10})_\infty
\]
\[
= H(q^4)(q^{10}; q^{10})_\infty \left( \frac{q^{20}; q^{20}}{q^{10}; q^{20}} \right)_\infty
\]
\[
= H(q^4)(q^{10}; q^{10})_\infty \psi(q^{10}). \tag{4.2.11}
\]
Combining (4.2.10) and (4.2.11), we complete the proof of Lemma 4.2.7. \( \square \)

**Lemma 4.2.8.** If \( \psi(q) \) and \( G(q) \) are defined by (1.3.11) and (2.1.16), respectively, we have
\[
\psi(q^{10})G(q^4) = -\frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} q^{12n+8} \frac{1}{1 - q^{20n+16}}. \tag{4.2.12}
\]

**Proof.** We again apply (4.2.2) with \( q \) replaced by \( q^4 \). Next, set \( z = q^8 \) and \( b = q^4 \). On the left side, we find that
\[
\sum_{n=-\infty}^{\infty} \frac{q^{8n}}{1 - q^{20n+16}} = \sum_{n=-\infty}^{\infty} \frac{q^{-12n-4}}{q^{-20n-4} - 1} = -\sum_{n=-\infty}^{\infty} \frac{q^{12m+8}}{1 - q^{20m+16}}, \tag{4.2.13}
\]
while on the right side, arguing as we did in (4.2.11) above, we arrive at
\[
\frac{(q^{12}, q^8, q^{20}, q^{20}; q^{20})_\infty}{(q^8, q^{12}, q^4, q^{16}; q^{20})_\infty} = G(q^4)(q^{20}; q^{20})^2 = G(q^4)(q^{10}; q^{10})_\infty \psi(q^{10}). \tag{4.2.14}
\]
Taking (4.2.13) and (4.2.14) together, we complete the proof. \( \square \)

**Lemma 4.2.9.** With \( G(q) \) and \( H(q) \) defined by (2.1.16) and (2.1.17), respectively,
\[
\frac{(q^5; q^5)_\infty G(q^2)H(q)}{H(q^2)} = \frac{1}{(q^5; q^5)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+1}}. \tag{4.2.15}
\]

**Proof.** Set \( z = -q \) and \( b = q \) in (4.2.2). Hence, by (2.1.16) and (2.1.17),
\[
\sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+1}} = \frac{(-q^2, -q^3, q^5; q^5)_\infty}{(-q, -q^4, q, q^4; q^5)_\infty}
\]
\[
= \frac{(q^4, q^6, q^{10})_\infty}{(q^2, q^5, q^{10})_\infty} \frac{(q^2, q^4, q^5; q^5)_\infty}{(q^2, q^4, q^5; q^5)_\infty} \frac{G(q^2)H(q)(q^5; q^5)_\infty^2}{H(q^2)},
\]
which is (4.2.15). \( \square \)
**Lemma 4.2.10.** For \( G(q) \) and \( H(q) \) defined by (2.1.16) and (2.1.17), respectively,

\[
\frac{(q^5; q^3)_\infty H(q^2)G(q)}{G(q^2)} = \frac{1}{(q^5; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 - q^{3n+2}}.
\]

**Proof.** Set \( z = -q^2 \) and \( b = q^2 \) in (4.2.2) to deduce that

\[
\sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 - q^{3n+2}} = \frac{(-q^4, -q, q^5; q^5)_\infty}{(-q^2, -q^3, q^3, q^9)_\infty} = \frac{(q^8, q^7; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty} \frac{(q^2, q^3, q^5, q^7; q^9)_\infty}{(q, q^4, q^2, q^3; q^9)_\infty} = \frac{H(q^2)G(q)(q^5; q^5)_\infty}{G(q^2)}
\]

by (2.1.16) and (2.1.17), and so the proof is complete. \( \square \)

**Lemma 4.2.11.** For \( H(q) \) defined in (2.1.17) and \( f(a, b) \) defined in (4.1.1),

\[
f(-q^9, -q)H(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{7n}}{1 - q^{10n+2}}.
\]

**Proof.** Replace \( q \) by \( q^2 \) in (4.2.2), and then put \( z = q^7 \) and \( b = q^2 \). By (2.1.17) and the Jacobi triple product identity (4.1.2),

\[
\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{1 - q^{10n+2}} = \frac{(q, q^9, q^{10}; q^{10})_\infty}{(q^2, q^7, q^8, q^{10})_\infty} = \frac{f(-q, -q^7)(q^{10}; q^{10})_\infty}{(q^2; q^5)_\infty (q^3; q^9)_\infty} = f(-q, -q^7)(q^{10}; q^{10})_\infty H(q),
\]

and so (4.2.17) readily follows. \( \square \)

**Lemma 4.2.12.** For \( G(q) \) defined in (2.1.16), we have

\[
f(-q^3, -q^7)G(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{10n+6}}.
\]

**Proof.** Apply (4.2.2) with \( q \) replaced by \( q^2 \), and then with \( z = q \) and \( b = q^6 \). By (2.1.16) and the Jacobi triple product identity (4.1.2),

\[
\sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{10n+6}} = \frac{(q^4, q^7, q^{10}; q^{10})_\infty}{(q, q^6, q^4, q^9; q^{10})_\infty} = \frac{f(-q^3, -q^7)(q^{10}; q^{10})_\infty}{(q, q^4; q^5)_\infty} = f(-q^3, -q^7)(q^{10}; q^{10})_\infty G(q).
\]

Hence, we have proved (4.2.18). \( \square \)
Lemma 4.2.13. Let \( \varphi(q) \) and \( G(q) \) be defined, respectively, by (1.3.1) and (2.1.16). Then

\[
\varphi(-q^5)G(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 - q^{5n+1}}.
\]

Proof. Employ (4.2.2) with \( z = -q^2 \) and \( b = q^2 \). Thus,

\[
\sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1 - q^{5n+1}} = \frac{(q^3, -q^2, q^5; q^5)_\infty}{(q^2, -q^3, q, q^4; q^5)_\infty} = G(q)(q^5; q^5)_\infty
\]

\[
= \frac{G(q)(q^{10}; q^{10})_\infty(q^5; q^5)_\infty}{(-q^5; q^5)_\infty} = G(q)(q^{10}; q^{10})_\infty \varphi(-q^5),
\]

by (2.1.16) and (1.3.1). Thus, (4.2.19) follows. \( \square \)

Lemma 4.2.14. If \( \varphi(q) \) and \( H(q) \) are defined, respectively, by (1.3.1) and (2.1.17), then

\[
\varphi(-q^5)H(q) = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+3}}.
\]

Proof. The proof is almost identical to the previous proof. Utilize (4.2.2) with \( z = -q \) and \( b = q^3 \). Therefore,

\[
\sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+3}} = \frac{(q^4, -q, q^5; q^5)_\infty}{(-q, -q^4, q^2; q^5)_\infty}
\]

\[
= H(q)(q^5; q^5)_\infty = \varphi(-q^5)H(q)(q^{10}; q^{10})_\infty,
\]

by (1.3.1) and (2.1.16), and so the proof is done. \( \square \)

4.3 The Relations Among \( M_i(q) \), \( 1 \leq i \leq 5 \)

Theorem 4.3.1. Recall that \( M_1(q), \ldots, M_5(q) \) are defined in (4.1.7)–(4.1.11). Then

\[
M_5(q) = 2M_3(q^2),
\]

\[
M_5(q) = -2M_2(q^2),
\]

\[
M_4(-q) = -M_3(q^2),
\]

\[
M_4(q^2) = -3M_3(q^2).
\]

Proof. We prove the four identities in order. We first prove (4.3.1). Using (4.1.11) and (4.1.9) and then Entry 2.4.7, we find that
where Lemmas 4.2.1 (with $q$ replaced by $q^2$), 4.2.9, and 4.2.13 were utilized in the ante-penultimate line, and the first and third sums were combined in the next line. This completes the proof of (4.3.1).

We next turn to the proof of (4.3.2). By (4.1.11), (4.1.8), and Entry 2.4.6, we establish (4.3.3). First using (4.1.10) and (4.1.9), second employing Entry 2.4.5, and third invoking Lemmas 4.2.1, 4.2.9 (with $q$ replaced by $q^2$), and 4.2.11 (with $q$ replaced by $q^5$), we find that

$$M_5(q) - 2M_2(q^2) = f_0(q) - 2\phi_0(-q^2) - \varphi(-q^5)G(q)$$

$$+ \frac{2(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}$$

$$= -\varphi(-q)G(q) - \varphi(-q^5)G(q) + \frac{2(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}$$

$$= -\frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 + q^{5n+1}} - \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - q^{5n+1}}$$

$$+ \frac{2}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - q^{10n+2}}$$

$$= \frac{2}{(q^{10}; q^{10})_\infty} \left( - \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - q^{10n+2}} + \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - q^{10n+2}} \right)$$

$$= 0,$$  \hspace{1cm} (4.3.5)

where in the ante-penultimate line we employed Lemmas 4.2.1, 4.2.13, and 4.2.9, with $q$ replaced by $q^2$, and in the penultimate line we combined the first two sums on the previous line. This completes the proof of (4.3.1).

We next turn to the proof of (4.3.2). By (4.1.11), (4.1.8), and Entry 2.4.6,

$$M_5(q) + 2M_2(q^2) = f_0(q) + 2F_0(q^2) - 2 + 2q^2\psi(q^{10})H(q^4) - \varphi(-q^5)G(q)$$

$$= \varphi(q)G(-q) + 2q^2\psi(q^{10})H(q^4) - \varphi(-q^5)G(q)$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - (-1)^nq^{5n+1}} - 2 \sum_{n=-\infty}^\infty \frac{q^{14n+8}}{1 - q^{20n+12}} \right)$$

$$- \sum_{n=-\infty}^\infty \frac{(-q^2)^n}{1 - q^{5n+1}}$$

$$= \frac{1}{(q^{10}; q^{10})_\infty} \left( -2 \sum_{n=-\infty}^\infty \frac{(-q^2)^{2n+1}q^{10n+6}}{1 - q^{20n+12}} - 2 \sum_{n=-\infty}^\infty \frac{q^{14n+8}}{1 - q^{20n+12}} \right)$$

$$= 0,$$  \hspace{1cm} (4.3.6)

where Lemmas 4.2.1 (with $q$ replaced by $-q$), 4.2.7, and 4.2.13 were utilized in the ante-penultimate line, and the first and third sums were combined in the next line.

We establish (4.3.3). First using (4.1.10) and (4.1.9), second employing Entry 2.4.5, and third invoking Lemmas 4.2.1, 4.2.9 (with $q$ replaced by $q^2$), and 4.2.11 (with $q$ replaced by $-q$), we find that

$$M_4(-q) + M_3(q^2)$$

$$= \psi_0(-q) + \phi_0(-q^2) - \frac{(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)} + qH(-q)f(q^6, q)$$
\[ \varphi(-q)G(q) - \frac{(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)} + qH(-q)f(q^9, q) \]
\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1+q^{5n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{10n+2}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{7n+1}}{1-q^{10n+2}} \right) \]
\[ = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n(1-q^{5n+1}) - (-q^2)^n + (-1)^n q^{7n+1}}{1-q^{10n+2}} = 0, \quad (4.3.7) \]

which completes the proof of (4.3.3).

Finally, we prove (4.3.4). First, use (4.1.7) and (4.1.9). Second, we replace \( q^2 \) in Entry 2.4.9, add Entry 2.4.6, and lastly subtract Entry 2.4.7. Third, we apply Lemmas 4.2.1 (twice, but with \( q^2 \) replaced by \(-q\) once), 4.2.5, and 4.2.12. Accordingly, we find that

\[ M_1(q^2) + 3M_2(q^2) \]
\[ = \chi_0(-q^2) + 3\phi_0(-q^2) - 2 + A(q^2) - \frac{3(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)} \]
\[ = \varphi(q)G(-q) + \varphi(-q)G(q) + A(q^2) - \frac{3(q^{10}; q^{10})_\infty G(q^4)H(q^2)}{H(q^4)} \]
\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1+q^{5n+1}} + \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1+(q^{10n+1})} \right) + \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+2}} - 3 \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n}{1-q^{10n+2}} \]
\[ = \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-q^2)^n(1+(-1)^n q^{5n+1}) + (1-q^{5n+1}) + (-1)^n - 3}{1-q^{10n+2}}. \]

We now observe that the even indexed terms vanish on the far right side above. Thus, replacing \( n \) by \( 2n+1 \), we conclude that

\[ M_1(q^2) + 3M_3(q^2) = - \frac{1}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}(-2-2q^{10n+6})}{1-q^{20n+12}} \]
\[ = \frac{2q^2}{(q^{10}; q^{10})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+6}} = 0, \quad (4.3.8) \]

by (4.2.2) with \( q \) replaced by \( q^2 \), and then with \( z = q^4 \) and \( b = q^6 \). This completes the proof of (4.3.4). \( \square \)

**Corollary 4.3.1.** If any one of \( M_i(q) \), \( 1 \leq i \leq 5 \), is identically equal to 0, then all are identically equal to 0.
4.3 The Relations Among $M_i(q)$, $1 \leq i \leq 5$

**Proof.** Corollary 4.3.1 follows immediately from (4.3.1)–(4.3.4).

In summary, note that the strategy above is to eliminate the presence of $\Phi(q)$. To obtain a more transparent proof of the mock theta conjectures, it would seem that we would need a better understanding of $\Phi(q)$.

**Theorem 4.3.2.** If $M_i(q)$, $6 \leq i \leq 10$, are defined by (4.1.14)–(4.1.18), then

\[
M_{10}(q) = \frac{2}{q} M_8(q^2),
\]

\[
M_{10}(q) = -\frac{2}{q} M_7(q^2),
\]

\[
M_9(q) = -\frac{1}{q} M_8(q),
\]

\[
M_6(q) = -3 M_8(q).
\]

**Proof.** We prove these identities in order. To prove (4.3.9), we first use (4.1.18) and (4.1.16), second employ Entry 2.4.3, and third utilize Lemmas 4.2.2, 4.2.14, and 4.2.10 (with $q$ replaced by $q^2$) to deduce that

\[
M_{10}(q) - \frac{2}{q} M_8(q^2)
\]

\[
= f_1(q) + \frac{2}{q} \phi_1(-q^2) - \varphi(-q^5)H(q) + \frac{2q(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)}
\]

\[
= \varphi(-q)H(q) - \varphi(-q^5)H(q) + \frac{2q(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)}
\]

\[
= \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n+1}}{1 + q^{5n+4}} - \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+3}} \right)
\]

\[
+ 2q \sum_{n=-\infty}^{\infty} \frac{(-q^4)^n}{1 - q^{10n+4}}
\]

\[
= \frac{1}{(q^4; q^4)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n}}{1 + q^{5n+4}} - \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+3}} \right)
\]

\[
+ 2q \sum_{n=-\infty}^{\infty} \frac{(-q^4)^n}{1 - q^{10n+4}}
\]

where we replaced $n$ by $-n - 1$ in the first sum in the penultimate line. We now note that (4.2.2) is symmetric in $b$ and $z$. Using this observation in the second and third sums on the far right side above, we find that
The Mock Theta Conjectures: Equivalence

\[ M_{10}(q) - \frac{2}{q} M_7(q^2) \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n}}{1 + q^{5n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1 + q^{5n+1}} + 2q \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1 + q^{10n+4}} \right) \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( -2 \sum_{n=-\infty}^{\infty} \frac{q^{6n+3}}{1 + q^{10n+6}} + 2q \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1 + q^{10n+4}} \right) \]

\[ = 0, \quad (4.3.13) \]

upon the replacement of \( n \) by \(-n-1\) in the first sum of the antepenultimate line. This proves (4.3.9).

For the proof of (4.3.10), we first use (4.1.18) and (4.1.15), then we apply Entry 2.4.2, and lastly we invoke Lemmas 4.2.2 (with \( q \) replaced by \(-q\)), 4.2.14, and 4.2.8. Accordingly,

\[ M_{10}(q) + \frac{2}{q} M_7(q^2) = f_1(q) + 2qF_1(q^2) - \varphi(-q^5)H(q) - 2q\psi(q^{10})G(q^4) \]

\[ = \varphi(q)H(-q) - \varphi(-q^5)H(q) - 2q\psi(q^{10})G(q^4) \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n+1}}{1 + (-1)^n q^{5n+4}} - \sum_{n=-\infty}^{\infty} \frac{(-q)^n}{1 - q^{5n+3}} \right) \]

\[ + 2 \sum_{n=-\infty}^{\infty} \frac{q^{12n+9}}{1 - q^{20n+16}} \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1 - (-1)^n q^{5n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1 + q^{5n+1}} \right) \]

\[ + 2 \sum_{n=-\infty}^{\infty} \frac{q^{12n+9}}{1 - q^{20n+16}} \],

where we replaced \( n \) by \(-n-1\) in the first sum and then used the symmetry in \( b \) and \( z \) from (4.2.2) in the second sum on the right-hand side of the next-to-last equality above. The odd indexed terms in the first two sums on the resulting right-hand side cancel. We then combine the first two sums and apply (4.2.2) twice, in the first time with \( q \) replaced by \( q^4 \), \( z = q^{16} \), and \( b = q^2 \), and in the second time with \( q \) replaced by \( q^4 \), \( z = q^{12} \), and \( b = q^{16} \). Hence,

\[ M_{10}(q) + \frac{2}{q} M_7(q^2) \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1 - q^{10n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{6n}}{1 + q^{10n+1}} + 2 \sum_{n=-\infty}^{\infty} \frac{q^{12n+9}}{1 - q^{20n+16}} \right) \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( 2 \sum_{n=-\infty}^{\infty} \frac{q^{16n+1}}{1 - q^{20n+2}} + 2 \sum_{n=-\infty}^{\infty} \frac{q^{12n+9}}{1 - q^{20n+16}} \right) \]
and so the proof of (4.3.10) is finished.

We turn now to the proof of (4.3.11). We use (4.1.17) and (4.1.16), then we employ Entry 2.4.1, and finally apply Lemmas 4.2.2 (with \( q = 2 \)), 4.2.12, and 4.2.10 (with \( q = 3 \)). Hence,

\[
M_9(q) + \frac{1}{q} M_8(q^2)
\]

\[
= \psi_1(q) - \frac{1}{q} \phi_1(-q^2) - G(q) f(-q^7, -q^3) - \frac{q(q^{10}; q^{10})_\infty G(q^2) H(q^4)}{G(q^4)}
\]

\[
= \varphi(q) H(-q) - G(q) f(-q^7, -q^3) - \frac{q(q^{10}; q^{10})_\infty G(q^2) H(q^4)}{G(q^4)}
\]

\[
= \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n+1}}{1 + (-1)^n q^{6n+4}} - \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{10n+6}} \right)
\]

\[
- q \sum_{n=-\infty}^{\infty} \frac{(-q^4)^n}{1 - q^{10n+4}}
\]

\[
= \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{4n+1}}{1 + q^{10n+4}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+3}}{1 - q^{10n+9}} \right)
\]

\[
- \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{10n+6}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+1}}{1 + q^{10n+4}} \right),
\]

where we separated the even and odd indexed terms in the first sum and applied the symmetry in \( b \) and \( z \) from (4.2.2) in the third sum in the penultimate equality. After cancellation, we apply (4.2.2) twice, the first time with \( q \) replaced by \( q^2 \), \( z = q^4 \), and \( b = q^9 \), and the second time with \( q \) replaced by \( q^2 \), \( z = q \), and \( b = q^6 \). Therefore,

\[
M_9(q) + \frac{1}{q} M_8(q^2)
\]

\[
= - \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{4n+3}}{1 - q^{10n+9}} + \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{10n+6}} \right)
\]

\[
= - \frac{1}{(q^{10}; q^{10})_\infty} \left( \frac{q^3 (q^{13}; q^{10})_\infty}{(q^4, q^6, q^9; q^{10})_\infty} + \frac{(q^7, q^3, q^{10}, q^{10}; q^{10})_\infty}{(q, q^2, q^6, q^9, q^{10}; q^{10})_\infty} \right)
\]

\[
= - \frac{1}{(q^{10}; q^{10})_\infty} \left( \frac{q^3 (q^3; q^{10})_\infty}{(q, q^2, q^6, q^9; q^{10})_\infty} + \frac{(q^3, q^7, q^{10}, q^{10}; q^{10})_\infty}{(q, q^2, q^6, q^9; q^{10})_\infty} \right).
\]
we find that invoke Entry 2.4.10, and third we apply (2.3.8) and Entry 2.4.1. Accordingly, the proof of (4.3.11) is thus complete.

Finally, we prove (4.3.12). First, we apply (4.1.14) and (4.1.16), second we invoke Entry 2.4.10, and third we apply (2.3.8) and Entry 2.4.1. Accordingly, we find that

\[ M_6(q^2) + 3M_8(q^2) = q^2 \psi_1(q^2) - 3q^2D(q^2) - \frac{3q^2(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)} \]

\[ = 2q^2F_1(q^2) - 2q^2D(q^2) - \frac{3q^2(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)} \]

\[ = 2(\psi_1(q) - \psi(q^2)G(q^4)) - 2(\psi_1(q) - \varphi(q)H(-q)) \]

\[ = 2q^2D(q^2) - \frac{3q^2(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)} \]

\[ = -2q^2\psi(q^2)G(q^4) + 2q\varphi(q)H(-q) - q^2D(q^2) - \frac{3q^2(q^{10}; q^{10})_\infty G(q^2)H(q^4)}{G(q^4)}. \]

We now apply Lemmas 4.2.4, 4.2.2 (with \( q \) replaced by \(-q\)), 4.2.6 (with \( q \) replaced by \( q^2 \)), and 4.2.10 (with \( q \) replaced by \( q^2 \)). After doing so, we rewrite the first sum and separate the even and odd indexed terms in the second sum. Hence,

\[ M_6(q^2) + 3M_8(q^2) = \frac{1}{(q^{10}; q^{10})_\infty} \left( -2 \sum_{n=-\infty}^{\infty} \frac{q^{2n+1}}{1 - q^{20n+6}} + 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n+2}}{1 + (-1)^n q^{10n+4}} \right) \]

\[ - \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1 - q^{10n+4}} - 3q^2 \sum_{n=-\infty}^{\infty} \frac{(-q^4)^n}{1 - q^{10n+4}} \]

\[ = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{n+1}(-1 - (-1)^n)}{1 - q^{10n+6}} + 2 \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1 + q^{10n+4}} \right) \]

\[ - 2 \sum_{n=-\infty}^{\infty} \frac{q^{4n+4}}{1 - q^{10n+9}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1 - q^{10n+4}} - 3q^2 \sum_{n=-\infty}^{\infty} \frac{(-q^4)^n}{1 - q^{10n+4}} \]

On the far right side above, we replace \( n \) by \(-n - 1\) in the first sum, and employ the symmetry in \( b \) and \( z \) from (4.2.2) in both the second and third sums. We therefore arrive at

\[ M_6(q^2) + 3M_8(q^2) = \frac{1}{(q^{10}; q^{10})_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{2n+4}(1 - (-1)^n)}{1 - q^{10n+4}} + 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n+2}}{1 - q^{10n+4}} \right) \]
\[-2 \sum_{n=-\infty}^{\infty} \frac{q^{9n+4}}{1-q^{10n+4}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n+2}}{1-q^{10n+4}} - 3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n+2}}{1-q^{10n+4}} \]
\[= \frac{1}{(q^{10}; q^{10})_{\infty}} \left( -2 \sum_{n=-\infty}^{\infty} \frac{q^{18n+4}}{1-q^{20n+4}} - 2 \sum_{n=-\infty}^{\infty} \frac{q^{8n+2}}{1-q^{20n+4}} \right), \]

where we combined the first and third sums and then combined the second, fourth, and fifth sums. Next we make two applications of (4.2.2), the first with \(q\) replaced by \(q^4\), \(z = q^{18}\), and \(b = q^8\), and the second with \(q\) replaced by \(q^4\), \(z = q^8\), and \(b = q^4\). Therefore,

\[M_6(q^2) + 3M_8(q^2)\]
\[= - \frac{2}{(q^{10}; q^{10})_{\infty}} \left( q^4(q^{22}, q^{20}; q^{20})_{\infty} + q^2(q^{12}, q^8, q^{20}, q^{20}; q^{20})_{\infty} \right)\]
\[= - \frac{2}{(q^{10}; q^{10})_{\infty}} \left( q^2(q^{20}, q^{20})_{\infty} + q^2(q^{16}, q^{20})_{\infty} \right)\]
\[= 0. \quad (4.3.16)\]

Replacing \(q^2\) by \(q\) above, we complete the proof of (4.3.12), and therefore also complete the proof of Theorem 4.3.2. \(\square\)

**Corollary 4.3.2.** If any one of the \(M_i(q)\), \(6 \leq i \leq 10\), is identically equal to 0, then all are identically equal to 0.

**Proof.** The desired result is an immediate consequence of (4.3.9)–(4.3.12). \(\square\)

### 4.4 Relations to Partitions

Entries 4.1.1 and 4.1.2 are all that Ramanujan stated concerning the relationship of the fifth order mock theta functions to \(\Phi(q)\) and \(\Psi(q)\). However, in [23], F. Garvan and the first author identified these results with assertions about partitions. We provide a brief account of these assertions.

Recall that the *rank* of a partition is equal to the largest part minus the number of parts; let \(N(b, m, n)\) denote the number of partitions of \(n\) with ranks congruent to \(b\) modulo \(m\).

Next, we observe that

\[\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_n} =: \sum_{n=0}^{\infty} \rho_0(n) q^n, \quad (4.4.1)\]

where \(\rho_0(n)\) is the number of partitions of \(n\) with unique smallest part, and all other parts do not exceed the double of the smallest part.

Furthermore,
The Mock Theta Conjectures: Equivalence

\[
\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}} = \frac{1}{1-q} + \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}} =: \frac{1}{1-q} + \sum_{n=1}^{\infty} \rho_1(n) q^n,
\]

where \(\rho_1(n)\) is the number of partitions of \(n\) with unique smallest part, and all other parts do not exceed the double of the smallest part plus 1.

**Theorem 4.4.1.** The assertion

\[ M_1(q) = 0 \]

is equivalent to

\[ \rho_0(n) = N(1, 5, 5n) - N(0, 5, 5n), \]

for \(n > 0\).

**Proof.** We rely on the work of Garvan [72] and Atkin and Swinnerton-Dyer [26]. We define

\[ R_{b,c}(d) := \sum_{n=0}^{\infty} (N(b, 5, 5n + d) - N(c, 5, 5n + d)) q^n. \]

Garvan [72, equation (2.7.39)] proved that

\[ R_{1,2}(0) = \Phi(q), \]

and Atkin and Swinnerton-Dyer [26, equation (6.12)] proved that

\[ R_{0,2}(0) + 2 R_{1,2}(0) = A(q) - 1, \]

where recall that \(A(q)\) is defined in (4.1.5). Hence, by (4.1.7), (4.4.5), and (4.4.6),

\[
M_1(q) = \chi_0(q) - 2 - 3 \Phi(q) + A(q) \\
= \chi_0(q) - 1 + R_{0,2}(0) - R_{1,2}(0) \\
= \chi_0(q) - 1 + R_{0,1}(0) \\
= \chi_0(q) - 1 - R_{1,0}(0),
\]

where we have used the facts that

\[ R_{b,c}(d) - R_{c,b}(d) = R_{b,c}(d) \]

and

\[ R_{b,c}(d) = -R_{c,b}(d). \]

Thus, by (4.4.7) and (4.4.4), the assertion \(M_1(q) = 0\) is equivalent to

\[ \chi_0(q) = 1 + \sum_{n=1}^{\infty} (N(1, 5, 5n) - N(0, 5, 5n)) q^n. \]

Comparing the coefficients of \(q^n\) on both sides above completes the proof. \(\square\)
**Theorem 4.4.2.** The assertion

\[ M_6(q) = 0 \]

is equivalent to the assertion

\[ 1 + \rho_1(n) = 2N(2, 5, 5n + 3) - N(1, 5, 5n + 3) - N(0, 5, 5n + 3), \quad (4.4.10) \]

for \( n > 0 \).

**Proof.** We note that Garvan [72, equation (2.7.40)] proved that

\[ R_{2,0}(3) = \frac{1}{q} \Psi(q), \quad (4.4.11) \]

and Atkin and Swinnerton-Dyer [26, equation (6.18)] proved that

\[ R_{0,1}(3) + R_{0,2}(3) = D(q) - 1, \quad (4.4.12) \]

where \( D(q) \) is defined in (4.1.12). Hence, by (4.1.14), (4.4.11), and (4.4.12),

\[
\frac{1}{q} M_6(q) = \chi_1(q) - \frac{3}{q} \Psi(q) - D(q) \\
= \chi_1(q) - 3R_{2,0}(3) - R_{0,1}(3) - R_{0,2}(3) - 1 \\
= \chi_1(q) - 1 - R_{2,0}(3) - R_{2,1}(3),
\]

(4.4.13)

where we invoked both (4.4.8) and (4.4.9). Hence, by (4.4.4) and (4.4.13), the assertion \( M_6(q) = 0 \) is equivalent to the statement

\[
\chi_1(q) = \frac{1}{1-q} + \sum_{n=1}^{\infty} \rho_1(n)q^n \\
= 1 + \sum_{n=0}^{\infty} (N(2, 5, 5n + 3) - N(0, 5, 5n + 3)) q^n \\
+ \sum_{n=0}^{\infty} (N(2, 5, 5n + 3) - N(1, 5, 5n + 3)) q^n.
\]

A comparison of the coefficients of \( q^n \) concludes the proof.

\[ \square \]

We offer two examples to illustrate each theorem. Let \( n = 1 \) in (4.4.3) and (4.4.10). Then, \( \rho_0(1) = 1 \), and of the 7 partitions of 5, two have rank congruent to 1 modulo 5, namely, 3 + 2 and 1 + 1 + 1 + 1 + 1, and one has rank congruent to 0 modulo 5, namely, 3 + 1 + 1. Thus, \( 1 = 2 - 1 \), in agreement with (4.4.3).

Also, \( \rho_1(1) = 1 \). We find that \( N(2, 5, 8) = 5 \), with the partitions of 8 having rank congruent to 2 modulo 5 being 8, 4 + 4, 5 + 2 + 1, 2 + 2 + 2 + 1 + 1, and 3 + 1 + 1 + 1 + 1 + 1. We also see that \( N(0, 5, 8) = N(1, 5, 8) = 4 \). The partitions of 8 with rank congruent to 0 modulo 5 are 7 + 1, 3 + 3 + 2, 4 + 2 + 1 + 1,
and \(2 + 1 + 1 + 1 + 1 + 1\), while those with rank congruent to 1 modulo 8 are \(4 + 3 + 1, 4 + 2 + 2, 5 + 1 + 1 + 1,\) and \(2 + 2 + 1 + 1 + 1 + 1\). Hence, the identity (4.4.10) is verified in this case by \(1 + 1 = 2 \times 5 - 4 = 2\).

Andrews and Garvan [23] calculated further examples. Let \(n = 5\) in (4.4.3). From their paper [23, p. 243], \(N(1, 5, 25) = 393, N(0, 5, 25) = 390,\) and \(\rho_0(5) = 3,\) with the relevant partitions being \(5, 3 + 2,\) and \(2 + 2 + 1.\) Hence, indeed, (4.4.3) holds when \(n = 5.\) Next, set \(n = 4\) in (4.4.10). Then \(N(2, 5, 23) = 252, N(1, 5, 23) = 250, N(0, 5, 23) = 251,\) and \(\rho_1(4) = 2,\) with the partitions that we want being \(4\) and \(3 + 1.\) Hence, the equation \(1 + 2 = 2 \times 252 - 250 - 251\) verifies (4.4.10) when \(n = 4.\)

In conclusion, we remark that it would be of enormous interest to find bijective proofs of Theorems 4.4.1 and 4.4.1.
5

Fifth Order Mock Theta Functions: Proof of the Mock Theta Conjectures

5.1 Introduction

In Chapter 2, Section 1, we defined Ramanujan’s ten fifth order mock theta functions, and in Chapter 4 we stated the ten mock theta conjectures. The point of the latter chapter was to reveal that the conjectures could be separated into two groups of 5 each and that the conjectures within each group are equivalent.

We shall therefore devote this chapter to proving one conjecture from each group. Namely, we shall prove (4.1.11) from the first group and (4.1.18) from the second. These are the mock theta conjectures related to

\[ f_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n}, \]  

and

\[ f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}, \]

respectively.

It should be pointed out that we have no hints whatsoever from Ramanujan on how to prove these conjectures. Indeed, the way in which he formulated them strongly suggests that the proofs given here may be very far from what Ramanujan had in mind.

The first part of this chapter is devoted to proving the following theorem.

**Theorem 5.1.1.** If \( f_0(q) \) and \( f_1(q) \) are defined by (5.1.1) and (5.1.2), respectively, then

\[ (q;q)\infty f_0(q) = \sum_{\substack{r,s,=\infty \atop \text{sg}(r) = \text{sg}(s) \atop r \equiv s \pmod{2}}} \text{sg}(r)(-1)^{(r-s)/2} q^{r+s+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)} \]  

(5.1.3)
and

\[(q; q) \infty f_1(q) = \sum_{r, s = -\infty \atop r \equiv s \mod 2}^{\infty} \text{sg}(r)(-1)^{(r-s)/2}q^{r+s+3n(5n+1)/2}, \quad (5.1.4)\]

where

\[\text{sg}(n) = \begin{cases} 1, & \text{if } n \geq 0, \\ -1, & \text{if } n < 0. \end{cases} \quad (5.1.5)\]

Note that if we wrote (5.1.3) and (5.1.4) in more conventional forms, i.e., without using \(\text{sg}(n)\) and the parity of the indices, we would need to express each double series as a sum of four double series.

We note that in [13] results comparable to (5.1.3) and (5.1.4) are proved for six of the remaining fifth order mock theta functions, and S. Zwegers [149] has given the related results for the mock theta functions \(\chi_0(q)\) and \(\chi_1(q)\).

There is nothing in Ramanujan’s writing to suggest that he knew either (5.1.3) or (5.1.4). These identities form the foundation of the subsequent proofs of (4.1.11) and (4.1.18). This is the main reason for our suspicion that Ramanujan’s discovery and probable proof of the mock theta conjectures must be quite different from the one presented here, which is based closely and entirely on D. Hickerson’s original proof [86]. While this is not the shortest proof, it is nonetheless most in line with Ramanujan’s other assertions in the Lost Notebook.

More recently, A. Folsom [68] proved the mock theta conjectures by realizing each side of the conjectured identities as the holomorphic projection of a harmonic weak Maass form. In addition, Hickerson and E. Mortenson [88] have placed the mock theta conjectures in a very general setting, and Mortenson has found numerous applications of their general theory [114], [115].

### 5.2 Hecke-type Series for \(f_0(q)\) and \(f_1(q)\)

First we require alternative formulations of (5.1.3) and (5.1.4).

**Lemma 5.2.1.** We have

\[\sum_{r, s = -\infty \atop r \equiv s \mod 2}^{\infty} \text{sg}(r)(-1)^{(r-s)/2}q^{r+s+3n(5n+1)/2} = \sum_{n=0}^{\infty} (-1)^{j}(1 - q^{4n+2})q^{n(5n+1)/2-j^2} \quad (5.2.1)\]
5.2 Hecke-type Series for \( f_0(q) \) and \( f_1(q) \)

\[
\begin{align*}
&\sum_{r,s,-\infty \atop \text{sg}(r) = \text{sg}(s) \atop r \equiv s \mod 2}^\infty \text{sg}(r)(-1)^{(r-s)/2} q^{r+s+\frac{3}{8}(r+s)^2+\frac{3}{4}(r+s)} \\
&= \sum_{n=0 \atop |j| \leq n}^\infty (-1)^j (1 - q^{2n+1})q^{n(5n+3)/2-j^2}. \quad (5.2.2)
\end{align*}
\]

**Proof.** We rewrite the right-hand sum of (5.2.1) as

\[
\begin{align*}
R(q) := \sum_{n=0 \atop |j| \leq n}^\infty (-1)^j (1 - q^{4n+2})q^{n(5n+1)/2-j^2} \\
= \sum_{n=0 \atop |j| \leq n}^\infty (-1)^j q^{n(5n+1)/2-j^2} - \sum_{n=0 \atop |j| \leq n}^\infty (-1)^j q^{(n+1)(5n+4)/2-j^2} \\
=: \sum_1 - \sum_2. \quad (5.2.3)
\end{align*}
\]

In \( \sum_1 \), set \( n = (r+s)/2 \) and \( j = (r-s)/2 \), where \( r \) and \( s \) have the same parity. Since, in fact, \( r = n + j \) and \( s = n - j \), we see that the condition \( -n \leq j \leq n \) is equivalent to

\[-(r+s)/2 \leq (r-s)/2 \leq (r+s)/2, \quad \text{i.e.,} \quad r \geq 0 \quad \text{and} \quad s \geq 0,
\]

which automatically implies that \( n = (r+s)/2 \geq 0 \). Hence,

\[
\sum_1 = \sum_{r,s=0 \atop r \equiv s \mod 2}^\infty (-1)^{(r-s)/2} q^{r+s+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)}. \quad (5.2.4)
\]

Next, in \( \sum_2 \), set \( n = -(r+s+2)/2 \) and \( j = (r-s)/2 \), where again \( r \) and \( s \) have the same parity. Since, in fact, \( r = j - n - 1 \) and \( s = -j - n - 1 \), we see that the condition \( -n \leq j \leq n \) is equivalent to

\[(r+s+2)/2 \leq (r-s)/2 \leq -(r+s+2)/2, \quad \text{i.e.,} \quad r \leq -1 \quad \text{and} \quad s \leq -1,
\]

which automatically implies that \( n = -(r+s+2)/2 \geq 0 \). Hence,

\[
\sum_2 = \sum_{r,s=-1 \atop r \equiv s \mod 2}^{-\infty} (-1)^{(r-s)/2} q^{r+s+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)}. \quad (5.2.5)
\]

Hence, putting (5.2.4) and (5.2.5) in (5.2.3), we complete the proof of (5.2.1).
The proof of (5.2.2) follows exactly the same steps as that of (5.2.1). The only change lies in the fact that the exponent on \( q \) in \( \sum_1 \) is increased by \( n \) and that in \( \sum_2 \) is decreased by \( n + 1 \). Consequently, the resulting exponent on \( q \) increases from \( (r + s)/4 \) to \( 3(r + s)/4 \). Thus, the proof of Lemma 5.2.1 is concluded.

To identify the expressions in (5.2.1) and (5.2.2) with \( f_0(q) \) and \( f_1(q) \), respectively, we require the weak form of Bailey’s Lemma [18, p. 582].

**Theorem 5.2.1 (Weak form of Bailey’s Lemma).** Two sequences \( \alpha_n \) and \( \beta_n \) satisfy

\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q; q)_{n-r}(aq; q)_{n+r}}
\]

if and only if

\[
\sum_{n=0}^{\infty} a^n q^2 \beta_n = \frac{1}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} a^n q^n \alpha_n.
\]

Moreover, the relationship between \( \beta_n \) and \( \alpha_n \) given by (5.2.6) can be inverted [10], i.e.,

\[
\alpha_n = \alpha_n(a, q) = \frac{1 - aq^{2a}}{1 - a} \sum_{j=0}^{n} (-1)^{n-j} (a; q)_{n+j} q^{(n-j)/2} \beta_j.
\]

The identity (5.2.7) is (5.2.2) of [20, p. 97] when we let \( \rho_1 \) and \( \rho_2 \) tend to infinity.

We now return to the two fifth order mock theta functions \( f_0(q) \) and \( f_1(q) \), defined in (5.1.1) and (5.1.2), respectively, which are central to the work in this chapter. We see immediately that if \( a = 1 \) and \( \beta_n = 1/(-q; q)_n \), then the left side of (5.2.7) becomes \( f_0(q) \), and when \( a = q \) and \( \beta_n = 1/(-q; q)_n \), then the left side of (5.2.7) becomes \( f_1(q) \). So, our next step is to find a useful formula for \( \alpha_n \) in these cases \( a = 1, q \) and \( \beta_n = 1/(-q; q)_n \). To accomplish this, we shall employ (5.2.8).

Comparing (5.1.3) with (5.2.1), we need to show that \( (q; q)_{\infty} f_0(q) \) is identical with the right side of (5.2.1). To do this, we need to show that when \( a = 1 \) and \( \beta_n = 1/(-q; q)_n \), then \( \alpha_0 = 1 \) and, for \( n \geq 1 \) in (5.2.7),

\[
\alpha_n = q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2} - q^{n(3n-1)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2}.
\]

In more detail, replacing \( n \) by \( m + 1 \) in the second sum on the right side below, we find that

\[
\sum_{n=0}^{\infty} q^n \alpha_n = \sum_{n=0}^{\infty} q^n \left\{ q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}
\right\}
\]
5.2 Hecke-type Series for $f_0(q)$ and $f_1(q)$

\[-q^{n(3n-1)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2}\]

\[= \sum_{n=0}^{\infty} q^{(5n^2+n)/2} \sum_{|j|\leq n} (-1)^j q^{-j^2} - \sum_{m=-1}^{\infty} q^{(5m^2+9m)/2} \sum_{|j|\leq m} (-1)^j q^{-j^2}\]

\[= \sum_{n=0}^{\infty} \sum_{|j|\leq n} (-1)^j (1 - q^{4n+2}) q^{n(5n+1)/2-j^2}.\]

Next, comparing (5.1.4) with (5.2.2), we see that to conclude that $(q; q)_{\infty} f_1(q)$ is identical with the right-hand side of (5.2.2), we need to show that when $a = q$ and $\beta_n = 1/(-q; q)_n$ in (5.2.7), then $\alpha_0 = 1$ and, for $n \geq 1$,

\[\alpha_n = \frac{1 - q^{2n+1}}{1 - q} q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}. \tag{5.2.10}\]

In more detail,

\[\sum_{n=0}^{\infty} q^{n^2+n} \beta_n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n} = \frac{1}{(q^2; q)_{\infty}} \sum_{n=0}^{\infty} q^{n^2+n} \alpha_n\]

\[= \frac{1}{(q^2; q)_{\infty}} \sum_{n=0}^{\infty} q^{n^2+n} \frac{1 - q^{2n+1}}{1 - q} q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}\]

\[= \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{|j|\leq n} (-1)^j (1 - q^{2n+1}) q^{n(5n+3)/2-j^2}.\]

We reiterate that we are required to show that when $a = 1$, then $\alpha_n$ given by (5.2.9) must be identical to that obtained from the inversion formula (5.2.8), and that when $a = q$, then $\alpha_n$ given by (5.2.10) must also be identical with (5.2.8).

We shall prove (5.2.9) and (5.2.10) via recurrences following [17]. First, we define

\[\beta_n := \frac{(bq; q)_n}{(q^2; q^2)_n}. \tag{5.2.11}\]

and

\[\alpha_n(a, b, q) := \frac{1 - aq^{2n}}{1 - a} \sum_{j=0}^{n} \frac{(-1)^{n-j} (a; q)_{n+j}(bq; q)_j q^{(n-j)}}{(q; q)_{n-j}(q^2; q^2)_j}. \tag{5.2.12}\]

If $b = 1$,

\[\alpha_n(a, 1, q) = \frac{1 - aq^{2n}}{1 - a} \sum_{j=0}^{n} \frac{(-1)^{n-j} (a; q)_{n+j}(q; q)_j q^{(n-j)}}{(q; q)_{n-j}(q^2; q^2)_j}.\]
\[
\frac{1 - aq^{2n}}{1 - a} \sum_{j=0}^{n} (-1)^{n-j} (a; q)_{n+j} \beta_j q^{\left(n, n\right)_j} = \alpha_n(a, q) = \alpha_n,
\]

by (5.2.8). As we shall see in what follows, we have introduced an extra parameter \(b\) as an aid in proving the necessary recurrence formula. Also let

\[
A_n(a, b) := \frac{1 - aq}{1 - aq^{2n+1}} \alpha_n(aq, bq).
\]

We claim that

\[
A_0(1, b) = 1, \quad A_1(1, b) = bq - b - q,
\]

and, for \(n > 1,
\]

\[
A_n(1, b) = (bq^{3n-1} - bq^n)A_{n-1}(1, b) + q^{4n-4}A_{n-2}(1, b).
\]

First, the verification of the first equality in (5.2.15) is trivial. Second, we note that

\[
A_1(1, b) = -1 + \frac{(q; q)_2(bq; q)_1}{(q^2; q^2)_1} = -1 + (1 - q)(1 - bq) = -q - bq + bq^2,
\]

and so the second equality in (5.2.15) has been verified. Third, we establish (5.2.16). We begin by noting that

\[
b(bq; q)_j = q^{-j-1}(- (bq; q)_{j+1} + (bq; q)_j).
\]

Hence, (5.2.16) may be rewritten as

\[
\sum_{j \geq 0} c(n, j)(bq; q)_j = \left(q^{3n-1} - q^n\right) \sum_{j \geq 0} c(n - 1, j)q^{-j-1}(- (bq; q)_{j+1} + (bq; q)_j)
\]

\[
+ q^{4n-4} \sum_{j \geq 0} c(n - 2, j)(bq; q)_j,
\]

where

\[
c(n, j) := \frac{(-1)^{n-j}(q; q)_{n+j} q^{\left(n, n\right)_j}}{(q; q)_{n-j}(q^2; q^2)_j}.
\]

Note that \((bq; q)_j, 0 \leq j \leq n\), is a polynomial in \(b\) of degree \(j\), and that these \(n + 1\) polynomials constitute a basis for the polynomials of degree \(\leq n\). Hence, (5.2.17) can be verified directly by comparing the coefficients of \((bq; q)_j\) on both sides. If we show that these coefficients, which are rational functions of \(q\), are equal, we complete the proof of (5.2.17) and therefore of (5.2.16) as well. To that end, if we equate coefficients of \((bq; q)_j\) in (5.2.17), we find that

\[
c(n, j) = -(q^{3n-1} - q^n)c(n - 1, j - 1)q^{-j} + (q^{3n-1} - q^n)c(n - 1, j)q^{-j-1}
\]

\[
+ q^{4n-4}c(n - 2, j).
\]
Using the definition (5.2.18), we divide out the common factor of
\[
\frac{(-1)^{n-j}(q)_n q^{n-j-2}}{(q)_n (q^2)_j}
\]
from each of the four expressions in (5.2.19) leaving us with the equation
\[
(1 - q^{n+j-1})(1 - q^n q^{n-j}) = -(q^{3n-1} - q^n)^{-j}(1 - q^{2j})q^{(n-j)}
\]
\[
- (q^{3n-1} - q^n)q^{j-1}(1 - q^{n+j-1})(1 - q^{n-j})q^{(n-j-1)}
\]
\[
+ q^{4n-4}(1 - q^{n-j-1})(1 - q^{n-j})q^{(n-2-j)}.
\]
(5.2.20)

We now leave (5.2.20) for the reader to verify using (most likely) computer algebra.

Specializing (5.2.15) and (5.2.16), we see that
\[
A_n(1,1) = \begin{cases} 
1, & \text{if } n = 0, \\
q^2 - 2q, & \text{if } n = 1, \\
(q^{3n-1} - q^n)A_{n-1}(1,1) + q^{4n-4}A_{n-2}(1,1), & \text{if } n > 1.
\end{cases}
\]
(5.2.21)

Now we note that if
\[
S_n := q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2},
\]
(5.2.22)

then
\[
S_0 = 1, \quad S_1 = q^2 - 2q,
\]
(5.2.23)
and, for \( n \geq 2 \),
\[
S_n - q^{3n-1}S_{n-1} = 2(-1)^n q^{n(n+1)/2}.
\]
(5.2.24)

Therefore, by (5.2.24),
\[
(S_n - q^{3n-1}S_{n-1}) + q^n (S_{n-1} - q^{3n-4}S_{n-2}) = 0.
\]
(5.2.25)

Thus, by (5.2.23) and (5.2.25), we deduce that
\[
S_n = \begin{cases} 
1, & \text{if } n = 0, \\
q^2 - 2q, & \text{if } n = 1, \\
(q^{3n-1} - q^n)S_{n-1} + q^{4n-4}S_{n-2}, & \text{if } n \geq 2.
\end{cases}
\]
(5.2.26)

Thus, by (5.2.21) and (5.2.26), \( A_n(1,1) \) and \( S_n \) satisfy the same initial conditions and recurrence relations. Therefore, \( A_n(1,1) = S_n, n \geq 0 \), and so by (5.2.14), (5.2.13), (5.2.12), and (5.2.8),
\[
S_n = A_n(1,1) = \frac{1 - q}{1 - q^{2n+1}} \alpha_n(q, q) = \frac{1 - q}{1 - q^{2n+1}} \alpha_n,
\]
(5.2.27)
or, by (5.2.27) and (5.2.22),

\[
\alpha_n = \frac{1 - q^{2n+1}}{1 - q} A_n(1, 1) = \frac{1 - q^{2n+1}}{1 - q} S_n
\]

\[
= \frac{1 - q^{2n+1}}{1 - q} q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2},
\]

(5.2.28)

in agreement with (5.2.10) in the case \( a = q \). Therefore, (5.2.10) follows from (5.2.28). In conclusion, we have shown that \( \alpha_n \) is given both by (5.2.8) and (5.2.28).

Lastly, the \( \alpha_n \) that is required in (5.2.9) is, by (5.2.8), given by

\[
\alpha_n = (1 - q^{2n}) \sum_{j=0}^{n} \frac{(-1)^{-j}(q; q)_{n+j-1}q^{n-j}}{(q; q)_{n-j}(-q; q)_{j}}.
\]

(5.2.29)

On the other hand, by (5.2.12) and (5.2.14),

\[
A_n(1, 1) - q^{2n-1}A_{n-1}(1, 1) = \sum_{j=0}^{n} \frac{(-1)^{-j}(q; q)_{n+j-1}q^{n-j}}{(q; q)_{n-j}(-q; q)_{j}}
\]

\[
- q^{2n-1} \sum_{j=0}^{n} \frac{(-1)^{-j}(q; q)_{n+j-1}q^{n-j}}{(q; q)_{n-j}(-q; q)_{j}}
\]

\[
= \sum_{j=0}^{n} \frac{(-1)^{-j}(q; q)_{n+j-1}q^{n-j}}{(q; q)_{n-j}(-q; q)_{j}}
\]

\[
\times \left( 1 - q^{n+j} + q^{2n-1-n+j+1}(1 - q^{n-j}) \right)
\]

\[
= (1 - q^{2n}) \sum_{j=0}^{n} \frac{(-1)^{-j}(q; q)_{n+j-1}q^{n-j}}{(q; q)_{n-j}(-q; q)_{j}}.
\]

(5.2.30)

Comparing (5.2.30) with (5.2.29) and noting (5.2.27), we see that \( \alpha_n \) prescribed in (5.2.9) has been established.

This then completes the objective of this section, which was to show that \( f_0(q) \) can be represented by (5.2.3), and \( f_1(q) \) can be represented by (5.2.4). Because of their importance and for the convenience of later use, we record portions of these conclusions as a separate theorem.

**Theorem 5.2.2.** With \( f_0(q) \) and \( f_1(q) \) defined by (5.1.1) and (5.1.2), respectively, and with \( \text{sg}(n) \) defined by (5.1.5),

\[
(q; q)_\infty f_0(q) = \sum_{r, s = \infty}^{\infty} \text{sg}(r)(-1)^{(r-s)/2} q^{rs + \frac{3}{8}(r+s)^2 + \frac{1}{2}(r+s)}
\]

(5.2.31)
\[(q; q)_\infty f_1(q) = \sum_{r, s = -\infty \atop sg(r) = sg(s)}^{\infty} r \equiv s (\text{mod } 2) \frac{sg(r)(-1)^{(r-s)/2} q^{(r+s)^2 + (r-s)/2}}{(r-s)^2 + (r+s)^2 + (r+s)} \] (5.2.32)

5.3 Theta Function Identities

In this section, we prove six identities related to Ramanujan’s theta functions, defined for \(|ab| < 1\) by

\[ f(a, b) := \sum_{n = -\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_{\infty}, \] (5.3.1)

by Jacobi’s triple product identity [19, p. 13, Lemma 1.2.2]. We note the special case

\[ f(-q) := f(-q, -q^2) = \sum_{n = -\infty}^{\infty} (-1)^n q^{(3n-1)/2} = (q; q)_{\infty}. \] (5.3.2)

The latter equality is known as Euler’s pentagonal number theorem. While we often use the notation \(f(a, b)\), we shall also normally write out representations in terms of the products \((a; q)_{\infty}\) for ease of application.

For our first identity, we restate the Quintuple Product Identity [19, p. 14, Lemma 1.2.3 with \(\lambda = qx^3\)].

Lemma 5.3.1 (Quintuple Product Identity). We have

\[ f(-qx^3, -q^2/x^3) + xf(-x^3q^2, -q/x^3) = \frac{f(-q)f(-x^2, -q/x^2)}{f(-x, -q/x)}, \]

or

\[ (qx^3, q^2/x^3, q^3; q^3)_{\infty} + x(x^3q^2, q/x^3, q^3, q^3)_{\infty} = \frac{(q; q)_{\infty}(x^3q^2, q/x^3, q)_{\infty}}{(x, q/x, q; q)_{\infty}}. \]

Lemma 5.3.2 ([11]). We have

\[ f(-x, -q/x)f(-y, -q/y) = f(xy, q^2/(xy))f(qy/x, qx/y) + xf(qxy, q/(xy))f(y/x, q^2x/y), \]

or

\[ (x, q/x, y, q/y, q; q)_{\infty} = (-xy, -q^2/(xy), -qy/x, -qx/y, q^2, q^2)_{\infty} - x(qxy, -q/(xy), -y/x, -q^2x/y, q^2q^2; q^2)_{\infty}. \]
Proof. To simplify our argument, we shall prove that

\[
\sum_{n=-\infty}^{\infty} (-1)^n z^n \alpha^n q^{n(n+1)/2} \sum_{m=-\infty}^{\infty} (-1)^m z^m \beta^m q^{m(m+1)/2} = \sum_{n=-\infty}^{\infty} \alpha^n \beta^{-n} q^{n^2} \sum_{N=-\infty}^{\infty} (\alpha \beta z^2)^N q^{N^2+N}
\]

or equivalently,

\[
f \left( -\frac{z \alpha q}{z \alpha} \right) f \left( -\frac{z \beta q}{z \beta} \right) = f \left( \frac{\alpha q}{\beta} , \frac{\beta q}{\alpha} \right) f \left( \alpha \beta z^2 q^2 , \frac{1}{\alpha \beta z^2} \right) - \frac{1}{z \beta} f \left( \frac{\alpha q^2}{\beta} , \frac{\beta}{\alpha} \right) f \left( \alpha \beta z^2 q^2 , \frac{q}{\alpha \beta z^2} \right),
\]

and we easily see that (5.3.4) is the assertion of Lemma 5.3.2 with \( \alpha = 1/y \), \( \beta = 1/x \), and \( z = 1 \).

To prove (5.3.3), we begin by replacing \( m \) by \( m - n \) in the second sum (which is admissible because of the absolute convergence of the double series).

Thus,

\[
\sum_{n=-\infty}^{\infty} (-1)^n z^n \alpha^n q^{n(n+1)/2} \sum_{m=-\infty}^{\infty} (-1)^m z^m \beta^m q^{m(m+1)/2} = \sum_{n=-\infty}^{\infty} (-1)^{m-n} z^{m-n} \alpha^{m-n} q^{m(m+1)/2} \sum_{m=-\infty}^{\infty} (-1)^{m-n} z^{m-n} \beta^{m-n} q^{n(n+1)/2}
\]

\[
= \sum_{m=-\infty}^{\infty} z^{2m} \beta^{2m} q^{m^2+m} \sum_{n=-\infty}^{\infty} \alpha^n \beta^{-n} q^{n^2-mn}
\]

\[
- \sum_{m=-\infty}^{\infty} z^{2m-1} \beta^{2m-1} q^{m^2} \sum_{n=-\infty}^{\infty} \alpha^n \beta^{-n} q^{(n-m)^2+(n-m)}
\]

\[
= \sum_{m=-\infty}^{\infty} z^{2m} \beta^{2m} \alpha^m q^{m^2+m} \sum_{n=-\infty}^{\infty} \alpha^n \beta^{-n} q^{n^2}
\]

\[
- \frac{1}{z \beta} \sum_{m=-\infty}^{\infty} z^{2m} \beta^{2m} \alpha^m q^{m^2} \sum_{n=-\infty}^{\infty} \alpha^n \beta^{-n} q^{n^2+n},
\]

where in the last step we replaced \( n - m \) by \( n \). This completes the proof of (5.3.3). \( \square \)
The next entry can be found in Ramanujan’s notebooks [127], [28, p. 45, Lemma 29(ii)].

**Lemma 5.3.3.** We have

\[ f(x, q/x)f(-y, -q/y) - f(-x, -q/x)f(y, q/y) = 2xf(-y/x, -q^2x/y)f(-qxy, -q/(xy)), \]

or

\[ (-x, -q/x, q, y, q/y, q; q)_\infty - (x, q/x, y, -q/y, q; q)_\infty = 2x(y/x, q^2x/y, q^2, qxy, q/(xy), q^2; q^2)_\infty. \quad (5.3.5) \]

**Proof.** Replace \( x \) by \(-x\) in Lemma 5.3.2, then subtract this result from Lemma 5.3.2 with \( y \) replaced by \(-y\). \( \square \)

The next three lemmas require Ramanujan’s \( 1_1 \psi_1 \) summation (4.2.1). If \( |b/a| < |t| < 1 \), then

\[ \sum_{n=-\infty}^{\infty} \frac{(a; q)_n t^n}{(b; q)_n} = \frac{(b/a; q)_\infty(a; q)_\infty(q/(at); q)_\infty(q; q)_\infty}{(a; q)_\infty(b; q)_\infty(t; q)_\infty(b/(at); q)_\infty}. \quad (5.3.6) \]

**Lemma 5.3.4.** If \( |q| < |x| < 1 \), then

\[ \sum_{r=-\infty}^{\infty} \frac{x^r}{1-yq^r} = \frac{f^3(q, q, xy, q/(xy); q)_\infty}{f(-x, -q/x)f(-y, -q/y)} = \frac{(q, q, xy, q/(xy); q)_\infty}{(x, q/x, y, q/y; q)_\infty}. \]

**Proof.** Set \( a = y \), \( b = yq \), and \( t = x \) in (5.3.6). Then multiply both sides by \((1 - y)^{-1}\) and recall that

\[ (x; q)_\infty(q/x; q)_\infty = \frac{1}{(q; q)_\infty} f(-x, -q/x). \]

\( \square \)

Lemma 5.3.4 is another version of (4.2.2). It is also a special case of a very general theorem of S.H. Chan [49]. The next result rewrites Lemma 5.3.4 in a more useful form.

**Lemma 5.3.5.** We have

\[ \sum_{r,s=-\infty}^{\infty} \frac{sg(r)x^r y^s q^{rs}}{sg(r)sg(s)} = \frac{f^3(q, q, xy, q/(xy); q)_\infty}{f(-x, -q/x)f(-y, -q/y)} = \frac{(q, q, xy, q/(xy); q)_\infty}{(x, q/x, y, q/y; q)_\infty}. \]
Proof. In light of the fact that the right-hand sides in Lemmas 5.3.4 and 5.3.5 are identical, we need only show that the identity of the left-hand sides. To that end,

\[
\sum_{r=-\infty}^{\infty} \frac{x^r}{1-y q^{r \tau}} = \sum_{r=0}^{\infty} x^r \sum_{s=0}^{\infty} y^s q^{r s} - \sum_{r=-\infty}^{-1} x^r q^{-r \tau} y^{-1}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \text{sg}(r) x^r y^s q^{r s} + \sum_{r=-\infty}^{-1} \text{sg}(r) x^r \sum_{s=1}^{\infty} y^{-s} q^{-r s}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \text{sg}(r) x^r y^s q^{r s} + \sum_{r=-\infty}^{-1} \sum_{s=-\infty}^{-1} \text{sg}(r) x^r y^s q^{r s}
\]

\[
= \sum_{r,s=-\infty}^{\infty} \text{sg}(r) x^r y^s q^{r s}.
\]

This completes the proof. \(\square\)

Finally in this section, we require a refinement of the previous lemma where parity is taken into account. Recall that Ramanujan’s function \(\varphi(q)\) is defined by

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^n.
\]

Lemma 5.3.6. We have

\[
\sum_{r,s=-\infty}^{\infty} \text{sg}(r) x^r y^s q^{r s} = \varphi(-q^2) f(-qxy, -q/(xy)) f(qx/y, qy/x) f\left(-x^2 y^2, -q^4/(x^2 y^2)\right)
\]

\[
\times f\left(-x^2, -q^2/x^2\right) f\left(-y^2, -q^2/y^2\right)
\]

\[
= \varphi(-q^2)(qxy, q/(xy), -qx/y, -qy/x; q^2_\infty)(x^2 y^2, q^4/(x^2 y^2), q^4; q^4_\infty).
\]

\[
\times (x^2 q^2/x^2, y^2, q^2/y^2; q^2_\infty).
\]

Proof. Applying Lemma 5.3.5 twice, we find that

\[
\sum_{r,s=-\infty}^{\infty} \text{sg}(r) x^r y^s q^{r s}
\]

\[
= \sum_{r,s=-\infty}^{\infty} \text{sg}(r) (x^2)^r (y^2)^s (q^4)^r s + qx y \sum_{r,s=-\infty}^{\infty} \text{sg}(r) (q^2 x^2)^r (q^2 y^2)^s (q^4)^r s
\]
Thus, we note that if we let $F$ then the foregoing equality. The residue of $F$ is what we wanted to prove.

We have used (4.1.3) and the Jacobi triple product identity (4.1.2) several times to rearrange reformulate the quotient of theta functions. The expression inside the large parentheses collapses to one term by Lemma 5.3.2, with $q$ replaced by $q^2$, $x$ replaced by $q/(xy)$, and $y$ replaced by $-qx/y$. Consequently,

$$\sum_{r,s=-\infty \atop \text{sg}(r) \equiv \text{sg}(s) \text{ (mod 2)}} \sum_{0}^{\infty} \text{sg}(r)x^ry^sq^{rs} = \frac{\varphi(-q^2)f(-qxy, -q/(xy))f(qx/y, qy/x)f(-x^2y^2, -q^4/(x^2y^2))}{f(-x^2, -q^2/x^2)f(-y^2, -q^2/y^2)},$$

which is what we wanted to prove.

\[\square\]

### 5.4 Partial Fractions and Appell–Lerch Series

#### Lemma 5.4.1

Suppose that $a$ and $b$ are arbitrary real non-zero numbers. If

$$F(z) := \frac{1}{f(-q^azb, -q^{m-a}/z^b)} = \frac{1}{(z^aq^m; q^m)_\infty(z^{-b}q^{m-a}; q^m)_\infty(q^m; q^m)_\infty},$$

then $F(z)$ is meromorphic for $z \neq 0$, with simple poles at all points $z_0$ such that $z_0 = q^{k-m-a}$ for some integer $k$, where we take the real positive root of the foregoing equality. The residue of $F(z)$ at $z_0$ is equal to

$$\frac{(-1)^{k-1}q^{mk(k-1)/2}z_0}{b(q^m; q^m)_\infty^3}.$$

**Proof.** We note that if we let $z = z_0x$, then

$$F(z) = \frac{1}{(x^aq^{km}; q^m)_\infty(x^{-b}q^{m-km}; q^m)_\infty(q^m; q^m)_\infty} = \frac{1}{(-1)^kq^{-mk(k-1)/2}x^{-bk}(x^b; q^m)_\infty(x^{-b}q^{m}; q^m)_\infty(q^m; q^m)_\infty}.$$
\[
\lim_{x \to 1} (x - 1) F(z) = \lim_{x \to 1} \frac{(x - 1)(-1)^k q^{mk(k-1)/2} x^b k}{(1 - x^b)(x^b q^m; q^m) \infty (x - x^b q^m; q^m) \infty (q^m; q^m) \infty}
\]
\[
= (-1)^k q^{mk(k-1)/2} \lim_{x \to 1} \frac{x - 1}{1 - x^b}
\]
\[
= (-1)^k q^{mk(k-1)/2} \frac{b(q^m; q^m) \infty}{(q^m; q^m) \infty}.
\]

If we remember that \(z = z_0 x\), we see that the desired result follows from above. \(\Box\)

**Lemma 5.4.2.** Suppose that \(F(z) = F(z, q)\) is analytic for all \(z \neq 0\), and assume that there is a constant \(C \neq 0\) and a positive integer \(n\) such that
\[
F(qz) = C z^{-n} F(z).
\]

Then,
\[
F(z) = \sum_{r=0}^{n-1} F_r z^r (C^{-1} q^r z^n, C q^{n-r} z^{-n})
\]
\[
= \sum_{r=0}^{n-1} F_r z^r (-C^{-1} q^r z^n, -C q^{n-r} z^{-n}, q^n; q^n) \infty.
\]

If, in addition, \(n\) is odd and exceeds 1, \(C = \pm 1\), and
\[
F(z^{-1}) = -C z^{-n} F(z),
\]

then
\[
F(z) = \sum_{r=0}^{(n-1)/2} F_r (z^r f(C q^r z^n, C q^{n-r} z^{-n})
\]
\[
- C z^{n-r} f(C q^r z^{-n}, C q^{n-r} z^n)) \infty.
\]

**Proof.** We write
\[
F(z) = \sum_{r=-\infty}^{\infty} F_r z^r, \quad 0 < |z| < \infty.
\]

Substituting (5.4.5) into (5.4.1) and equating coefficients of \(z^r\), \(-\infty < r < \infty\), we deduce that
\[
F_{r+n} = C^{-1} q^r F_r.
\]

It follows by mathematical induction that, for all positive integers \(n\) and integers \(k\),
\[
F_{r+kn} = C^{-k} q^{k+n(k-1)/2} F_r.
\]
Hence,
\[
F(z) = \sum_{r=0}^{n-1} \sum_{k=-\infty}^{\infty} F_{r+kn} z^{r+kn}
= \sum_{r=0}^{n-1} \sum_{k=-\infty}^{\infty} C^{-k} q^{r k+n(k-1)/2} F_r z^{r+kn}
= \sum_{r=0}^{n-1} F_r z^r \sum_{k=-\infty}^{\infty} q^{nk(k-1)/2} (C^{-1} q^r z^n)^k
= \sum_{r=0}^{n-1} F_r z^r f(C^{-1} q^r z^n, C q^{n-r} z^{-n}), \tag{5.4.8}
\]
which is (5.4.2).

To establish (5.4.4), we substitute (5.4.5) into (5.4.3). Then, equating coefficients of \(z^{-r}, -\infty < r < \infty\), we find that
\[
F_r = -CF_{n-r}. \tag{5.4.9}
\]
Hence, by (5.4.9) and (5.4.6),
\[
F_0 = -CF_n = -C C^{-1} q^0 F_0 = -F_0;
\]
so \(F_0 = 0\). Hence, replacing \(r\) by \(n-r\) in the second sum on the right-hand side below, we find from (5.4.8) that
\[
F(z) = \left( \sum_{r=1}^{(n-1)/2} + \sum_{r=(n+1)/2}^{n-1} \right) F_r z^r f(C^{-1} q^r z^n, C q^{n-r} z^{-n})
= \sum_{r=1}^{(n-1)/2} (F_r z^r f(C^{-1} q^r z^n, C q^{n-r} z^{-n}) + F_{n-r} z^{n-r} f(C^{-1} q^{n-r} z^n, C q^{r} z^{-n}))
= \sum_{r=1}^{(n-1)/2} (F_r z^r f(C q^r z^n, C q^{n-r} z^{-n}) - CF_r z^{n-r} f(C q^{n-r} z^n, C q^{r} z^{-n}))
\]
by (5.4.9) and the fact that \(C = C^{-1}\). Hence, we deduce (5.4.4) from the last equality above.

\[\square\]

**Lemma 5.4.3.** If we define
\[
g(x,q) := \frac{1}{x} \left( -1 + \sum_{n=0}^{\infty} \frac{q^n}{(x;q)_{n+1}(q/x;q)_n} \right) , \tag{5.4.10}
\]
then
\[
g(x,q) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x;q)_n(q/x;q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - x q^n}. \tag{5.4.11}
\]
Proof. The second of these assertions is a restatement of equation (1.3.13). As for the first, we see that

\[
\frac{1}{x} \left( -1 + \sum_{n=0}^{\infty} \frac{q^n}{(x;q)_{n+1}(q/x;q)_n} \right) - \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x;q)_n(q/x;q)_n} = \frac{1}{1-x} + \sum_{n=1}^{\infty} \frac{(x^{-1}q^n - (1-xq^n))q^{n(n-1)}}{(x;q)_{n+1}(q/x;q)_n}
\]

\[
= \frac{1}{1-x} + \sum_{n=1}^{\infty} \frac{((1-xq^n)(1-x^{-1}q^n) + q^{2n})q^{n(n-1)}}{(x;q)_{n+1}(q/x;q)_n}
\]

\[
= \frac{1}{1-x} - \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x;q)_{n+1}(q/x;q)_n} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(x;q)_{n+1}(q/x;q)_n}
\]

\[
= - \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x;q)_{n+1}(q/x;q)_n} + \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x;q)_{n+1}(q/x;q)_n}
\]

\[
= 0.
\]

Thus, the proof is complete. \(\square\)

Lemma 5.4.4. The function \( g(x, q) \) is the coefficient of \( z^0 \) in the Laurent series expansion of

\[
A(z) := A(z, x, q) := \frac{(xz, q/(xz), q; q)_\infty (z, q^3/z, q^3; q^3)_\infty}{(x, q/x, z, q/z; q)_\infty}
\]

in the annulus \(|q| < |z| < 1\).

Proof. We apply Lemma 5.3.4. Therefore, for \(|q| < |z| < 1\),

\[
(q; q)_\infty A(z) = \sum_{r=\infty}^{\infty} \frac{z^r}{1-xq^r} \sum_{s=\infty}^{\infty} (-1)^s q^{3s(3s-1)/2} z^s,
\]

and the coefficient of \( z^0 \) above is clearly equal to

\[
\sum_{r=\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)/2}}{1-xq^r} = (q; q)_\infty g(x, q),
\]

by Lemma 5.4.3. Dividing both sides above by \((q; q)_\infty\), we obtain the desired result. \(\square\)

Lemma 5.4.5. For \( g(x, q) \) defined in Lemma 5.4.3 and \( A(z) \) defined in Lemma 5.4.4,

\[
A(z) = \frac{(x^3 z, q^3/(x^3 z), q^3; q^3)_\infty g(x, q)}{1 - zq^{k+1}} - \sum_{k=\infty}^{\infty} \frac{(-1)^k q^{3k(k+1)/2} x^{3k+1} z^{k+1} - 1}{}
\]

\[
- \sum_{k=\infty}^{\infty} \frac{(-1)^k q^{3k(k+1)/2} x^{3k+1} z^{k+1}}{1 - z^{-1}x^{3k-1} z^{-k-1}}.
\]
5.4 Partial Fractions and Appell–Lerch Series

Proof. Let

\[ L(z) := \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)/2} x^{3r+1} z^{r+1}}{1 - zq^{3r+1}} \]

and

\[ M(z) := \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r(r+3)/2+1} x^{-3r-1} z^{-r-1}}{1 - z^{-1}q^{3r+1}}. \]

Next, we define

\[ F(z) := A(z) + L(z) + M(z). \]  \hspace{1cm} (5.4.12)

It follows that

\[ F(q^3z) = -x^{-3}z^{-1}F(z), \]  \hspace{1cm} (5.4.13)

because each of \( A(z) \), \( L(z) \), and \( M(z) \) is easily shown to satisfy the functional equation (5.4.13).

Our next step is to show that \( F(z) \) is analytic for all \( z \neq 0 \) (so that we may apply Lemma 5.4.2 to \( F(z) \)). We note that \( L(z) \) and \( M(z) \) are meromorphic for \( z \neq 0 \). The function \( L(z) \) has simple poles at \( z = q^{3k-1} \) and that \( M(z) \) has simple poles at \( z = q^{3k+1} \) for every integer \( k \). If we now calculate the residues, we find that at \( z = q^{3k-1} \), the residue of \( L(z) \) is given by

\[ \lim_{z \to q^{3k-1}} (z - q^{3k-1})L(z) = \lim_{z \to q^{3k-1}} (z - q^{3k-1}) \frac{(-1)^k q^{3k(k-1)/2} x^{-3k+1} z^{-k+1}}{1 - zq^{-3k+1}} = (-1)^{k-1} q^{(-3k^2 + 11k - 4)/2} x^{1-3k}. \]

Also, the residue of \( A(z) \) at \( z = q^{3k-1} \) is given by

\[ \lim_{z \to q^{3k-1}} (z - q^{3k-1})A(z) = (-1)^k q^{(-3k^2 + 11k - 4)/2} x^{1-3k}. \]

Next, we calculate the residues of \( M(z) \) and \( A(z) \) at \( z = q^{3k+1} \). First, for \( M(z) \), this residue is given by

\[ \lim_{z \to q^{3k+1}} (z - q^{3k+1})M(z) = \lim_{z \to q^{3k+1}} (z - q^{3k+1}) \frac{(-1)^k q^{3k(k+3)/2+1} x^{-3k-1} z^{-k-1}}{1 - z^{-1}q^{3k+1}} = (-1)^k q^{(-3k^2 + 7k + 2)/2} x^{-3k-1}. \]

Furthermore, the residue of \( A(z) \) at \( z = q^{3k+1} \) is seen to be

\[ \lim_{z \to q^{3k+1}} (z - q^{3k+1})A(z) = (-1)^{k+1} q^{(-3k^2 + 7k + 2)/2} x^{-3k-1}. \]

Thus, we see that all the possible poles of \( F(z) \) are, in fact, removable. I.e., \( F(z) \) is analytic for all \( z \) except for \( z = 0 \). In addition, \( F(z) \) satisfies equation
and so we may apply Lemma 5.4.2 with $n = 1$, $C = -x^{-3}$, and $q$ replaced by $q^3$. Consequently,

$$F(z) = F_0(x^3 z, q^3/(x^3 z), q^3; q^3)_\infty,$$  

(5.4.14)

where $F_0$ is the coefficient of $z^0$ in the Laurent expansion of $F(z)$ around $z = 0$.

Examining $L(z)$, we see that $z^0$ can only possibly arise from the term with $r = -1$, namely,

$$-\frac{x^{-2}}{1 - z/q^2} = \frac{x^{-2} z^{-1} q^2}{1 - q^2/z} = x^{-2} \sum_{m=1}^{\infty} q^{2m} z^{-m}.$$

Since the coefficient of $z^0$ is absent here as well, we see that the coefficient of $z^0$ in $L(z)$ is 0.

Similarly, for $M(z)$, we see that $z^0$ can only possibly arise from the term with $r = -1$, to wit

$$-\frac{q^{-2} x^2}{1 - 1/(q^2 z)} = \frac{x^2 z}{1 - q^2 z} = \frac{x^2}{q^2} \sum_{m=1}^{\infty} (q^2 z)^m,$$

and so we see that the coefficient of $z^0$ in $M(z)$ is also equal to 0. Hence, by (5.4.12), we see that $F_0$, the coefficient of $z^0$ in $F(z)$ must be the coefficient of $z^0$ in $A(z)$, which by Lemma 5.4.4, is $g(x, q)$. In summary,

$$F_0 = g(x, q).$$  

(5.4.15)

The conclusion for Lemma 5.4.5 now follows from (5.4.15), (5.4.12), (5.4.14), and the definitions of $L(z)$ and $M(z)$. $\square$

### 5.5 Proof of the Mock Theta Conjectures

The six results in this section (five lemmas and the main theorem) tie the functions $f_0(q)$ and $f_1(q)$ to a variety of theta functions and expansions involving instances of $g(x, q)$.

Define

$$B(z) := B(z, q) := \frac{z^2 (-z, -q/z, q; q)_\infty (z, q^3/z, q^3; q^3)_\infty}{(z, q^2/z; q^2)_\infty}.$$  

(5.5.1)

We note that $B(z)$ is meromorphic for $z \neq 0$ with simple poles at $z = q^{6k+2}$ for each integer $k$. Furthermore,

$$B(q^6 z) = -z^{-5} B(z),$$  

(5.5.2)

$$B(z^{-1}) = z^{-5} B(z).$$  

(5.5.3)
Our goal is to derive two representations for $B(z)$. One is given in Lemma 5.5.2 and involves the two fifth order mock theta functions $f_0(q)$ and $f_1(q)$. The second is found in Lemma 5.5.4 and involves the function $A(z)$, first defined in Lemma 5.4.4, and the representation for $A(z)$ from Lemma 5.4.5. We note that this representation for $B(z)$ involves the function $g(x, q)$. As we shall see later, two special cases for $g(x, q)$ yield the functions $\Phi(q)$ and $\Psi(q)$ appearing in the mock theta conjectures. Each of the two representations for $B(z)$ readily yield 5-dissections for $B(z)$, each dissection involving the same two generalized Lambert series. Of course, these dissections must be the same. In particular, we equate the functions of $z^0$ that are multiplied by the powers $z$ and $z^2$, namely, $B_1(z^5)$ and $B_2(z^5)$, respectively. We then equate the coefficients of $z^0$ in the two representations for $B_1(z^5)$ to deduce the first mock theta function identity (4.1.11), and we equate the coefficients of $z^0$ in the two representations for $B_2(z^5)$ to derive the second mock theta function identity (4.1.18). Readers will now see why the method used to prove the mock theta function conjectures is called the “constant term method.”

Lemma 5.5.1. In the annulus, $|q^2| < |z| < 1$, the coefficient of $z$ in the Laurent series for $B(z)$ is equal to $qf_0(q)$, and the coefficient of $z^2$ is equal to $f_1(q)$.

Proof. By Lemma 5.3.6, with $x = -\sqrt{z}$ and $y = \sqrt{z}$, and by (1.3.1), we see that

$$
\sum_{r,s=-\infty}^{\infty} \text{sgn}(r)(-1)^r z^{(r+s)/2} q^s
\left.\begin{array}{c}
\frac{(q^2; q^2)_\infty}{z^2, q^4/z^2, q^4; q^4)_\infty}
\frac{(-q^2; q^2)_\infty}{z, q^3/z, q^3; q^3)_\infty}
\end{array}\right)
$$

Therefore, by (5.5.1) and (5.5.4),

$$
(q; q)_\infty B(z) = \frac{z^2(q; q)_\infty}{z^2, q^3/z, q^3; q^3}_\infty
\left.\begin{array}{c}
\frac{(-q^2; q^2)_\infty}{z, q^3/z, q^3; q^3)_\infty}
\end{array}\right)
$$

In the sequel, we use the notation $[z^n]$ to denote the coefficient of $z^n$. The coefficient of $z$ in (5.5.5) is obtained by taking $t = -(r + s + 2)/2$ and is thus

$$
[z](q; q)_\infty B(z) = \sum_{r,s=-\infty}^{\infty} \text{sgn}(r)(-1)^{(r-s)/2+1} q^{r+s+\frac{3}{2}(r+s)^2+\frac{9}{4}(r+s)+3}.
$$
Replacing \( r \) and \( s \) by \(-1-r\) and \(-1-s\), respectively, and observing that \( \text{sg}(-1-r) = -\text{sg}(r) \), we see that we can write (5.5.6) as

\[
[z] (q; q)_\infty B(z) = \sum_{r, s = -\infty}^{\infty} \text{sg}(r)(-1)^{(r-s)/2} q^{rs+\frac{3}{8}(r+s)^2+\frac{1}{4}(r+s)+1} = (q; q)_\infty f_0(q),
\]

by (5.2.31) of Theorem 5.2.2. This last equality is equivalent to the first assertion of Lemma 5.5.1.

Similarly, to obtain the coefficient of \( z^2 \) in (5.5.5), we take \( t = -(r+s)/2 \) and find immediately that

\[
[z^2](q; q)_\infty B(z) = \sum_{r, s = -\infty}^{\infty} \text{sg}(r)(-1)^{(r-s)/2} q^{rs+\frac{3}{8}(r+s)^2+\frac{3}{2}(r+s)} = (q; q)_\infty f_1(q),
\]

by (5.2.32) in Theorem 5.2.2. The foregoing equality is equivalent to the second assertion in Lemma 5.5.1.

Our next lemma provides a representation of \( B(z) \) that isolates each pole.

**Lemma 5.5.2.** If \( B(z) \) is defined in (5.5.1), then

\[
B(z) = qf_0(q) \left( z(q^6 z^5, q^{24}/z^5, q^{30}; q^{30})_\infty + z^4(q^{24} z^5, q^{6}/z^5, q^{30}; q^{30})_\infty \right) + f_1(q) \left( z^2(q^{12} z^5, q^{18}/z^5, q^{30}; q^{30})_\infty + z^3(q^{18} z^5, q^{12}/z^5, q^{30}; q^{30})_\infty \right) + L(z) + M(z),
\]

where

\[
L(z) = 2 \sum_{r = -\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3 z^5 r+5}}{1 - q^{6r+2 z^5}}, \quad (5.5.7)
\]

\[
M(z) = 2 \sum_{r = -\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3 z^5 r-5 r}}{1 - q^{6r+2 z^5-1}}. \quad (5.5.8)
\]

**Proof.** We define

\[
F(z) = B(z) - L(z) - M(z). \quad (5.5.9)
\]

It follows directly that

\[
F(q^6 z) = -z^{-5} F(z), \quad (5.5.10)
\]

because each of \( B(z) \), \( L(z) \), and \( M(z) \) satisfies this functional equation. In addition,
Thus, the coefficients of $z$ only term that could produce coefficients for $z$ and $z^2$ is for $r = -1$, and this term is

$$
- \frac{2q^3}{1 - z/q^3} = 2q^7 z^{-1} \sum_{n=0}^{\infty} q^{4n} z^{-n}.
$$

Thus, the coefficients of $z$ and $z^2$ in $L(z)$ are both equal to 0. Similarly, the only term that could produce coefficients for $z$ and $z^2$ in $M(z)$ arises from the term with $r = 0$, which is

$$
2q^3 \sum_{n=0}^{\infty} q^{2n} z^{-n}.
$$

Thus, these coefficients of $z$ and $z^2$ are 0 as well.

Hence, $F_1$ and $F_2$ equal the coefficients of $z$ and $z^2$, respectively, in $B(z)$, and by Lemma 5.5.1, these are $q f_0(q)$ and $f_1(q)$, respectively. This completes the proof of Lemma 5.5.2. \qed
We now turn to a result which was stated as Lemma 2.3.1 in [21, p. 18]. We did not prove this result there because we were attempting to avoid proofs that seemed rather distant from the way that Ramanujan did things. However, we truly have no evidence of how or whether Ramanujan proved the mock theta conjectures. (We have faith that he had a proof, but we have no evidence.) This lemma of Atkin and Swinnerton-Dyer [26] will greatly facilitate our work; so we include it here with a proof.

**Lemma 5.5.3.** Let \( f(z) \) be an analytic function of \( z \), except possibly for a finite number of poles in every annulus \( 0 < r_1 \leq |z| \leq r_2 \). Suppose that for certain constants \( A \neq 0 \) and \( w \), with \( 0 < |w| < 1 \), and for some integer \( n \) (possibly positive, 0, or negative) that

\[
f(wz) = Az^n f(z),
\]

identically in \( z \). Then either \( f(z) \) has \( n \) more poles than zeros in \( |w| < |z| \leq 1 \), or \( f(z) \) vanishes identically.

**Proof.** Suppose that \( f(z) \) is not identically equal to 0. Then \( f(z) \) must have only a finite number of zeros in any region \( 0 < r_1 \leq |z| < r_2 \). We now choose \( \epsilon > 0 \) so that \( f(z) \) has no poles or zeros in \( 1 < |z| \leq 1 + \epsilon \), and so therefore none in \( |w| < |z| \leq |w|(1 + \epsilon) \). Let \( C \) and \( C' \) denote the positively oriented circles \( |z| = 1 + \epsilon \) and \( |z| = |w|(1 + \epsilon) \), respectively. Then \( f(z) \) has the same numbers of poles and zeros in \( |w| < |z| \leq 1 \) as it has in the region between \( C \) and \( C' \). Let \( N_Z \) and \( N_P \) denote the number of zeros and the number of poles, respectively, of \( f(z) \) in the region between these two circles. Thus, by the argument principle [145, p. 119], we see that the excess of the number of poles of \( f(z) \) in \( |w| < |z| \leq 1 \) over zeros is

\[
N_P - N_Z = -\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz + \frac{1}{2\pi i} \int_{C'} \frac{f'(z)}{f(z)} \, dz
= -\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz + \frac{1}{2\pi i} \int_C \frac{f'(wz)}{f(wz)} \, dz
= -\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz + \frac{1}{2\pi i} \int_C \left( \frac{f'(z)}{f(z)} + \frac{n}{z} \right) \, dz
= \frac{1}{2\pi i} \int_C \frac{n}{z} \, dz = n,
\]

where in the penultimate step we used (5.5.14). \( \Box \)

**Corollary 5.5.1.** Suppose that \( F(z) \) is analytic for \( z \neq 0 \) and satisfies (5.4.1) with \( C \neq 0 \). Then either \( F(z) \) has exactly \( n \) zeros in \( |q| < |z| \leq 1 \) or \( F(z) \equiv 0 \) for all \( z \).

**Proof.** In Lemma 5.5.3, take \( w = q \) and \( A = C \), replace \( n \) by \(-n\), and note that \( F(z) \) does not have any poles. \( \Box \)
Lemma 5.5.4. For $1 \leq r \leq 4$, we define

$$\begin{align*}
G_r(z) &:= q^{(r-2)^2/2}z^r \phi(-q^5)(q^{2r}, q^{5-2r}, q^5; q^5)_{\infty} (q^{6r}z^5, q^{30-6r}/z^5, q^{30}; q^{30})_{\infty} \\
&= \frac{q^{(r-2)^2}z^r \phi(-q^5)(q^{2r}z^5, q^{10-2r}/z^5, q^{10}; q^{10})_{\infty} (z^5, q^{30}/z^5, q^{30}; q^{30})_{\infty}}{(q^2, q^{10-2r}; q^{10})_{\infty} (z, q^{10}/z^5, q^{10})_{\infty} (q^{2r}, q^{10}; q^{10})_{\infty}} \tag{5.5.15}
\end{align*}$$

and

$$\begin{align*}
H_r(z) &:= -2q^3z^r A(z^5, q^{2r}, q^{10}) \\
&= -2q^3z^r \frac{(q^{10}; q^{10})_{\infty} (q^{2r}z^5, q^{10-2r}/z^5, q^{10}; q^{10})_{\infty} (z^5, q^{30}/z^5, q^{30}; q^{30})_{\infty}}{(q^2, q^{10-2r}; q^{10})_{\infty} (z, q^{10}/z^5, q^{10})_{\infty} (q^{2r}, q^{10}; q^{10})_{\infty}} \tag{5.5.16}
\end{align*}$$

where $A(z, x, q)$ is defined in Lemma 5.4.4. Furthermore, define

$$H_0(z) := \frac{2q^3(q^{10}; q^{10})_{\infty} (z^{10}, q^{30}/z^{10}, q^{30}; q^{30})_{\infty}}{(z^5, q^{10}/z^5, q^{10})_{\infty} (z, q^{10}/z^5, q^{10})_{\infty} (q^{2r}, q^{10}; q^{10})_{\infty}} \tag{5.5.17}$$

If $z \neq 0, \omega q^{2k}$, where $\omega$ is an arbitrary fifth root of unity and $k$ is any integer, then

$$B(z) = G_1(z) + G_2(z) - G_3(z) - G_4(z) + \sum_{r=0}^{4} H_r(z),$$

where $B(z)$ is defined in (5.5.1).

Proof. Let

$$V(z) := B(z) - G_1(z) - G_2(z) + G_3(z) + G_4(z) - \sum_{r=0}^{4} H_r(z). \tag{5.5.18}$$

Our lemma is equivalent to showing that $V(z)$ is identically equal to 0.

Now each of $B(z)$, $G_r(z)$, and $H_r(z)$ satisfies the functional equation

$$f(q^{6}z) = -z^{-5}f(z).$$

Also, $B(z)$ and $H_0(z)$ satisfy the equation

$$f(z^{-1}) = z^{-5}f(z).$$

Furthermore, for $1 \leq r \leq 4$,

$$G_r(z^{-1}) = -z^{-5}G_{5-r}(z),$$

while, for $0 \leq r \leq 4$,

$$H_r(z^{-1}) = z^{-5}H_{5-r}(z).$$

Therefore,

$$V(q^{6}z) = -z^{-5}V(z) \tag{5.5.19}$$

and
Next, we observe that \( V(z) \) is meromorphic for \( z \neq 0 \) with, at most, simple poles at the points \( z = \omega q^{6k \pm 2} \), where \( \omega \) is any fifth root of unity and \( k \) is any integer. We will now show that the residue at each possible pole is actually equal to 0. This, in turn, reveals that \( V(z) \) is analytic for all \( z \neq 0 \).

We begin with \( z = \omega q^2 \). By (5.5.12), the residue of \( B(z) \) at \( q^2 \) is equal to \( 2q^5 \), and \( B(z) \) is analytic at \( z = \omega q^2 \) if \( \omega \neq 1 \). The functions \( G_r(z) \), \( 1 \leq r \leq 4 \), are analytic for \( z \neq 0 \). For \( 1 \leq r \leq 4 \), the residue of \( H_r(z) \) at \( z = \omega q^2 \) equals, by (5.5.16) and Lemma 5.4.5,

\[
\lim_{z \to \omega q^2} (z - \omega q^2) H_r(z) = \lim_{z \to \omega q^2} (z - \omega q^2) \frac{2q^3 z^r q^{10-2r} z^{-5}}{1 - q^{10} z^{-5}} = 2\omega^{r+1} q^5.
\]

When \( r = 0 \), we see that the residue of \( H_0(z) \), given by (5.5.17), at \( z = \omega q^2 \) is

\[
\lim_{z \to \omega q^2} (z - \omega q^2) H_0(z) = \frac{2q^3 (q^{10}; q^{10})_\infty (q^{10}; q^{50}; q^{30})_\infty}{(q^{10}; q^{10}; q^{10})_\infty} \lim_{z \to \omega q^2} \frac{(z - \omega q^2)}{(1 - q^{10}/z^5)} = \frac{2q^3}{5q^{-2} \omega^{-1}} = \frac{2\omega q^5}{5}.
\]

Therefore, the residue of \( V(z) \) at \( z = q^2 \) is equal to

\[
2q^5 - \sum_{r=0}^{4} \frac{2}{5} q^{5r} = 0,
\]

and at \( z = \omega q^2 \), with \( \omega \neq 1 \), is equal to

\[
0 - \sum_{r=0}^{4} \frac{2}{5} \omega^{r+1} q^{5} = 0.
\]

Hence, \( V(z) \) is analytic at \( z = \omega q^2 \), for any fifth root of unity, and by the functional equations (5.5.19) and (5.5.20), \( V(z) \) is analytic at all \( z = \omega q^{6k \pm 2} \) and thus analytic for all \( z \neq 0 \).

Now we shall invoke Corollary 5.5.1 with \( C = -1 \), \( n = 5 \), and \( q \) replaced by \( q^6 \). By Corollary 5.5.1, we see that \( V(z) \) is identically 0 (as desired), provided that we can find six zeros of \( V(z) \) inside \( |q^6| < |z| \leq 1 \). We shall show that, in fact, \( V(z) = 0 \) for \( z = -q, -q^2, \pm q^3, -q^4, \) and \( -q^5 \). This will then complete the proof of our lemma.

By (5.5.19) and (5.5.20),

\[
V(q^6 z^{-1}) = -z^5 V(z^{-1}) = -V(z).
\]
5.5 Proof of the Mock Theta Conjectures

Hence,

\[ V(\pm q^3) = 0, \quad (5.5.21) \]
\[ V(-q^4) = V(-q^2), \quad (5.5.22) \]
\[ V(-q^5) = -V(-q). \quad (5.5.23) \]

Hence, we will have the six desired zeros provided that we can show that

\[ V(-q) = V(-q^2) = 0. \quad (5.5.24) \]

By (5.5.16), with \(1 \leq r \leq 4\), and by (4.1.3),

\[
\frac{H_{5-r}(z)}{H_r(z)} = \frac{z^{5-2r} f(-q^{10-2r} z^5, -q^{2r} / z^5)}{f(-q^{2r} z^5, -q^{10-2r} / z^5)} = -\frac{q^{2r} z^{-2r} f(-q^{-2r} z^5, -q^{2r+10} / z^5)}{f(-q^{2r} z^5, -q^{10-2r} / z^5)}.
\]

Thus,

\[
\frac{H_{5-r}(-q^k)}{H_r(-q^k)} = -\frac{q^{2r-2r} k f(q^{5k-2r}, q^{2r+10-5k})}{f(q^{5k+2r}, q^{10-2r-5k})}.
\]

If \(k = 1\), the identity above gives

\[
\frac{H_{5-r}(-q)}{H_r(-q)} = -\frac{f(q^{5-2r}, q^{2r+5})}{f(q^{5+2r}, q^{5-2r})} = -1,
\]

while if \(k = 2\), we obtain, upon an application of (4.1.3),

\[
\frac{H_{5-r}(-q^2)}{H_r(-q^2)} = -\frac{q^{-2r} f(q^{10-2r}, q^{2r})}{f(q^{10+2r}, q^{-2r})} = -\frac{f(q^{-2r}, q^{2r+10})}{f(q^{10+2r}, q^{-2r})} = -1.
\]

The last two equations imply that

\[
\sum_{r=1}^{4} H_r(-q^k) = 0.
\]

We note that, by (5.5.1),

\[ B(-q^k) = 0. \]

It thus remains to examine

\[ V(-q^k) = -G_1(-q^k) - G_2(-q^k) + G_3(-q^k) + G_4(-q^k) - H_0(-q^k) \quad (5.5.25) \]
\[ = (-G_1(-q^k) + G_4(-q^k)) + (-G_2(-q^k) + G_3(-q^k)) - H_0(-q^k) \]
\[ = \frac{\varphi(-q^5)}{(q; q)_\infty} \{ q^{k+1} (q^2, q^3, q^5; q^5)_\infty \{ (-q^{5k+6}, -q^{24-5k}, q^{30}, q^{30})_\infty \}
\]
\[ - q^{3k} (-q^{5k+24}, -q^{6-5k}, q^{30}, q^{30})_\infty \}
\[ - q^{2k} (q, q^4, q^5; q^5)_\infty \{ (-q^{5k+12}, -q^{18-5k}, q^{30}, q^{30})_\infty \}
\]
\[ - q^k (-q^{5k+18}, -q^{12-5k}, q^{30}, q^{30})_\infty \} - H_0(-q^k). \]
In the two cases of interest here, namely, \( k = 1, 2 \), the two product differences reduce to one product by Lemma 5.3.1, the quintuple product identity. In particular, if we replace \( q \) by \( q^{10} \) and then set \( x = -q^3 \) in Lemma 5.3.1, we find that

\[
(q^{10}; q^{10})_\infty (q^6; q^4; q^{10}; q^{10})_\infty \\
= (q^3; q^3)_{\infty} (q^6; q^2; q^{10}; q^{10})_\infty.
\] (5.5.26)

If we replace \( q \) by \( q^{10} \) and then set \( x = -q \) in Lemma 5.3.1, we deduce that

\[
(q^{10}; q^{10})_\infty (q^{2}; q^{8}; q^{10}; q^{10})_\infty \\
= (q; q^9; q^{10}; q^{10})_\infty.
\] (5.5.27)

Replacing \( q \) by \( q^{10} \), then setting \( x = -q^2 \) in Lemma 5.3.1, and invoking (4.1.3), we deduce that

\[
(q^{10}; q^{10})_\infty (q^6; q^2; q^{10}; q^{10})_\infty \\
= (q^2; q^8; q^{10}; q^{10})_\infty.
\] (5.5.28)

Lastly, replacing \( q \) by \( q^{10} \), then setting \( x = -q^4 \) in Lemma 5.3.1, and applying (4.1.3), we arrive at

\[
(q^{10}; q^{10})_\infty (q^8; q^2; q^{10}; q^{10})_\infty \\
= (q^4; q^6; q^{10}; q^{10})_\infty.
\] (5.5.29)

Consequently, returning to (5.5.25) when \( k = 1 \) and invoking (5.5.26) and (5.5.27), we see that

\[
V(-q) = \frac{q^2 \varphi(-q^5)}{(q; q)_\infty} \left\{ \frac{f(-q^2; -q^3; q^{10}; q^{10})_\infty f(-q^4; -q^6)}{f(q^3; q^7)} - \frac{f(-q; -q^2; q^{10}; q^{10})_\infty f(-q^2; -q^4)}{f(q; q^3)} \right\} - H_0(-q).
\] (5.5.30)

We shall put the two quotients in curly brackets above under one common denominator. To that end, by the Jacobi triple product identity (4.1.2),

\[
f(q^3; q^7) f(q; q^9) = (q^3; q^7; q^{10}; q; -q; -q^9; q^{10}; q^{10})_\infty \\
= \frac{(q; q^2)_\infty (q^{10}; q^{10})_\infty^2}{(-q^2; q^{10})_\infty} = \frac{(q^2; q^4)_\infty (q^{10}; q^{10})_\infty^2 (q^2; q^{10})_\infty}{(q; q^2)_\infty (q^{10}; q^{20})_\infty}.
\] (5.5.31)
In the numerator of (5.5.30), we have
\[ f(-q^2, -q^3) f(-q^4, -q^6) f(q, q^9) - f(-q, -q^4) f(-q^2, -q^8) f(q^3, q^7) \]
\[ = (q^2, q^4, q^7, q^8, q^{10}, q^{14}, q^{16}, q^{20})_\infty f(q, q^9) \]
\[ - (q, q^4, q^6, q^8, q^{10}, q^{12}, q^{16}, q^{20})_\infty f(q^3, q^7) \]
\[ = (q^5; q^{10})_\infty (q^2, q^4, q^6, q^8, q^{10}, q^{14}, q^{16}, q^{20})_\infty \]
\[ \times \{ (q^3, q^7, q^{10}, q^{14})_\infty f(q, q^9) - (q, q^9, q^{10}, q^{14})_\infty f(q^3, q^7) \} \]
\[ = (q^5; q^{10})_\infty (q^2; q^7)_\infty \{ f(-q^3, -q^7) f(q, q^9) - f(-q, -q^9) f(q^3, q^7) \} . \]
(5.5.32)

We now apply Lemma 5.3.3 with \( q \) replaced by \( q^{10} \), and then with \( x = q \) and \( y = q^3 \) to deduce, via the Jacobi triple product identity (4.1.2), that
\[ f(-q^3, -q^7) f(q, q^9) - f(-q, -q^9) f(q^3, q^7) = 2q f(-q^2, -q^{18}) f(-q^{14}, -q^6) \]
\[ = 2q (q^2; q^{18}, q^{20}, q^{24}, q^{26}, q^{29})_\infty \]
\[ = 2q (q^2; q^{24})_\infty (q^{20}, q^{29})_\infty \]
(5.5.33)

Now put (5.5.33) into (5.5.32), and then put (5.5.31) and (5.5.32) into (5.5.30) to finally conclude that
\[ V(-q) = \frac{q^2 \varphi(-q^5) (q^{10}; q^{10})_\infty (q^{10}; q^{20})_\infty (q^2; q^2)_\infty (q^5; q^{10})_\infty (q^2; q^2)_\infty}{(q; q)_\infty (q^2; q^4)_\infty (q^{10}; q^{14})_\infty (q^{10}; q^{16})_\infty (q^5; q^{10})_\infty} \]
\[ \times \frac{2q (q^2; q^{10})_\infty (q^{20}; q^{20})_\infty}{(q^2; q^2)_\infty} - H_0(-q) \]
\[ = 2q^3 \frac{(q^5; q^{10})_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{20})_\infty} - H_0(-q) \]
\[ = 2q^3 \left\{ \frac{(q^5; q^{10})_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{20})_\infty} - \frac{(q^{10}; q^{10})_\infty (q^5; q^{10})_\infty}{(q^{10}; q^{20})_\infty} \right\} \]
\[ = 2q^3 \left\{ \frac{(q^5; q^{10})_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{20})_\infty} - \frac{(q^5; q^{10})_\infty (q^5; q^{10})_\infty}{(q^{20}; q^{20})_\infty} \right\} = 0. \]

where in the third equality above we used (1.3.1) and the definition of \( H_0(z) \) from (5.5.17). Thus, we have demonstrated that the first portion of (5.5.24) holds.

For \( k = 2 \), we return to (5.5.25). Invoking (5.5.28) and (5.5.29) below and also employing (4.1.3), we find that
\[ V(-q^2) = \frac{q^2 \varphi(-q^5)}{(q; q)_\infty} \{ f(-q^2, q^9) - f(-q, q^5) \} f(q^{16}, q^{14}) - q^6 f(q^{34}, q^{31}) \]
We now put (5.5.35) and (5.5.36) into (5.5.34). Using (1.3.1) and the definition
of \(-q^2\) and \((-q^2, -q^6)\), we get
\[
\frac{q^3 \varphi(-q^5)(q^{10}; q^{10})_\infty \left\{ f(-q^2, -q^3) f(-q^4, -q^6) \right\}}{q^2 f(q^2, q^6)} - q f(-q, -q^4) f(-q^2, -q^6) = H_0(-q^2). 
\tag{5.5.34}
\]

As in the case when \(k = 1\), we combine the two quotients in curly brackets on the right-hand side above. We first examine the denominator, which is given by
\[
f(q^2, q^6) f(q^4, q^6) = (q^2, -q^8)(q^{10}, q^{10})_\infty \quad \quad \tag{5.5.35}
\]

We next examine the numerator. Applying below Lemma 5.3.2 with \(q\) replaced by \(-q^6\), and then with \(x = q\) and \(y = -q^2\) and the Jacobi triple product identity (4.1.2) several times, we find that
\[
f(-q^2, -q^3) f(-q^4, -q^6) f(q^4, q^6) = (-q^2, -q^8)(q^{10}, q^{10})_\infty \quad \quad \tag{5.5.36}
\]

We now put (5.5.35) and (5.5.36) into (5.5.34). Using (1.3.1) and the definition of \(H_0(z)\) from (5.5.17), we find that
\[
V(-q^2) = \frac{q^3 \varphi(-q^5)(q^{10}, q^{10})_\infty (q^2, q^2)_\infty (q^{20}, q^{20})_\infty}{(q^2, q^2)_\infty (q^{10}, q^{10})_\infty (q^{20}, q^{20})_\infty} \times \frac{(q; q^2)_\infty (q^4; q^4)_\infty (q^{10}, q^{10})_\infty (q^{20}, q^{20})_\infty}{(q^6; q^{10})_\infty (q^{20}, q^{20})_\infty} - H_0(-q^2)
\]

\[
= q^3 \varphi(-q^5)(q^{10}; q^{10})_\infty (q^{10}, q^{20})_\infty - H_0(-q^2)
\]
5.5 Proof of the Mock Theta Conjectures

\[
\frac{q^3(q^5; q^5)_\infty(q^{10}; q^{10})_\infty(q^{20}; q^{20})_\infty}{(-q^3; q^3)_\infty(q^5; q^{10})_\infty} - H_0(-q^2)
\]

\[
\frac{q^3(q^5; q^5)_\infty(q^{10}; q^{10})_\infty(q^{20}; q^{20})_\infty}{(q^5; q^{10})_\infty} - H_0(-q^2)
\]

\[
q^3(q^{10}; q^{10})_\infty(q^{20}; q^{20})_\infty - \frac{2q^3(q^{10}; q^{10})_\infty(q^{20}; q^{10}, q^{30}; q^{30})_\infty}{(-q^{10}, -1; q^{10})_\infty}
\]

\[
q^3\varphi^2(-q^{10}) - q^3 \frac{(q^{10}; q^{10})_\infty^2}{(-q^{10}; q^{10})_\infty^2}
\]

\[
q^3\varphi^2(-q^{10}) - q^3\varphi^2(-q^{10}) = 0.
\]

Hence, the second part of (5.5.24) holds. We have therefore proved that
\[V(z) = 0\] for six distinct values of \(z\). Hence, \(V(z)\) is identically equal to 0, as was to be shown.

\[\square\]

**Theorem 5.5.1 (The Mock Theta Conjectures).** Equations (4.1.11) from the first group and (4.1.18) from the second group in Chapter 4 are valid.

**Proof.** Lemmas 5.5.2 and 5.5.4 each provide a 5-dissection of the function \(B(z)\). Equating the coefficients of \(z,\) and \(z^2\) in these two different representations for \(B(z)\) enables us to complete our proofs of the Mock Theta Conjectures.

We see that Lemma 5.5.2 allows us to decompose \(B(z)\) in the form

\[B(z) = \sum_{j=0}^{4} z^j B_j(z^5),\]

where, as we shall demonstrate,

\[B_1(z^5) = qf_0(q)(q^6 z^5, q^24/z^5, q^{30}; q^{30})_\infty + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+21r+5} z^{5r+5}}{1 - q^{30r+10} z^5} + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+39r+11} z^{-5r-5}}{1 - q^{30r+10} z^{-5}} \tag{5.5.37}\]

and

\[B_2(z^5) = f_1(q)(q^{12} z^5, q^{18}/z^5, q^{30}; q^{30})_\infty + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+27r+7} z^{5r+5}}{1 - q^{30r+10} z^5} + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+33r+9} z^{-5r-5}}{1 - q^{30r+10} z^{-5}} \tag{5.5.38}\]

The identities (5.5.37) and (5.5.38) follow from Lemma 5.5.2 once we observe that
and
\[
\frac{1}{1 - q^{6r+2}z} = \frac{1}{1 - q^{30r+10}z^5} \sum_{j=0}^{4} z^j q^{(6r+2)j}
\]

and
\[
\frac{1}{1 - q^{6r+2}z^{-1}} = \frac{z^{-5}}{1 - q^{30r+10}z^{-5}} \sum_{j=1}^{5} z^j q^{(6r+2)(5-j)}.
\]

On the other hand, by Lemma 5.5.4,
\[
B_1(z^5) = z^{-1} (G_1(z) + H_1(z))
= \frac{q\varphi(-q^5)(q^2, q^3, q^5; q^5)_\infty (q^6 z^5, q^{24}/z^5, q^{30}; q^3)_\infty}{(q; q)_\infty} - 2q^3 A(z^5, q^2, q^{10})
\]

and
\[
B_2(z^5) = z^{-2} (G_2(z) + H_2(z))
= \frac{\varphi(-q^5)(q^4, q^5; q^5)_\infty (q^{12} z^5, q^{18}/z^5, q^{30}; q^3)_\infty}{(q; q)_\infty} - 2q^3 A(z^5, q^4, q^{10}).
\]

By Lemma 5.4.5,
\[
A(z^5, q^2, q^{10}) = (q^6 z^5, q^{24}/z^5, q^{30}; q^3)_\infty g(q^2, q^{10})
- \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+21r+2} z^{5r+5}}{1-q^{30r+10}z^5}
- \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+39r+8} z^{-5r-5}}{1-q^{30r+10}z^{-5}}
\]

and
\[
A(z^5, q^4, q^{10}) = (q^{12} z^5, q^{18}/z^5, q^{30}; q^3)_\infty g(q^4, q^{10})
- \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+27r+4} z^{5r+5}}{1-q^{30r+10}z^5}
- \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+33r+6} z^{-5r-5}}{1-q^{30r+10}z^{-5}}.
\]

Put (5.5.41) into (5.5.39) and put (5.5.42) into (5.5.40) to deduce, respectively, that
\[
B_1(z^5) = \frac{q\varphi(-q^5)(q^6 z^5, q^{24}/z^5, q^{30}; q^3)_\infty}{(q; q)_\infty (q^6, q^5)_\infty} - 2q^3 (q^6 z^5, q^{24}/z^5, q^{30}; q^3)_\infty g(q^2, q^{10})
+ 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+21r+5} z^{5r+5}}{1-q^{30r+10}z^5}
+ 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+39r+11} z^{-5r-5}}{1-q^{30r+10}z^{-5}}
\]

and
5.5 Proof of the Mock Theta Conjectures 91

\[ B_2(z^5) = \frac{\varphi(-q^5)(q^{12}z^5, q^{18}/z^5, q^{30}; q^{30})}{(q^2; q^4)_\infty(q^3; q^5)_\infty} \]  
\[ - 2q^3(q^{12}z^5, q^{18}/z^5, q^{30}; q^{30}) g(q^4, q^{10}) \]  
\[ + 2 \sum_{r=0}^{\infty} \frac{(-1)^r q^{15r^2 + 27r + 7} z^5}{1 - q^{30r + 10} z^5} + 2 \sum_{r=0}^{\infty} \frac{(-1)^r q^{15r^2 + 33r + 9} z^{-5}}{1 - q^{30r + 10} z^{-5}}. \]  

(5.5.44)

Observe from (5.5.41) that the contributions of the generalized Lambert series to (5.5.37) and (5.5.43) are identical, and also observe from (5.5.42) that the contributions of the generalized Lambert series to (5.5.38) and (5.5.44) are also identical.

Extracting the coefficient of \( z^0 \) in \( B_1(z^5) \) first from (5.5.37) and second from (5.5.43), and dividing both sides by \( q \), we see that

\[ f_0(q) = \frac{\varphi(-q^5)}{(q^2; q^4)_\infty(q^3; q^5)_\infty} - 2q^2 g(q^2, q^{10}). \]  

(5.5.45)

From (5.4.10), we see that

\[ g(q^2, q^{10}) = \frac{1}{q^2} \left( -1 + \sum_{n=0}^{\infty} \frac{q^{10n^2}}{(q^2; q^{10})_{n+1}(q^8; q^{10})_{n}} \right) = \frac{1}{q^2} \Phi(q^2), \]  

(5.5.46)

by (4.1.6). Putting (5.5.46) into (5.5.45) and using (2.1.16), we see that (5.5.45) is equivalent to equation (4.1.11) of Chapter 4.

Extracting the coefficient of \( z^0 \) first from (5.5.38) and second from (5.5.44), we deduce that

\[ f_1(q) = \frac{\varphi(-q^5)}{(q^2; q^4)_\infty(q^3; q^5)_\infty} - 2q^3 g(q^4, q^{10}). \]  

(5.5.47)

From (5.4.10), we see that

\[ g(q^4, q^{10}) = \frac{1}{q^4} \left( -1 + \sum_{n=0}^{\infty} \frac{q^{10n^2}}{(q^4; q^{10})_{n+1}(q^6; q^{10})_{n}} \right) = \frac{1}{q^4} \Psi(q^2), \]  

(5.5.48)

by (4.1.13). Substituting (5.5.48) in (5.5.47) and employing (2.1.17), we observe that (5.5.47) yields (4.1.18) in Chapter 4.

As we stated in the Introduction of this chapter, the foregoing proof is the original one due to Hickerson [86]. □
Sixth Order Mock Theta Functions

6.1 Introduction

Unlike the fifth order mock theta functions, the sixth order mock theta functions (so named in [24]) seemingly do not yield to any elementary considerations. Consequently, this chapter will, of necessity, be somewhat long in order to include not only analogues of the mock theta conjectures (cf. Chapter 5), but also the various relations between these functions (cf. Chapter 4).

We shall be treating two principal functions,

\[
\phi_6(q) := \sum_{n=0}^\infty \frac{(-1)^n(q; q^2)_n q^{n^2}}{(-q; q)_{2n}} \tag{6.1.1}
\]

and

\[
\psi_6(q) := \sum_{n=0}^\infty \frac{(-1)^n(q; q^2)_n q^{(n+1)^2}}{(-q; q)_{2n+1}}. \tag{6.1.2}
\]

Ramanujan wrote \(\phi(q)\) for \(\phi_6(q)\) and \(\psi(q)\) for \(\psi_6(q)\); however, we must add the subscript 6 to avoid confusion with our more standard use of \(\phi(q)\) and \(\psi(q)\) throughout these volumes.

There are five further relevant functions. We have chosen notation for them; Ramanujan did not. These functions are defined by

\[
\rho_6(q) := \sum_{n=0}^\infty \frac{(-q; q)_n q^{\frac{n+1}{2}}}{(q; q^2)_{n+1}}, \tag{6.1.3}
\]

\[
\sigma_6(q) := \sum_{n=0}^\infty \frac{(-q; q)_n q^{\frac{n+2}{2}}}{(q; q^2)_{n+1}}, \tag{6.1.4}
\]

\[
\lambda_6(q) := \sum_{n=0}^\infty \frac{(-1)^n(q; q^2)_n q^n}{(-q; q)_n}, \tag{6.1.5}
\]
We note that the series for $\mu_6(q)$ does not converge. However, the sequence of even-indexed partial sums converges as does that of the odd-indexed partial sums. We define $\mu_6(q)$ as the average of these two values. Furthermore, Ramanujan used the notation $\phi(q)$ for both $\phi_6(q)$ and $\Phi_6(q)$, and so we have distinguished these two functions using the upper and lower cases.

Ramanujan found several Appell–Lerch series expansions for $\phi_6(q)$ and $\psi_6(q)$ and several identities connecting the functions listed in (6.1.1)–(6.1.8). To our surprise, we were unable to find elementary proofs (in the sense of Chapters 2 and 4) for any of these results.

The remainder of this chapter is organized along the lines similar to Chapter 5. The next section is devoted to lemmas regarding standard theta functions. Section 6.3 provides representations of these functions analogous to those provided for $f_0(q)$ and $f_1(q)$ in Section 5.2 of Chapter 5. Section 6.4 is devoted to identities for $\phi_6(q)$ and $\psi_6(q)$ analogous to the mock theta conjectures in Section 5.5 in Chapter 5. Once these results are in hand, we are in a position to prove all of the remaining identities asserted by Ramanujan that relate $\phi_6(q)$ and $\psi_6(q)$ to $\rho_6(q)$, $\sigma_6(q)$, $\lambda_6(q)$, $\mu_6(q)$, and $\gamma_6(q)$. All of this material is based on [24]. In Section 6.6 we consider two outlier identities, one of which was proved in [32]; the other has not been heretofore proved.

### 6.2 Theta Function Identities

We shall require many of the results from Section 5.3, Chapter 5. In addition, we need the following results, each of which is proved by classical means.

**Lemma 6.2.1.** Let $|q| < 1$ and let $\omega$ be a primitive cube root of unity. Then

$$
\sum_{n=-\infty}^{\infty} (-\omega)^n q^{n(n-1)/2} = (1 - \omega)(q^3; q^3)_\infty,
$$

(6.2.1)

and

$$
\sum_{n=-\infty}^{\infty} \omega^n q^{n(n-1)/2} = \frac{(1 + \omega)\varphi(-q)(q^6; q^6)_\infty}{(q^3; q^3)_\infty}.
$$

(6.2.2)
and
\[
\sum_{n=-\infty}^{\infty} (-1)^n(\omega q^2)^{n(n-1)/2} q^n \sum_{n=-\infty}^{\infty} (-1)^n(\omega^2 q^2)^{n(n-1)/2} q^n = \frac{(q^6; q^6)_\infty(q, q^5, q^6; q^6)_\infty}{(q^2; q^2)_\infty(q^3, q^{15}; q^{18})_\infty}. \tag{6.2.3}
\]

**Proof.** We first examine (6.2.1). In each of the three identities of Lemma 6.2.1 we require Jacobi’s triple product identity (4.1.2). First,
\[
\sum_{n=-\infty}^{\infty} (-\omega)_n q^n = (\omega, q\omega^{-1}, q; q)_\infty
\]
\[
= (1 - \omega) \prod_{n=1}^{\infty} (1 - \omega q^n)(1 - \omega^{-1} q^n)(1 - q^n)
\]
\[
= (1 - \omega) \prod_{n=1}^{\infty} (1 - q^{3n}) = (1 - \omega)(q^3; q^3)_\infty, \tag{6.2.4}
\]
which proves (6.2.1).

Second, by (6.2.4) and (1.3.1),
\[
\sum_{n=-\infty}^{\infty} \omega q^n = (-\omega, -\omega^2 q, q; q)_\infty
\]
\[
= (1 + \omega) \prod_{n=1}^{\infty} (1 + \omega q^n)(1 + \omega^2 q^n)(1 - q^n)
\]
\[
= (1 + \omega) \prod_{n=1}^{\infty} \frac{(1 + q^{3n})(1 - q^n)}{(1 + q^n)}
\]
\[
= (1 + \omega) \frac{\varphi(-q)(q^6; q^6)_\infty}{(q^3; q^3)_\infty},
\]
and so (6.2.2) has been proved.

Finally, by (6.2.4),
\[
\sum_{n=-\infty}^{\infty} (-1)^n(\omega q^2)^{n(n-1)/2} q^n \sum_{n=-\infty}^{\infty} (-1)^n(\omega^2 q^2)^{n(n-1)/2} q^n
\]
\[
= (q, \omega q, \omega q^2; \omega q^2)_\infty(q, \omega^2 q, \omega^2 q^2; \omega^2 q^2)_\infty
\]
\[
= (q, \omega q^3, \omega^2 q^5; q^6)_\infty(\omega q, \omega^2 q^3, \omega^5; q^6)_\infty
\]
\[
\times (\omega^2 q^2, \omega^4 q^4, q^6; q^6)_\infty(q, \omega^2 q^3, \omega^5; q^6)_\infty
\]
\[
\times (\omega^2 q, \omega^3 q^3, q^5; q^6)_\infty(\omega^2 q^2, \omega^4 q^4, q^6; q^6)_\infty
\]
\[
= (q, q^5, q^6)^2(\omega q, \omega q^2, \omega^3 q, \omega^4, q^5; q^6)_\infty
\]
Lemma 6.2.2. If \( H \) is the sum on the right side of (6.2.6), we define

\[
L(q) = \frac{(q, q^6; q^6)_{\infty}}{(q^2; q^2)_{\infty}(q^3, q^{15}, q^{18}; q^{18})_{\infty}}.
\]

Hence, (6.2.3) has been established. \( \Box \)

Lemma 6.2.2. If \( n > 0 \) and \( 0 < |q| < 1 \), then

\[
\frac{(q; q^3)_{\infty}(xz, q/(xz), q; q)_{\infty}(x^n, q^n/x^n, q^n; q^n)_{\infty}}{(q^n; q^n)_{\infty}(x, q/x, q; q)_{\infty}(z, q/z, q; q)_{\infty}} = \sum_{k=0}^{n-1} x^k(x^n z, q^{n-k}/(x^n z), q^n; q^n)_{\infty},
\]

where neither \( x \) nor \( z \) is an integral power of \( q \).

Proof. Letting \( L(z) \) denote the left side of (6.2.6) and \( R_k(z) \) denote the \( k \)th term in the sum on the right side of (6.2.6), we define

\[
V(z) := L(z) - \sum_{k=0}^{n-1} R_k(z). \tag{6.2.7}
\]

We shall prove the lemma by showing that \( V(z) \) is identically equal to 0.

By the manipulation of \( q \)-products, we see that

\[
R_k(qz) = x^{-1}R_{k+1}(z), \quad 0 \leq k \leq n-2,
\]

while

\[
R_{n-1}(qz) = \frac{x^{n-1}(q^n x^n z, 1/(x^n z), q^n; q^n)_{\infty}}{(q^n z, 1/z, q^n; q^n)_{\infty}} = \frac{(1-z)(1-1/(x^n z))x^{n-1}(x^n z, q^n/(x^n z), q^n; q^n)_{\infty}}{(1-z)(1-x^n z)(z, q^n/z, q^n; q^n)_{\infty}} = x^{-1}R_0(z). \tag{6.2.8}
\]

Also,

\[
L(qz) = \frac{(q; q^3)_{\infty}(xqz, 1/(xz), q; q)_{\infty}(x^n, q^n/x^n, q^n; q^n)_{\infty}}{(q^n; q^n)_{\infty}(x, q/x, q; q)_{\infty}(qz, 1/z, q; q)_{\infty}} = \frac{(1-1/(xz))(1-z)(q; q^3)_{\infty}(xz, q/(xz), q; q)_{\infty}(x^n, q^n/x^n, q^n; q^n)_{\infty}}{(1-xz)(1-1/z)(q^n; q^n)_{\infty}(x, q/x, q; q)_{\infty}(z, q/z, q; q)_{\infty}} = x^{-1}L(z). \tag{6.2.9}
\]
Hence, by (6.2.7), (6.2.8), and (6.2.9),
\[ V(qz) = x^{-1}V(z). \] (6.2.10)

We also see that \( V(z) \) is meromorphic with possible simple poles at \( z = q^m \), for each integer \( m \). For \( z = 1 \), we see, by Lemma 5.4.1, that both \( L(z) \) and \( R_0(z) \) have residue
\[ -\frac{(x^n, q^n/x^n, q^n; q^n)_\infty}{(q^n; q^n)_\infty^3} \]
at \( z = 1 \). On the other hand, \( R_k(z) \) is analytic at \( z = 1 \) for \( 1 \leq k \leq n - 1 \). Hence, \( V(z) \) has residue 0 at \( z = 1 \), i.e., \( V(z) \) is analytic at \( z = 1 \). Therefore, by (6.2.10), \( V(z) \) is analytic at all integral powers of \( q \) and thus for all \( z \neq 0 \). We may then expand \( V(z) \) in a Laurent series around \( z = 0 \), say,
\[ V(z) = \sum_{n=-\infty}^{\infty} V_n z^n, \]
and (6.2.10) implies that, for each integer \( n \),
\[ V_n q^n = x^{-1}V_n. \]

But \( x \) is required not to be an integral power of \( q \). Hence, \( V_n = 0 \) for every integer \( n \). Thus, \( V(z) \equiv 0 \), and consequently Lemma 6.2.2 is proved. \( \Box \)

**Lemma 6.2.3.** Let \( n \) be a positive integer, \( 0 < \vert q \vert < 1 \), and \( x \neq 0 \). Then
\[ (x, q/x, q; q)_\infty = \sum_{k=0}^{n-1} (-1)^k x^k q^{k(k-1)/2} \]
\[ \times ((-1)^{n+1} q^{n(n-1)/2+kn} x^n, (-1)^{n+1} q^{n(n+1)/2-kn} x^{-n}, q^n; q^{n^2})_\infty. \] (6.2.11)

**Proof.** If we rewrite each of the products in (6.2.11) using Jacobi’s triple product identity (4.1.2), we see that we must prove that
\[ \sum_{N=\infty}^{\infty} (-1)^N x^N q^{N(N-1)/2} \]
\[ = \sum_{k=0}^{n-1} (-1)^k x^k q^{k(k-1)/2} \sum_{s=-\infty}^{\infty} (-1)^{ns} x^s q^{n^2 s(s-1)/2+ns(n(n-1)/2+kn).} \]
This last assertion follows by setting \( N = ns + k, 0 \leq k \leq n - 1, -\infty < s < \infty \), in the sum on the left side above. \( \Box \)

**Lemma 6.2.4.** Let \( n \) be a positive integer, \( 0 < \vert q \vert < 1 \), and \( x \) and \( y \) both non-zero. Then
\[
(x, q/x, q; q)_{\infty}(y, q^n/y, q^n; q^n)_{\infty} = \sum_{k=0}^{n} (-1)^{k} x^k y^{(k-1)/2} \times ((-1)^{n} q^{(n-1)/2 + kn} x^ny^{n+1}; q^{n+1})_{\infty} \times ((-q^{1-k}x^{-1}y, -q^{k+n}x^{-1}y^{n+1}; q^{n+1})_{\infty}. \tag{6.2.12}
\]

**Proof.** The left-hand side of (6.2.12) reduces via Jacobi’s triple product identity (4.1.2) to

\[
\sum_{r=-\infty}^{\infty} (-1)^{r} x^{r} q^{(r-1)/2} \sum_{s=-\infty}^{\infty} (-1)^{s} y^{s} q^{n(s-1)/2} = \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} x^{r} y^{s} q^{(r-1)/2 + ns(s-1)/2}. \tag{6.2.13}
\]

We now dissect this double sum into \(n+1\) parts according to the residue class of \(r+s\) modulo \((n+1)\). When \(r+s \equiv k \pmod{(n+1)}\), we write \(r=nu-v+k\) and \(s=u+v\). This makes \(u\) and \(v\) independent integers, each running from \(-\infty\) to \(\infty\), because

\[
u = \frac{s + r - k}{n + 1} \quad \text{and} \quad v = \frac{sn - r + k}{n + 1}.
\]

Hence, the sum on the right-hand side of (6.2.13) becomes

\[
\sum_{k=0}^{n} \sum_{u,v=-\infty}^{\infty} (-1)^{(nu-v+k)+(u+v)} x^{nu-v+k} y^{n+1} q^{(nu-v+k) + (u+v)/(2 + n^2)}
\]

\[
= \sum_{k=0}^{n} (-1)^{k} x^k y^{(k-1)/2} \times \sum_{u=-\infty}^{\infty} (-1)^{(n+1)u} x^u y^{u} q^{(n+1)u(u-1)/2 + (n(n-1)/2 + nk)u}
\]

\[
\times \sum_{v=-\infty}^{\infty} x^{-v} y^{v} q^{(n+1)v(v-1)/2 + (1-k)u}.
\]

Applying Jacobi’s triple product identity (4.1.2) to both the \(u\)-sum and the \(v\)-sum, we deduce the right-hand side of (6.2.12). \(\square\)

**Lemma 6.2.5.** We have

\[
(q^9, q^9, q^{18}; q^{18})_{\infty} - 2(q^3, q^{15}; q^{18})_{\infty} = (q, q^2; q^2)_{\infty} = \varphi(-q), \tag{6.2.14}
\]

\[
2(-q^3, -q^6, q^9; q^9)_{\infty} + q(-1, -q^9, q^9; q^9)_{\infty} = (-1, -q, q; q)_{\infty}. \tag{6.2.15}
\]
6.3 Hecke-type Series for the Sixth Order Mock Theta Functions

\[(q^9, q^9, q^18; q^{18})\propto + q(q^3, q^{15}, q^{18}; q^{18})\propto = \frac{(q^2; q^2)\propto (q^6; q^6)\propto (q, q^6, q^6)\propto}{(q, q^5, q^6)\propto}, \quad (6.2.16)\]

and

\[(q, q^2; q^2)\propto + 3(q^9, q^9, q^{18}; q^{18})^2 = \frac{4(q^6; q^6)^3 (q, q^5, q^6, q^6)\propto}{(q^2; q^2)\propto (q^3, q^{15}, q^{18}; q^{18})\propto}, \quad (6.2.17)\]

Proof. Identities (6.2.14) and (6.2.15) are special cases of Lemma 6.2.3 with \(n = 3\) and the help of (4.1.3). More precisely, for (6.2.14), replace \(q\) by \(q^2\) and then \(x\) by \(q\); for (6.2.15), set \(x = -1\). The identity (6.2.16) is a special case of the quintuple product identity; namely, replace \(q\) by \(q^6\) and then set \(x = q\) in Lemma 5.3.1 of Chapter 5 and employ (4.1.2). For (6.2.17), we begin with Lemma 6.2.3 setting \(n = 3\), replacing \(q\) by \(\omega q^2\), where \(\omega\) is a primitive cube root of unity, and then putting \(x = q\). Thus,

\[
(q, \omega q, \omega q^2; q^2)\propto = (q^9, q^9, q^{18}; q^{18})\propto - (1 + \omega)q(q^3, q^{15}, q^{18}; q^{18})\propto
\]

\[= \frac{1}{2} \left(-\omega^2(q, q^2; q^2)\propto + (1 - \omega)(q^9, q^9, q^{18}; q^{18})\propto\right), \quad (6.2.18)\]

where in the last step we used (6.2.14).

We now replace \(\omega\) by \(\omega^2\) in (6.2.18) and then multiply the two equalities together to deduce that

\[
(q, \omega q, \omega q^2; q^2)\propto (q, \omega^2 q, \omega^2 q^2; \omega^2 q^2)\propto
\]

\[= \frac{1}{4} \left((q, q^2; q^2)\propto + 3(q^9, q^9, q^{18}; q^{18})^2\propto\right). \quad (6.2.19)\]

We now refer to the calculation (6.2.5) in order to simplify the left-hand side of (6.2.19). This then completes the proof of (6.2.17).

\(\square\)

6.3 Hecke-type Series for the Sixth Order Mock Theta Functions

We shall be using Bailey’s Lemma as in Section 5.2. However, we do not have analogues of the recurrence proofs utilized in Section 5.2. Consequently, the developments in this section will be much more intricate.

In Section 5.2, we only required a weak form of Bailey’s Lemma (namely, (5.2.7)). Now we require the full lemma. Recall that two sequences \(\{\alpha_n\}_{n \geq 0}\) and \(\{\beta_n\}_{n \geq 0}\) are called a Bailey pair (relative to \(a\)) provided that

\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r} (aq)_{n+r}}. \quad (6.3.1)
\]

This relation may be inverted [10, page 278, equation (4.1)]. Thus, equivalent to (6.3.1),
\[ \alpha_n = \frac{1 - a q^{2n}}{1 - a} \sum_{j=0}^{n} \frac{(-1)^{n-j}(a)_{n+j}q^{(n-j)} \beta_j}{(q)_{n-j}}. \]  

(6.3.2)

**Lemma 6.3.1 (Bailey’s Lemma).** If \( \{\alpha_n\} \) and \( \{\beta_n\} \) are a Bailey pair, then so are \( \{\alpha'_n\} \) and \( \{\beta'_n\} \), where

\[
\alpha'_n = \frac{(p_1)_n(p_2)_n(aq/(p_1 p_2))\alpha_n}{(aq/p_1)_n(aq/p_2)_n} \quad (6.3.3)
\]

and

\[
\beta'_n = \frac{1}{(aq/p_1)_n(aq/p_2)_n} \sum_{j=0}^{n} \frac{(p_1)_j(p_2)_j(aq/(p_1 p_2))_{n-j}(aq/p_1 p_2)^j \beta_j}{(q)_{n-j}}. \quad (6.3.4)
\]

We refer the reader to [14, p. 25] for a proof of this standard theorem.

If we now replace \( \beta_n \) by \( \beta'_n \) and \( \alpha_n \) by \( \alpha'_n \) in (6.3.1) and then let \( n \to \infty \), we obtain the following lemma.

**Lemma 6.3.2.** If \( \alpha_n \) and \( \beta_n \) form a Bailey pair relative to \( a \), then (assuming the absolute convergence of the two infinite series below)

\[
\sum_{n=0}^{\infty} \frac{(p_1)_n(p_2)_n(aq/(p_1 p_2))^n \alpha_n}{(aq/p_1)_n(aq/p_2)_n} = \frac{(aq)_{\infty}(aq/(p_1 p_2))_{\infty}}{(aq/p_1)_{\infty}(aq/p_2)_{\infty}} \sum_{n=0}^{\infty} \frac{(p_1)_n(p_2)_n(aq/(p_1 p_2))^n \beta_n}{(aq/p_1)_n(aq/p_2)_n}. \quad (6.3.5)
\]

Our use of Bailey’s Lemma is somewhat complicated. We first obtain a Bailey pair with several parameters and then we insert the result into Lemma 6.3.1 to obtain the Bailey pair that will provide the required information for the 6th order mock theta functions.

**Lemma 6.3.3.** The sequences \( \{A_n\} \) and \( \{B_n\} \) form a Bailey pair, where

\[
A_n = A_n(a, b, c, q) := \frac{(1 - a q^{2n})a^n q^{n^2 - n}}{1 - a} \sum_{j=0}^{n} \frac{(-1)^j(1 - a q^{2j-1})(a)_{j-1}(b)_j(c)_j}{(q)_j(a/b)_j(a/c)_j(bc)_j q^{j(j-1)/2}} \quad (6.3.6)
\]

and

\[
B_n = B_n(a, b, c, q) := \frac{(a/bc)_n}{(a/b)_n(a/c)_n(q)_n}. \quad (6.3.7)
\]

**Proof.** We replace \( \beta_j \) with \( B_j \) in (6.3.2). Thus,
\[
(1 - aq^{2n}) \sum_{j=0}^{n} \frac{(-1)^{n-j}(a)_{n+j-1}q^{jn}}{(q)_n} B_j
= \frac{(-1)^n(1 - aq^{2n})(aq)_n^{-1}q^{n(n-1)/2}}{(q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j(aq^n)_j(a/(bc))_j q^j}{(q)_j(a/b)_j(a/c)_j}. \tag{6.3.8}
\]

We must show that this latter expression equals \( A_n \). To see this, we refer to Watson’s \( q \)-analogue of Whipple’s theorem \cite[p. 81, equation (4.1.3)]{20} with \( \alpha, \beta, \gamma, \) and \( \delta \) replaced by \( a/q, b, c, \) and \( dq^N \), respectively. We then let \( d \to a \) and \( \epsilon \to 0 \). We thus obtain

\[
\sum_{j=0}^{n} \frac{(-1)^j(1 - aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{(q)_j(a/b)_j(a/c)_j(bc)^{j(j-1)/2}}
= \frac{(-1)^n(a)_n}{(q)_n a^n q^{n(n-1)/2}} \sum_{j=0}^{n} \frac{(q^{-n})_j(aq^n)_j(a/(bc))_j q^j}{(q)_j(a/b)_j(a/c)_j}, \tag{6.3.9}
\]

and this last expression allows us to reduce (6.3.8) to \( A_n \) as required. \qed

\textbf{Theorem 6.3.1.} The sequences \( \{A'_n\} \) and \( \{B'_n\} \) form a Bailey pair, where

\[
A'_n = A'_n(a, b, c, q) := \frac{(1 - aq^{2n})(a/b)_n(a/c)_n(bc)^n q^{n^2}}{(1 - a)(qb)_n(qc)_n} \sum_{j=0}^{n} \frac{(-1)^j(1 - aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{(q)_j(a/b)_j(a/c)_j(bc)^{j(j-1)/2}}, \tag{6.3.10}
\]

and

\[
B'_n = B'_n(a, b, c, q) := \frac{1}{(qb)_n(qc)_n}. \tag{6.3.11}
\]

\textbf{Proof.} We apply Lemma 6.3.1 with \( p_1 = a/b \) and \( p_2 = a/c \) to the Bailey pair in Lemma 6.3.3, and denote the resulting Bailey pair by \( \{A'_n\} \) and \( \{B'_n\} \). It is immediate that \( \{A'_n\} \) is given by (6.3.10). For \( \{B'_n\} \), we see that the sum in question is

\[
B'_n = \frac{1}{(qb)_n(qc)_n} \sum_{j=0}^{n} \frac{(a/(bc))_j(qbc/a)_{n-j}(qbc/a)^j}{(q)_j(q)_j^{-n-j}}
= \frac{(qbc/a)_n}{(qb)_n(qc)_n(q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j(a/(bc))_j q^j}{(q)_j(aq^n/(bc))_j}
= \frac{(qbc/a)_n(q^{-n})_n(a/(bc))_n}{(qb)_n(qc)_n(qc)_n(qaq^n/(bc))_n}
= \frac{1}{(qb)_n(qc)_n},
\]

where in the penultimate line we used the Chu-Vandermonde theorem \cite[p. 11, equation (1.3.3)]{20}. This completes the proof. \qed
It should be noted that we will subsequently require \( A'_n(1, b, c, q) \), and this appears to be problematic because of the factor \((1 - a)\) in the denominator. However, this is only an apparent singularity. The numerator of the jth summand in \( A'_n \) contains \((1 - a)\) if \( j \geq 2 \). Furthermore, when the terms for \( j = 0 \) and \( j = 1 \) are added together, their sum contains \((a - 1)\) in the numerator. This leads directly to

\[
A'_n(1, b, c, q) = \frac{(1/b)_n(1/c)_n(bc)^n(1 - q^{2n})q^n}{(qb)_n(qc)_n} \left\{ \frac{b + c + q - 2 - qbc}{(1 - q)(1 - b)(1 - c)} \right. \\
+ \sum_{j=2}^{n} \left( -1 \right)^j (b)_j(c)_j (1 - q^{2j-1}) \left( 1 - q^j \right)^{j+1} q^{j(j-1)/2} \right\} \tag{6.3.12}
\]

We now rewrite (6.3.12) by breaking the sum into two parts using the identity

\[
\frac{1 - q^{2j-1}}{(1 - q^j)(1 - q^{j-1})} = \frac{1}{1 - q^j} + \frac{q^{j-1}}{1 - q^{j-1}},
\]

and then replacing \( j \) by \( j + 1 \) in the second sum. This yields for the sum in (6.3.12)

\[
\sum_{j=2}^{n} \left( -1 \right)^j (b)_j(c)_j (1 - q^{2j-1}) \left( 1 - q^j \right)^{j+1} q^{j(j-1)/2} = \\
\sum_{j=2}^{n} \left( -1 \right)^j (b)_j(c)_j \left( 1 - q^j \right)^{j+1} q^{j(j-1)/2} = \\
b_1(c)_1 \left( 1 - q \right)bc + \sum_{j=1}^{n-1} \left( -1 \right)^j (b)_j(c)_j \left( 1 - q^j \right)^{j+1} q^{j(j-1)/2} = \\
\frac{(b)_1(c)_1}{(1/b)_1(1/c)_1(1 - q)bc} + \sum_{j=1}^{n-1} \frac{-1}{(1 - q^j)(1/b)_j(1/c)_j(bc)^j q^{j(j-1)/2}} = \\
\frac{(b)_1(c)_1}{(1/b)_1(1/c)_1(1 - q)bc} + \frac{-1}{(1 - q^n)(1/b)_n(1/c)_n(bc)^n q^{n(n-1)/2}} = \\
\frac{(b)_1(c)_1}{(1/b)_1(1/c)_1(1 - q)bc} + \frac{-1}{(1 - q^n)(1/b)_n(1/c)_n(bc)^n q^{n(n-1)/2}}
\]
6.3 Hecke-type Series for the Sixth Order Mock Theta Functions

\[ + \sum_{j=1}^{n-1} \frac{(-1)^j (b)_j (c)_j (bc-1)(1+q^j)}{(1/b)_{j+1}(1/c)_{j+1}(bc)^{j+1}q^{(j-1)/2}}. \]

Using this latter expression in (6.3.12), we find that

\[ A'_n(1,b,c,q) = \frac{(1/b)_n(1/c)_n(bc)^n(1-q^{2n})q^{n^2}}{(qb)_n qc_n} \left\{ \frac{b+c+q-2-qbc}{(1-q)(1-b)(1-c)} \right\} \]

\[ + \frac{(b)_1(c)_1}{(1/b)_1(1/c)_1(1-q)bc} + \frac{(-1)^n(b)_n(c)_n}{(1-q^n)(1/b)_n(1/c)_n(bc)^n q^{n(n-1)/2}} \]

\[ + \sum_{j=1}^{n-1} \frac{(-1)^j (b)_j (c)_j (bc-1)(1+q^j)}{(1/b)_{j+1}(1/c)_{j+1}(bc)^{j+1}q^{(j-1)/2}} \right\} \}

Now,

\[ \frac{b+c+q-2-qbc}{(1-q)(1-b)(1-c)} + \frac{(b)_1(c)_1}{(1/b)_1(1/c)_1(1-q)bc} \]

\[ = \frac{bc-1}{(1-q)(1-b)(1-c)} = \frac{bc-1}{(1-b)(1-c)}. \]

Consequently,

\[ A'_n(1,b,c,q) = \frac{(-1)^n(1+q^n)(1-b)(1-c)q^{(n^2)}}{(1-bq^n)(1-cq^n)} \]

\[ + \frac{(1/b)_n(1/c)_n(bc-1)(bc)^n(1-q^{2n})q^{n^2}}{(qb)_n qc_n} \]

\[ \times \left\{ \frac{1}{(1-b)(1-c)} + \sum_{j=1}^{n-1} \frac{(-1)^j (b)_j (c)_j (1+q^j)}{(1/b)_{j+1}(1/c)_{j+1}(bc)^{j+1}q^{(j-1)/2}} \right\} \}

We are now in a position to list the eight Bailey pairs required to prove Ramanujan's assertions about the sixth order mock theta functions. The following table provides the values of the parameters \(a, b, c,\) and \(d\) along with the associated values of the Bailey pairs \(A'_n\) and \(B'_n,\) where here \(d = q\) or \(d = q^2.\) It should be remarked that \(A'_0\) always has the value 1. For \(n \geq 0,\) we furthermore require the definitions

\[ S_n := \sum_{j=-n}^{n} (-1)^j q^{-j^2} \]  

(6.3.14)

and

\[ T_n := \sum_{j=0}^{n} q^{-j(j+1)/2}. \]
Lastly, \( \omega = e^{2\pi i/3} \). We now extend the definition (6.3.14) to negative values of \( n \). If \( n \) is negative and \( N > -n \), define

\[
S_n := \sum_{j=-N}^{n}(-1)^j q^{-j^2} - \sum_{j=-N}^{-n-1}(-1)^j q^{-j^2}.
\] (6.3.15)

Therefore,

\[
S_{-n} = \sum_{j=-N}^{-n}(-1)^j q^{-j^2} - \sum_{j=-N}^{-n-1}(-1)^j q^{-j^2} = -S_{n-1}.
\] (6.3.16)

**Theorem 6.3.2.** The following are eight Bailey pairs \( A'_n \) and \( B'_n \):

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
<th>( B'_n )</th>
<th>( A'_n )</th>
<th>no.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1/q</td>
<td>( q^2 )</td>
<td>( -1/(q^2)_{2n} )</td>
<td>( q^{2n^2+n}S_n - q^{2n^2-n}S_{n-1}, \ n \geq 1 )</td>
<td>(1)</td>
</tr>
<tr>
<td>( q^2 )</td>
<td>-1</td>
<td>-q</td>
<td>( q^2 )</td>
<td>( -1/(q^2)_{2n} )</td>
<td>( q^{2n^2+n}(1 - q^{2n+1})/1 - q )</td>
<td>(2)</td>
</tr>
<tr>
<td>( q \sqrt{-1} )</td>
<td>( -q \sqrt{-1} )</td>
<td>( q )</td>
<td>( (q^2; q^2)_n )</td>
<td>( (-1)^n q^{n^2+n}(T_n + T_{n-1}) ), ( n \geq 1 )</td>
<td>(3)</td>
<td></td>
</tr>
<tr>
<td>( q \sqrt{-1} )</td>
<td>( q \sqrt{-1} )</td>
<td>( q )</td>
<td>( (q^2; q^2)_n )</td>
<td>( (-1)^n q^{n^2+n}(1 - q^{2n+2})T_n/1 - q^2 )</td>
<td>(4)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>( q )</td>
<td>( -1/(q^2)_{n} )</td>
<td>( q^{(3n^2+n)/2}S_n - q^{(3n^2-n)/2}S_{n-1}, \ n \geq 1 )</td>
<td>(5)</td>
</tr>
<tr>
<td>( q )</td>
<td>-1</td>
<td>0</td>
<td>( q )</td>
<td>( -1/(q^2)_{n} )</td>
<td>( q^{(3n^2+n)/2}(1 - q^{2n+1})/1 - q )</td>
<td>(6)</td>
</tr>
<tr>
<td>( q^2 )</td>
<td>-1</td>
<td>0</td>
<td>( q )</td>
<td>( -1/(q^2)_{n} )</td>
<td>( \frac{1 - q^{2n+2}}{(1 - q)(1 - q^2)} \left{ q^{3(n+1)/2}S_n + q^{(n+1)(3n+2)/2}S_{n+1} \right} )</td>
<td>(7)</td>
</tr>
<tr>
<td>1</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
<td>( q )</td>
<td>( \frac{(q)_n}{(q^3; q^3)_n} )</td>
<td>( \frac{3(-1)^n q^{n(n+1)/2}(1 + q^n)}{1 + q^n + q^{2n}}, \ n \geq 1 )</td>
<td>(8)</td>
</tr>
</tbody>
</table>

**Table 6.1.** Values for Bailey pairs \( A'_n \) and \( B'_n \)

**Proof.** Direct substitutions of \( a, b, \) and \( c \), as given in Theorem 6.3.1, yield the values of \( A'_n \) and \( B'_n \) for the second, third, fourth, fifth, sixth, and eighth entries in Table 6.1. Since the first and seventh are more difficult to prove, we provide details.

For the first, we see that direct substitutions into (6.3.13) yield
right-hand side of (6.3.18) satisfies the equality

It is a trivial (but tedious) matter of algebraic simplification to show that the
and keeping (6.3.17) in mind, we see that we must prove the equivalent identity

and this, along with the initial value

Table 6.1 as

Referring to the first line of Table 6.1, noting that

or

By (6.3.10), we must show that the right side of (6.3.19) is equal to

It is a trivial (but tedious) matter of algebraic simplification to show that the
right-hand side of (6.3.19) is equal to

and this, along with the initial value

establishes the first entry of Table 6.1 by mathematical induction.

Similarly, we can rewrite the expression for \( A'_n \) in the seventh entry of Table 6.1 as

By (6.3.10), we must show that the right side of (6.3.19) is equal to

and this, along with the initial value

establishes the first entry of Table 6.1 by mathematical induction.

Similarly, we can rewrite the expression for \( A'_n \) in the seventh entry of Table 6.1 as

By (6.3.10), we must show that the right side of (6.3.19) is equal to

and this, along with the initial value

establishes the first entry of Table 6.1 by mathematical induction.

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and this, along with the initial value

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and this, along with the initial value

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By (6.3.10), we must show that the right side of (6.3.19) is equal to

and this, along with the initial value

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By (6.3.10), we must show that the right side of (6.3.19) is equal to

and this, along with the initial value

establishes the first entry of Table 6.1 by mathematical induction.

Similarly, we can rewrite the expression for \( A'_n \) in the seventh entry of Table 6.1 as

By (6.3.10), we must show that the right side of (6.3.19) is equal to

and this, along with the initial value

establishes the first entry of Table 6.1 by mathematical induction.
If we now solve (6.3.19) for \( S_n \), it is again a matter of algebra to deduce that \( S_n \) and the right-hand side so obtained satisfy the same recurrence. One then checks the case for \( n = 0 \), and the required identity follows by mathematical induction.

Our next theorem gives Hecke-type identities for products of Ramanujan’s theta functions

\[
\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad \text{and} \quad \varphi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}
\]

and the sixth order mock theta functions (6.1.3)–(6.1.8) that are analogous to those found for \( f_0(q) \) and \( f_1(q) \) in Section 5.2.

**Theorem 6.3.3.** We have the identities

\[
\psi(q) \varphi(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \quad \text{(6.3.20)}
\]

\[
\psi(q) \psi(q) = \sum_{n=0}^{\infty} (-1)^n q^{3n^2+1} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \quad \text{(6.3.21)}
\]

\[
\varphi(-q) \rho_0(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-(j+1)/2}, \quad \text{(6.3.22)}
\]

\[
\varphi(-q) \sigma_0(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n+1)(3n+2)/2} \sum_{j=-n}^{n} (-1)^j q^{-(j+1)/2}, \quad \text{(6.3.23)}
\]

\[
\psi(q) \lambda_0(q) = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \quad \text{(6.3.24)}
\]

\[
2\psi(q) \mu_0(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \quad \text{(6.3.25)}
\]

\[
(q) \nu_0(q) = 3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 + q^n + q^{2n}}. \quad \text{(6.3.26)}
\]

**Proof.** We proceed by taking the Bailey pairs from Theorem 6.3.2 and inserting them into (6.3.5) with appropriate choices of \( \rho_1 \) and \( \rho_2 \). The most difficult of these is (6.3.25), owing to the fact that Ramanujan’s original series does not converge. So we shall provide complete proofs of (6.3.21) and (6.3.25). The remaining five identities are proved in exactly the same way that (6.3.21) is proved. Therefore, for these five identities, we shall only list the appropriate Bailey pairs, the power of \( q \), and the values of \( a, b, c, d, \rho_1, \) and \( \rho_2 \).

We now prove (6.3.21). First, we replace \( q \) by \( q^2 \), then we set \( a = q^2 \), \( b = -1 \), \( c = -q \), and \( \rho_1 = q \), and lastly let \( \rho_2 \to \infty \) in (6.3.5). Then we insert the Bailey pair from Theorem 6.3.2 (2) to arrive at
\[ \sum_{n=0}^{\infty} (-1)^n q^{3n^2+3n} S_n = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2+2n}}{(-q; q)_{2n+1}}. \]

Multiplying both sides by \( q \) and employing (1.3.11) and (6.1.2), we obtain (6.3.21).

We next examine (6.3.25). As we remarked at the beginning of this chapter, the even-indexed partial sums (6.1.6) converge as do the odd-indexed partial sums. We therefore define \( \mu_6(q) \) to be the average of these two values. Consequently, adding together the \( k \)th and \((k + 1)\)st partial sums, we find that

\[
2\mu_6(q) = \lim_{k \to \infty} \left( 1 + \sum_{n=0}^{k} \left\{ \frac{(-1)^n (q; q^2)_n}{(-q; q)_n} + \frac{(-1)^{n+1} (q; q^2)_{n+1}}{(-q; q)_{n+1}} \right\} \right)
\]

\[
= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n (q^{n+1} + q^{2n+1})}{(-q; q)_{n+1}}
\]

\[
= 1 + \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (q; q^2)_n q^{2n+1}}{(-q; q)_n} - \frac{(-1)^{n+1} (q; q^2)_{n+1} q^{n+1}}{(-q; q)_{n+1}} \right\}
\]

\[
= 2 + \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{2n+1}}{(-q; q)_n} - \lambda_0(q),
\]

by (6.1.5).

Now in (6.3.5), we set \( a = q^2, b = -1, \rho_1 = \sqrt{q}, \) and \( \rho_1 = -\sqrt{q}, \) let \( c \to 0, \) and insert the Bailey pair from Theorem 6.3.2 (7). We then multiply both sides by \( q(1 + q)(1 - q^2)/(1 - q^3) \) and employ (1.3.11) to deduce that

\[
\psi(q) \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{2n+1}}{(-q; q)_n}
\]

\[
= -\sum_{n=0}^{\infty} \frac{(-1)^n (1 + q)(1 - q^{2n+2})}{(1 - q^{2n+1})(1 - q^{2n+3})} \left\{ q^{(3n^2+7n+2)/2} S_n + q^{(3n^2+9n+4)/2} S_{n+1} \right\}
\]

\[
= -\sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{1 - q^{2n+1}} + \frac{q}{1 - q^{2n+3}} \right\}
\]

\[
\times \left\{ q^{(3n^2+7n+2)/2} S_n + q^{(3n^2+9n+4)/2} S_{n+1} \right\}. 
\]

We note that the series on the left side in (6.3.28) appears on the right side of (6.3.27). We divide the far right-hand side of (6.3.28) into two sums, replace \( n \) by \( n - 1 \) in the second sum, and lastly recombine the series. In doing so, we note below that we have added the term \((S_{-1} + S_0)/(1 - q)\) to the second sum. However, by the functional equation (6.3.16), this expression is equal to 0. Hence, by (6.3.27),
\[ \psi(q)(2\mu_6(q) + \lambda_6(q) - 2) = \sum_{n=0}^{\infty} \frac{(-1)^n(q; q^2)_n q^{2n+1}}{(-q; q)_n} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{1 - q^{2n+1}} \left\{ q^{(3n^2+n)/2} S_{n-1} + q^{(3n^2+3n)/2} S_n - q^{(3n^2+7n+2)/2} S_n - q^{(3n^2+9n+4)/2} S_{n+1} \right\} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{1 - q^{2n+1}} \left\{ (q^{(3n^2+n)/2}(1 - q^{2n+1}) + q^{(3n^2+3n)/2}(1 - q^{2n+1}) \right. \]
\[ + q^{(3n^2+5n+2)/2}(1 - q^{2n+1}) S_n - 2q^{(n^2+n)/2} + 2q^{(n^2+5n+2)/2} \right\} \]
\[ = \sum_{n=0}^{\infty} \left\{ (-1)^n \left( q^{(3n^2+n)/2} + q^{(3n^2+3n)/2} + q^{(3n^2+5n+2)/2} \right) S_n - 2q^{(n^2+n)/2} \right\} \]
\[ = \sum_{n=0}^{\infty} (-1)^n \left( q^{(3n^2+n)/2} + q^{(3n^2+5n+2)/2} \right) S_n + \psi(q)\lambda_6(q) - 2\psi(q), \]

by (6.3.24). (Note that (6.3.24) is proved independently from (6.3.25).) Therefore,

\[ 2\psi(q)\mu_6(q) = \sum_{n=0}^{\infty} (-1)^n q^{(3n^2+n)/2} S_n + \sum_{n=0}^{\infty} (-1)^n q^{(3n^2+5n+2)/2} S_n, \]

and replacing \( n \) by \(-n-1\) in the latter sum above gives (6.3.25) upon the use of (6.3.16).

We now indicate the appropriate substitutions for the remaining five assertions.

For (6.3.20), replace \( q \) by \( q^2 \) in (6.3.5), then set \( a = 1, b = -1, c = -1/q, \)
and \( \rho_1 = q, \) and lastly let \( \rho_2 \to \infty. \) We then insert the Bailey pair (1)
from Theorem 6.3.5.

For (6.3.22), set \( a = q, \) \( b = \sqrt{q}, \) \( c = -\sqrt{q}, \) and \( \rho_1 = -q \) in (6.3.5),
and then let \( \rho_2 \to \infty. \) We then insert the Bailey pair (3) from Theorem 6.3.5.
Recall that in (6.3.15) we extended the definition of \( S_n \) to negative \( n \) and
derived the functional equation (6.3.16). We can extend the definition of \( T_n \)
in an analogous fashion and so derive the associated functional equation

\[ T_n = -T_{-n-1}. \]  
(6.3.29)

For (6.3.23), set \( a = q^2, \) \( b = \sqrt{q}, \) \( c = -\sqrt{q}, \) and \( \rho_1 = -q \) in (6.3.5),
and then let \( \rho_2 \to \infty. \) We then insert the Bailey pair (4) from Theorem 6.3.5,
multiply both sides by \( q(1 - q), \) and invoke (6.3.29) with \( n \) replaced by \( n + 1. \)

For (6.3.24), set \( a = q, \) \( b = -1, \) \( \rho_1 = \sqrt{q}, \) and \( \rho_2 = 1/\sqrt{q}, \) and let \( c \to 0 \) in
(6.3.5). We then insert the Bailey pair (6) from Theorem 6.3.5.

Finally, for (6.3.26), set \( a = 1, b = \omega, \) and \( c = \omega^2, \) and let \( \rho_1 \) and \( \rho_2 \) tend
to \( \infty \) in (6.3.5). We then insert the Bailey pair (8) from Theorem 6.3.5. .
6.4 Entries for $\phi_6(q)$ and $\psi_6(q)$

There are six entries in the lost notebook that involve only $\phi_6(q)$ and $\psi_6(q)$. We shall prove these results in this section. The proofs are analogous to those used in Chapter 5 to prove the mock theta conjectures.

We begin by rewriting the identities (6.3.20) and (6.3.21) from Theorem 6.3.3.

**Theorem 6.4.1.** Recall that $\phi_6(q)$ and $\psi_6(q)$ are defined in (6.1.1) and (6.1.2), respectively. Then

$$\psi(q)\phi_6(q) = \sum_{r,s=-\infty}^{\infty} \text{sg}(r)(-1)^r q^{r+s+\left(\frac{r+s+1}{2}\right)}, \tag{6.4.1}$$

and

$$2\psi(q)\psi_6(q) = \sum_{r,s=-\infty}^{\infty} \text{sg}(r)(-1)^r q^{r+s+\left(\frac{r+s+2}{2}\right)}. \tag{6.4.2}$$

**Proof.** First, we recall some basic facts about $\text{sg}(n)$, which is defined by

$$\text{sg}(n) = \begin{cases} 1, & \text{if } n \geq 0, \\ -1, & \text{if } n < 0. \end{cases}$$

This has (as previously noted) several simplifying uses. Namely,

$$\sum_{n=a}^{b} c_n = \sum_{\text{sg}(n-a) = \text{sg}(b-n)} \text{sg}(n-a)c_n. \tag{6.4.3}$$

Also, we note that

$$\frac{1}{1-z} = \sum_{\text{sg}(n) = \epsilon(z)} \text{sg}(n)z^n \tag{6.4.4}$$

holds for all $z$ with $|z| \neq 1$, where

$$\epsilon(z) = \begin{cases} 1, & \text{if } |z| < 1, \\ -1, & \text{if } |z| > 1. \end{cases}$$

Identity (6.4.4) may now be combined with Lemma 5.3.4 to yield
\[
\frac{(q, q, xy, q/(xy); q)_\infty}{(x, q/x, y, q/y; q)_\infty} = \sum_{r, s = -\infty}^{\infty} \text{sg}(r)x^ry^sq^{rs}, \quad (6.4.5)
\]

provided that \( |q| < |x| < 1 \) and \( |q| < |y| < 1 \).

Applying (6.4.3) to the inner sum in (6.3.20) and letting \( r = n + j \) and \( s = n - j \), we find that

\[
\psi(q)\phi_6(q) = \sum_{n, j = -\infty}^{\infty} \text{sg}(n + j)(-1)^{n+j}q^{3n^2+n-j^2}
\]

where it is permissible to drop the condition \( r \equiv s \pmod{2} \), because if \( r \) and \( s \) are of opposite parity, then the term with indices \( r, s \) cancels the term with indices \( s, r \).

To obtain (6.4.2), we first note that if \( m = -1 - n \), then

\[
(-1)^m q^{3m^2+3m+1} \sum_{j = -m}^{m} (-1)^j q^{-j^2} = (-1)^n q^{3n^2+3n+1} \sum_{j = -n}^{n} (-1)^j q^{-j^2}.
\]

Hence, replacing the sum over nonnegative \( n \) in (6.3.21) by a sum over all \( n \) doubles the value. So,

\[
2\psi(q)\psi_6(q) = \sum_{n = -\infty}^{\infty} (-1)^n q^{3n^2+3n+1} \sum_{|j| \leq |n|} (-1)^j q^{-j^2}.
\]

Now rearranging as we did for \( \phi_6(q) \), we find that

\[
2\psi(q)\psi_6(q) = \sum_{r, s = -\infty}^{\infty} \text{sg}(r)(-1)^{r+s+1}q^{r+s+\left(\frac{r+s+1}{2}\right)}.
\]

In order to obtain the fundamental identities connected with \( \phi_6(q) \) and \( \psi_6(q) \), we introduce the function

\[
A(z) = A(z, q) := \frac{z^2(-q)(-z^2, -q/z^2, q; q)_\infty}{(z, q/z, q; q)_\infty},
\]

whose partial fraction decomposition provides the necessary framework.
\textbf{Theorem 6.4.2.} For $|q| < |z| < 1$, in the Laurent expansion of $A(z)$,

\[
[z^0]A(z) = 2\psi_6(q),
\]
\[
[z^{-1}]A(z) = \phi_6(q).
\]

\textit{Proof.} By (6.4.5), with $x = z$ and $y = -z$, (4.1.2), (1.3.1), (1.3.11), and Euler’s theorem, for $|q| < |z| < 1$,

\[
\psi(q)A(z) = \frac{z(q; q)^3 (z^2, -q/z^2; q; q)_{\infty}}{(-z; q; q)_{\infty} (z, q/z, q; q; q)_{\infty}} (-z, q/z; q; q)_{\infty}
\]
\[
= z \sum_{r, s = -\infty}^{\infty} \text{sg}(r)(-1)^{-r^2} z^{r+s} \sum_{t = -\infty}^{\infty} q^{t(t+1)/2} z^{-t}.
\]

The coefficients of $z^0$ and $z^{-1}$ are obtained by setting $t = r + s$ and $t = r + s + 1$, respectively. By (6.4.6) and (6.4.7), these coefficients are $\psi(q)\phi_6(q)$ and $2\psi(q)\psi_6(q)$, respectively. Dividing both sides above by $\psi(q)$ completes the proof of the theorem. \hfill \Box

We note that, for each integer $n$,

\[
(zq^n, z^{-1}q^{1-n}, q; q)_{\infty} = \sum_{m = -\infty}^{\infty} (-1)^m z^m q^{mn + (m-1)/2}
\]
\[
= \sum_{m = -\infty}^{\infty} (-1)^m z^m q^{m(n+1)} (\binom{n}{z})
\]
\[
= (-1)^n z^n q^{-\binom{n}{z}} (z, q/z, q; q)_{\infty}. \tag{6.4.8}
\]

(See also (4.1.3).) Applying (6.4.8) to $A(z)$, we see that

\[
A(qz) = \frac{zq\varphi^2(-q)(-q^2z^2, -1/(qz^2), q; q)_{\infty}}{(qz, 1/z, q; q)_{\infty}}
\]
\[
= \frac{zq\varphi^2(-q)(1 + 1/(qz^2))(1 + 1/z^2)(-q^2z^2, -q/(qz^2), q; q)_{\infty}}{(1 - 1/z)(qz, q/z, q; q)_{\infty}}
\]
\[
= \frac{zq \cdot z^{-4}q^{-1} \varphi^2(-q)(-z^2, -q/z^2, q; q)_{\infty}}{-z^{-1}(z, q/z, q; q)_{\infty}}
\]
\[
= -z^{-3}A(z), \tag{6.4.9}
\]

and also that

\[
A(z^{-1}) = \frac{z^{-1} \varphi^2(-q)(-1/z^2, -qz^2, q; q)_{\infty}}{(1/z, qz, q; q)_{\infty}}
\]
\[
= \frac{z^{-1} \varphi^2(-q)(1 + 1/z^2)(-q/z^2, -qz^2, q; q)_{\infty}}{(1 - 1/z)(q/z, qz, q; q)_{\infty}}
\]
= -z^{-3}\varphi^2(-q)(-q/z^2, -z^2, q; q)_{\infty}

= -z^{-3}A(z). \quad (6.4.10)

We still have some distance to go in identifying Ramanujan’s identities, owing to the fact that $A(z)$ has poles at $z = q^k$ for each nonzero integer $k$. Thus, our next step is to proceed with what amounts to an isolation of the poles of $A(z)$ through effectively employing an analogue of partial fractions.

Toward this goal, we define

$$L(z) := 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(3r+1)/2} z^{3r+2}}{1 - q^rz}. \quad (6.4.11)$$

It is easily verified that $L(z)$ satisfies both (6.4.9) and (6.4.10). Now define

$$V(z) := A(z) - L(z). \quad (6.4.12)$$

From the previous comments, it is clear that $V(z)$ is meromorphic for $z \neq 0$, with at most simple poles at $z = q^k$, $-\infty < k < \infty$, $k \neq 0$, and is easily seen to satisfy the functional equations (6.4.9) and (6.4.10).

First, we compute the residue of $V(z)$ at $z = 1$, namely,

$$\lim_{z \to 1} (z - 1)A(z) = -\varphi^2(-q)(-1, -q, q; q)_{\infty}

= -2 \frac{(q; q)_3^3(-q; q)^2_{\infty}}{(-q; q)^2_{\infty}(q; q)^3_{\infty}} = -2,$$

by (1.3.1), and, by inspection,

$$\lim_{z \to 1} (z - 1)L(z) = -2.$$

Hence, the residue of $V(z)$ at $z = 1$ equals 0, i.e., $V(z)$ is analytic at $z = 1$. Consequently, the functional equation (6.4.9) (which, as was noted, is valid for $L(z)$ as well) shows that $V(z)$ is analytic for $z = q^k$ for each nonzero integer $k$. Hence, $V(z)$ is analytic for all $z \neq 0$.

We define the associated Laurent expansion

$$V(z) := \sum_{n=-\infty}^{\infty} V_n z^n. \quad (6.4.13)$$

Now by Lemma 5.4.2 of Chapter 5 with $n = 3$ and $C = -1$, we see that

$$V(z) = V_0(z^3, q^3/z^3, q^3; q^3)_{\infty} + z V_1(qz^3, q^2/z^3, q^3; q^3)_{\infty}

+ z^2 V_2(q^2 z^3, q/z^3, q^3; q^3)_{\infty}.$$

If we apply (6.4.10) to $V(z)$, we find that $V_2 = -V_1$. Consequently,
for $\psi$ we deduce from (6.4.15) and (6.4.16) that

$$ V(z) = V_0(z^3, q^3/z^3, q^3; q^3)_{\infty} + V_1(z(qz^3, q^2/z^3, q^3; q^3)_{\infty} - z^2(q^2z^3, q/z^3, q^3; q^3)_{\infty}) $$

$$ = -z^{-3}V_0(1/z^3, q^3z^3, q^3; q^3)_{\infty} - z^{-2}V_1(q/z^3, q^2z^3, q^3; q^3)_{\infty} + V_1 z^{-1}(q^2/z^3, q^3z^3, q^3; q^3)_{\infty}, $$

(6.4.14)

where we applied (6.4.10) once again to $V(z)$. Now we restrict $z$ to the annulus $|q| < |z| < 1$. By Theorem 6.4.2, the coefficients of $z^0$ and $z^{-1}$ in $A(z)$ are $2\psi_0(q)$ and $\phi_0(q)$, respectively. Also, $|q^rz| < 1$ if and only if $r \geq 0$. Hence, by (6.4.11),

$$ L(z) = 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(3r+1)/2} z^{3r+2}}{1 - q^r} $$

$$ = 2 \sum_{r=0}^{\infty} (-1)^r q^{r(3r+1)/2} z^{3r+2} \sum_{s=0}^{\infty} q^{rs} z^s $$

$$ - 2 \sum_{r=-\infty}^{-1} (-1)^r q^{r(3r+1)/2} z^{3r+2} \sum_{s=-\infty}^{-1} q^{rs} z^s $$

$$ = 2 \sum_{r,s=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} z^{3r+2} \sum_{s=0}^{\infty} q^{rs} z^s $$

Note that when $\text{sg}(r) = \text{sg}(s)$ either $3r + s + 2 \geq 2$ or $3r + s + 2 \leq -2$; so the coefficients of $z^0$, $z^1$, and $z^{-1}$ in $L(z)$ are equal to 0. Hence, by Theorem 6.4.2 and (6.4.12), $V_0 = 2\psi_0(q)$ and $V_1 = \phi_0(q)$. Substituting these into (6.4.14), we obtain the following theorem.

**Theorem 6.4.3.** If $0 < |q| < 1$ and $z$ is not an integral power of $q$, then

$$ A(z) = \phi_0(q) \left( z(qz^3, q^2/z^3, q^3; q^3)_{\infty} - z^2(q^2z^3, q/z^3, q^3; q^3)_{\infty} \right) $$

(6.4.15)

$$ + 2\psi_0(q)(z^3, q^3z^3, q^3; q^3)_{\infty} + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(3r+1)/2} z^{3r+2}}{1 - q^r z}. $$

Next, we dissect the identity (6.4.15) in order to obtain separate identities for $\psi_0(q)$ and $\phi_0(q)$. To this end, we uniquely define $A_0(z^3)$, $A_1(z^3)$, and $A_2(z^3)$ by

$$ A(z) := A_0(z^3) + zA_1(z^3) + z^2A_2(z^3). $$

(6.4.16)

Furthermore, noting that

$$ \frac{1}{1 - q^r z} = \frac{1 + q^rz + q^{2r}z^2}{1 - q^{3r}z^3}, $$

we deduce from (6.4.15) and (6.4.16) that
and

\[
A_2(z^3) = -\phi_0(q)(q^2z^3, q/z^3, q^3; q^3)_\infty + 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r+1/2} z^3 r^3}{1 - q^{3r} z^3}. \tag{6.4.18}
\]

On the other hand, applying Lemma 6.2.2 with \(n = 3\) and with \(x\) and \(z\) replaced by \(1/z\) and \(-z^3\), respectively, and multiplying by

\[
\frac{z^3(q^3; q^3)_\infty \varphi^2(-q)(-z^3, -q/z^3, q; q)_\infty}{(q; q)_\infty^3 (z^3, q^3/z^3, q^3; q^3)_\infty},
\]

we find that

\[
A(z) = \frac{(q^3; q^3)_\infty^3 \varphi^2(-q)(-z^3, -q/z^3, q; q)_\infty}{(q; q)_\infty^3 (z^3, q^3/z^3, q^3; q^3)_\infty} \left( \frac{z^3(-1, -q^3, q^3; q^3)_\infty}{(-z^3, -q^3z^3, q^3; q^3)_\infty} \right) + \frac{z(-q, q^2, q^3; q^3)_\infty}{(-q^2 z^3, -q/z^3, q^3; q^3)_\infty}. \tag{6.4.19}
\]

Comparing (6.4.19) with (6.4.18), we see that, by (1.3.1),

\[
A_0(z^3) = \frac{(q^3; q^3)_\infty^3 \varphi^2(-q)(-z^3, -q/z^3, q; q)_\infty}{(q; q)_\infty^3 (z^3, q^3/z^3, q^3; q^3)_\infty} \left( \frac{z^3(-1, -q^3, q^3; q^3)_\infty}{(-z^3, -q^3z^3, q^3; q^3)_\infty} \right) + (q^3; q^3)_\infty(z^3, q^3/z^3, q^3; q^3)_\infty \tag{6.4.20}
\]

and

\[
A_2(z^3) = \frac{(q^3; q^3)_\infty^3 \varphi^2(-q)(-z^3, -q^2z^3, q; q)_\infty}{(q; q)_\infty^3 (z^3, q^3/z^3, q^3; q^3)_\infty} \left( \frac{z(-q, q^2, q^3; q^3)_\infty}{(-q^2 z^3, -q/z^3, q^3; q^3)_\infty} \right). \tag{6.4.21}
\]

If we note that

\[
(-z^3, -q^2z^3, q^3; q^3)_\infty(-z^3, -q^2z^3, q^3; q^3)_\infty(-q^2 z^3, -q/z^3, q^3; q^3)_\infty = (-z^3 q)_\infty(-q/z^3 q)_\infty(q^3; q^3)_\infty
\]

and that the terms independent of \(z\) in (6.4.21) are, by (2.1.14),

\[
\frac{(q^3; q^3)_\infty^3 \varphi^2(-q)(q; q)_\infty(-q, -q^2z^3, q^3; q^3)_\infty}{(q; q)_\infty^3 (q^3; q^3)_\infty^2 (-q^2; q^3)_\infty}\]

\[
= \frac{(q^3; q^3)_\infty^2 (q^3; q^3)_\infty(-q; q^3)_\infty(-q^2; q^3)_\infty}{(q; q)_\infty^3 (-q^3; q^3)_\infty}
\]

\[
= \frac{(q^3; q^3)_\infty(q^3; q^3)_\infty(-q; q^3)_\infty}{(-q^3; q^3)_\infty(-q^3; q^3)_\infty}
\]

\[
= \frac{(q^3; q^3)_\infty(q; q)_\infty(-q^3; q^3)_\infty}{(q^2; q^2)_\infty^2 \varphi(-q^3)}.
\]
then (6.4.21) takes the shape
\[
A_2(z^3) = \frac{(q; q)_\infty \varphi(-q^3)(-z^3, -q^3/z^3, q^3; q^3)_\infty (-q^2z^3, -q/z^3, q^3; q^3)_\infty}{(q^2; q^2)_\infty (z^3, q^3/z^3, q^3; q^3)_\infty}. \tag{6.4.22}
\]
Comparing (6.4.17) with (6.4.20), and (6.4.18) with (6.4.22), we observe that replacing \(z^3\) by \(z\), respectively, by \(qz\) yields the following theorem.

**Theorem 6.4.4.** If \(|q| < 1\) and \(z\) is neither 0 nor an integral power of \(q^3\), then
\[
(z, q^3/z, q^3; q^3)_\infty \psi_6(q) = \frac{z(q; q)_\infty (q, q^5, q^6)\varphi(-q^3)(-qz, -q^2/z, q^3)_\infty (-q^2z, -q/z, q^3; q^3)_\infty}{(q^2; q^2)_\infty (qz, q^2/z, q^3; q^3)_\infty} - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)/2} z^{r+1}}{1 - q^{3r+1} z}, \tag{6.4.23}
\]
and if \(z\) is neither 0 nor of the form \(q^{3k+2}, -\infty < k < \infty\), then
\[
(z, q^3/z, q^3; q^3)_\infty \phi_6(q) = \frac{(q; q)_\infty \varphi(-q^3)(-z, -q^3/z, q^3)_\infty (-qz, -q^2/z, q^3; q^3)_\infty}{(q^2; q^2)_\infty (qz, q^2/z, q^3; q^3)_\infty} - 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)/2} z^{r+1}}{1 - q^{3r+1} z}. \tag{6.4.24}
\]

We now specialize the results of Theorem 6.4.4 to prove six entries from the lost notebook.

First, taking \(z = -1, -q, -q^2\), and \(q\) in (6.4.23), we deduce that, respectively,
\[
(-1, -q^3, q^3; q^3)_\infty \psi_6(q) = -\frac{1}{2} q^2(-q) + \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}}, \tag{6.4.25}
\]
\[
(-q, -q^2, q^3; q^3)_\infty \psi_6(q) = \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(3r+2)/2}}{1 + q^{3r+1}}, \tag{6.4.26}
\]
\[
(-q, -q^2, q^3; q^3)_\infty \psi_6(q) = \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(3r+4)/2}}{1 + q^{3r+2}}, \tag{6.4.27}
\]
and
\[
(q; q)_\infty \psi_6(q) = 2q(q^6, q^5)^3_\infty \frac{(q^2; q^2)_\infty}{(q^2; q^2)_\infty} - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{(r+1)(3r+2)/2}}{1 + q^{3r+1}}. \tag{6.4.28}
\]

Next, letting \(z = -1/q, -1, -q, \) and \(q\) in (6.4.24) and using (2.1.14), we find that, respectively,
\[ (-q, -q^2, q^3) \phi_6(q) = 2 \sum_{r=-\infty}^{\infty} \frac{q^{(3r+1)/2}}{1+q^{3r}}, \quad (6.4.29) \]

\[ (-1, -q^3, q^3) \phi_6(q) = 2 \sum_{r=-\infty}^{\infty} \frac{q^{r(3r+1)/2}}{1+q^{3r+1}}, \quad (6.4.30) \]

\[ (-q, -q^2, q^3) \phi_6(q) = \varphi^2(-q) + 2 \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(3r+2)/2}}{1+q^{3r+2}}, \quad (6.4.31) \]

and

\[ (q; q) \phi_6(q) = \frac{\varphi^3(-q^3)}{\varphi(-q)} - 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{(r+1)(3r+2)/2}}{1-q^{3r+2}}. \quad (6.4.32) \]

We are now prepared to prove several items from the lost notebook.

**Entry 6.4.1 (p. 2, equation 2; p. 4, equations 6b, 4b=5b,3b).** We have

\[ (q; q) \phi_6(q) = 1 - 2 \sum_{n=-\infty}^{\infty} \frac{q^{(2n+1)(3n+1)+1}}{1+q^{3n+1}} + 2 \sum_{n=-\infty}^{\infty} \frac{q^{n(6n+1)}}{1-q^n+q^{2n}}, \quad (6.4.33) \]

\[ (-q, -q^2, q^3) \phi_0(q) = 2 \sum_{n=-\infty}^{\infty} \frac{q^{n(3n+1)/2}}{1+q^{3n}}, \quad (6.4.34) \]

\[ \psi(q^3) \phi_0(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n(n+1)/2}}{1+q^{3n+1}}, \quad (6.4.35) \]

and

\[ (-q, -q^2, q^3) \psi_0(q) = \sum_{n=-\infty}^{\infty} \frac{q^{(n+1)(3n+2)/2}}{1+q^{3n+1}}. \quad (6.4.36) \]

**Proof.** Three of these assertions have been proved already. Equation (6.4.34) is (6.4.29). Equation (6.4.35) reduces to (6.4.30) once one notes that, by (1.3.11),

\[ (-1, -q^3, q^3) = 2 \psi(q^3). \]

Lastly, equation (6.4.36) is identical with (6.4.26).

It is rather more arduous to obtain (6.4.33) from (6.4.32). To prove this last result, we begin by noting that

\[
\frac{(q^3, q^3, q^6; q^6)_\infty}{(q, q^2; q^2)_\infty} = \frac{2(q^3; q^3)_\infty(-q, -q^2, q^3; q^3)_\infty}{(q; q)_\infty(-1, -q^3, q^3; q^3)_\infty} \]

\[ = 2 \sum_{r=-\infty}^{\infty} \frac{q^r}{1+q^{3r}}. \]
which follows from \[19, p. 279, equation (12.4.26)\] with \(q\) replaced by \(q^3\), and then with \(t = q\), \(a = -1\), and \(b = -q^3\). Consequently, by (6.4.32), we arrive at

\[
(q; q)_\infty \phi_6(q) = 2 \sum_{r=-\infty}^{\infty} \frac{q^r}{1 + q^{3r}} - 2 \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{(r+1)(3r+2)/2}}{1 - q^{3r+2}}. \tag{6.4.37}
\]

We now consider the first sum on the right-hand side of (6.4.37) and write

\[
2 \sum_{r=-\infty}^{\infty} \frac{q^r}{1 + q^{3r}} = 1 + 2 \sum_{r=-\infty}^{\infty} \frac{q^r}{1 + q^{3r}} - 2 \sum_{r=-\infty}^{\infty} \frac{q^r(1 - q^{6r})}{1 + q^{3r}}. \tag{6.4.38}
\]

We examine separately the two sums on the right-hand side of (6.4.38). First,

\[
\sum_{r=\infty}^{\infty} \frac{q^r(6r+1)}{1 + q^{3r}} = \sum_{r=1}^{\infty} \frac{q^r(6r+1)}{1 + q^{3r}} + \sum_{r=1}^{\infty} \frac{q^r(6r-1)}{1 + q^{-3r}}
\]

\[
= \sum_{r=1}^{\infty} \frac{q^r(6r+1)(1 + q^r)}{1 + q^{3r}}
\]

\[
= \sum_{r=1}^{\infty} \frac{q^r(6r+1)}{1 - q^r + q^{2r}}. \tag{6.4.39}
\]

Next,

\[
1 - q^{6r^2} = \begin{cases} \sum_{s=0}^{2r-1} (-1)^s q^{3rs}, & \text{if } r \geq 0, \\ \sum_{s=0}^{2r-1} (-1)^s q^{3ps + 3\rho}, & \text{if } \rho = -r > 0, \end{cases}
\]

\[
= \begin{cases} \sum_{s=0}^{2r-1} (-1)^s q^{3rs}, & \text{if } r \geq 0, \\ - \sum_{s=2r}^{2r-1} (-1)^s q^{-3ps}, & \text{if } \rho = -r > 0, \end{cases}
\]

\[
= \sum_{sg(s) = sg(2r-1-s)} sg(s)(-1)^s q^{3rs}.
\]

Hence, putting the calculation above into the last sum in (6.4.38), we find that
\[
\sum_{r=-\infty}^{\infty} \frac{q^r(1-q^{6r^2})}{1+q^{3r}} = \sum_{sg(s)=sg(2r-1-s)} sg(s)(-1)^s q^{3r+s+r}. \tag{6.40}
\]

In the right-hand side of (6.40), we set \( r = n + 1 + \left[ \frac{1}{2} s \right] \) and note that \( sg(2r-1-s) = sg(r-1-\left[ \frac{1}{2} s \right]) \); consequently,

\[
\sum_{sg(s)=sg(2r-1-s)} sg(s)(-1)^s q^{3r+s+r} = \sum_{sg(s)=sg(n)} sg(s)(-1)^s q^{(n+1+\left[ \frac{1}{2} s \right])(3s+1)}
\]

\[
= \sum_{s=-\infty}^{\infty} (-1)^s q^{(1+\left[ \frac{1}{2} s \right])(3s+1)} \sum_{sg(n)=sg(s)} sg(s)q^{(3s+1)n}
\]

\[
= \sum_{s=-\infty}^{\infty} \left( -1 \right)^s q^{(1+\frac{1}{2} s)(3s+1)} \frac{1}{1-q^{3s+1}},
\]

by (6.4.4). We now split this last sum into the even-indexed terms \( s = 2r \) and the odd-indexed terms \( s = 2r + 1 \). Hence, from (6.4.40),

\[
\sum_{r=-\infty}^{\infty} \frac{q^r(1-q^{6r^2})}{1+q^{3r}} = \sum_{r=-\infty}^{\infty} \frac{q^{r+1}(6r+1)}{1-q^{6r+1}} - \sum_{r=-\infty}^{\infty} \frac{q^{r+1}(6r+4)}{1-q^{6r+4}}. \tag{6.41}
\]

Second, we consider the latter sum on the right-hand side of (6.437) and split it according to the parity of \( r \), with \( r \) replaced by \(-2r - 2\) in the even case and by \(-2r - 1\) in the odd case. Thus,

\[
\sum_{r=-\infty}^{\infty} \left( -1 \right)^r q^{(r+1)(3r+2)/2} = -\sum_{r=-\infty}^{\infty} \frac{q^{(2r+3)(3r+2)}}{1-q^{6r+4}} + \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(6r+1)}}{1-q^{6r+1}}. \tag{6.42}
\]

We now use our results from (6.438)–(6.442) in (6.437) to obtain

\[
(q;q)_{\infty} \phi_6(q) = 1 + 2 \sum_{r=1}^{\infty} \frac{q^{6r+1}}{1-q^{r} + q^{2r}} - 2 \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(6r+4)} - q^{(2r+3)(3r+2)}}{1-q^{6r+4}}
\]

\[
= 1 + 2 \sum_{r=1}^{\infty} \frac{q^{6r+1}}{1-q^{r} + q^{2r}} - 2 \sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(6r+4)}}{1+q^{3r+2}}.
\]

Replacing \( r \) by \(-r - 1\) in the final sum above, we see that (6.433) has been established. \( \square \)

**Entry 6.4.2** (p. 13, equations 6b, 5b). If \( \omega = e^{2\pi i/3} \), then

\[
\frac{\psi_6(\omega q) - \psi_6(\omega^2 q)}{(\omega - \omega^2)q} = \frac{\psi(q)\psi(q^9)(q^9; q^6)_{\infty}}{\psi(q^3)(q^3; q^6)_{\infty}}. \tag{6.43}
\]

and

\[
\phi_6(q^9) - \phi_6(q) - q^{-3} \psi_6(q^9) = \frac{\psi(q^3)(q^9; q^6)_{\infty}}{\psi(q)\psi(q^3)} \tag{6.44}
\]
6.4 Entries for $\phi_6(q)$ and $\psi_6(q)$

Proof. We derive (6.4.43) from (6.4.25). By (6.4.25) and (6.2.14),

$$
(-1, -q^3, q^3; q^3)_\infty \psi_6(q) = -\sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} - \frac{1}{2} \varphi^2(-q) \quad (6.4.45)
$$

$$
= \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} - \frac{1}{2} \left\{ (q^9, q^9, q^{18}; q_{\infty}^{18}) - 2q(q^3, q^{15}, q^{18}; q_{\infty}^{18}) \right\}^2
$$

$$
= \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} - \frac{1}{2} (q^9, q^9, q^{18}; q_{\infty}^{18})^2
$$

$$
+ 2q(q^3, q^{15}, q^{18}; q_{\infty}^{18}) - 2q^2(q^3, q^{15}, q^{18}; q_{\infty}^{18})^2.
$$

In (6.4.45), we first replace $q$ by $\omega q$, and second replace $q$ by $\omega^2 q$. Subtracting the second from the first yields

$$
(-1, -q^3, q^3; q^3)_\infty (\psi_6(\omega q) - \psi_6(\omega^2 q))
$$

$$
= 2(\omega - \omega^2)(q^3, q^{15}, q^{18}; q_{\infty}^{18}) - 2q(q^3, q^{15}, q^{18}; q_{\infty}^{18})
$$

$$
= 2(\omega - \omega^2)q(q^3, q^{15}, q^{18}; q_{\infty}^{18}) - (q^2, q^2, q^6; q_{\infty}^{6})
$$

$$
- (q, q^2, q^6, q^6; q_{\infty}^{6}),
$$

by (6.2.16). Hence,

$$
\frac{\psi_6(\omega q) - \psi_6(\omega^2 q)}{(\omega - \omega^2)q} = \frac{(q^3, q^{15}, q^{18}; q_{\infty}^{18})(q^2, q^2, q^6; q_{\infty}^{6})}{(-1, -q^3, q^3; q_{\infty}^{3})}(q, q^2, q^6; q_{\infty}^{6}),
$$

which, by (4.1.2) and (1.3.11), is equivalent to (6.4.43).

We now turn to (6.4.44). We begin with (6.4.25), and two lines later note that replacing $r$ by $-r$ in the third sum identifies it with the first sum. To that end,

$$
(-1, -q^3, q^3; q^3)_\infty \psi_6(q) = -\frac{1}{2} \varphi^2(-q) \quad (6.4.46)
$$

$$
= \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}(1 - q^{3r} + q^{6r})}{1 + q^{3r}}
$$

$$
= \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} - \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+3)/2}}{1 + q^{3r}} + \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+5)/2}}{1 + q^{3r}}
$$

$$
= 2 \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} - \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+3)/2}}{1 + q^{3r}}
$$

$$
= 2 \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1 + q^{3r}} + 2q^3 \sum_{r=-\infty}^{\infty} \frac{q^{27r(r+1)/2}}{1 + q^{27r+9}} + 2 \sum_{r=-\infty}^{\infty} \frac{q^{9(r+1)(3r+2)/2}}{1 + q^{27r+18}}
$$

$$
- \sum_{r=-\infty}^{\infty} \frac{q^{27r(r+1)/2}}{1 + q^{27r}} - q^{-3} \sum_{r=-\infty}^{\infty} \frac{q^{9(r+1)(3r+2)/2}}{1 + q^{27r+9}} - q^{-3} \sum_{r=-\infty}^{\infty} \frac{q^{9(r+1)(3r+4)/2}}{1 + q^{27r+18}}.
$$
where in the last step we subdivided the two series in the previous equality into residue classes modulo 3, i.e., we replaced the index of summation \( r \) by \( 3r \), \( 3r + 1 \), and \( 3r + 2 \) in each of the two sums. Now, miraculously, these six sums appear in the six identities (6.4.29)–(6.4.31) and (6.4.25)–(6.4.27). Consequently, from (6.4.46),

\[
\begin{align*}
(-1, -q^3, q^3; q^3)_\infty \psi_6(q) + & \frac{1}{2} \varphi^2(-q) \\
= (-q^9, -q^{18}, q^{27}; q^{27})_\infty \phi_6(q^9) + & q^3(-1, -q^{27}, q^{27}; q^{27})_\infty \phi_6(q^9) \\
+ ((-q^9, -q^{18}, q^{27}; q^{27})_\infty \phi_6(q^9) - & \varphi^2(-q^9)) \\
- (1, -q^{27}, q^{27}; q^{27})_\infty \psi_6(q^9) + & \frac{1}{2} \varphi^2(-q^9) \\
- q^{-3}(-q^9, -q^{18}, q^{27}; q^{27})_\infty \psi_6(q^9) - & q^{-3}(-q^9, -q^{18}, q^{27}; q^{27})_\infty \psi_6(q^9) \\
= 2(-q^9, -q^{18}, q^{27}; q^{27}) + & q^3(-1, -q^{27}, q^{27}; q^{27})_\infty (\phi_6(q^9) - q^{-3} \psi_6(q^9)) \\
- & \frac{3}{2} \varphi^2(-q^9). \\
& (6.4.47)
\end{align*}
\]

We now apply (6.2.15) with \( q \) replaced by \( q^3 \) to the first factor of the first term on the right-hand side of (6.4.47) to deduce that

\[
(-1, -q^3, q^3; q^3)_\infty \psi_6(q) + \frac{1}{2} \varphi^2(-q) \\
= (-1, -q^3, q^3; q^3)_\infty (\phi_6(q^9) - q^{-3} \psi_6(q^9)) - \frac{3}{2} \varphi^2(-q^9).
\]

Hence, by (6.2.17),

\[
\phi_6(q^9) - \psi_6(q) - q^{-3} \psi_6(q^9) = \frac{1}{2(-1, -q^3, q^3; q^3)_\infty} \cdot \frac{4(q^6; q^6)_\infty^3(q, q^5, q^6; q^6)_\infty}{(q^3; q^3)_\infty(q^3, q^{15}; q^{18})_\infty},
\]

which is equivalent to (6.4.44). This concludes the proof of Entry 6.4.5. \( \square \)

### 6.5 Entries for the Remaining Functions

The method developed in detail in Section 6.4 can now be applied to examine the identities related to the functions listed in (6.1.3)–(6.1.8). The proofs are somewhat abbreviated, being similar to those in Section 6.4. We begin with \( \rho_6(q) \), \( \sigma_6(q) \), and \( \Phi_6(q) \).

**Entry 6.5.1 (p. 13, equations 4b, 3b; p. 2, equation 1).** We have

\[
q^{-1} \psi_6(q^2) + \rho_6(q) = (-q; q^2)_\infty^2(-q, -q^5, q^6; q^6)_\infty, \quad (6.5.1)
\]
\begin{align*}
\phi_6(q^2) + 2\sigma_6(q) &= (-q; q^2)\infty (-q^3, -q^3, q^6; q^6)\infty, \quad (6.5.2) \\
\text{and} \\
\rho_6(q) &= 2q^{-1}\Phi_6(q^3) + \frac{\psi^2(q)}{\varphi(-q^3)}. \quad (6.5.3)
\end{align*}

**Proof.** We begin by noting that
\[
\sum_{j=0}^{n} q^{-j(j+1)/2} = \frac{1}{2} \sum_{j=-n-1}^{n} q^{-j(j+1)/2}. \quad (6.5.4)
\]
Now with \(r = -n - j - 2\) and \(s = -n + j - 1\) in (6.3.22), we find that
\[
2\varphi(-q)\rho_6(q) = \sum_{r, s = -\infty \atop \text{sg}(r) = \pm \text{sg}(s) \atop r \not\equiv s \pmod{2}} \text{sg}(r)(-1)^{(r+s+1)/2} q^{rs/2 + (r+s+3)/4 - 1}. \quad (6.5.5)
\]
Similarly, with \(r = n + j + 1\) and \(s = n - j\) in (6.3.23), we obtain
\[
2\varphi(-q)\sigma_6(q) = \sum_{r, s = -\infty \atop \text{sg}(r) = \pm \text{sg}(s) \atop r \not\equiv s \pmod{2}} \text{sg}(r)(-1)^{(r+s+1)/2} q^{rs/2 + (r+s+1)/2}. \quad (6.5.6)
\]
Next, we define
\[
B(z) := \frac{z^2\psi^2(q)(qz^2, q/z^2; q^2)\infty}{(z, q^2/z, z^2; q^2)\infty}. \quad (6.5.7)
\]
By Lemma 5.3.6 of Chapter 5 with \(q, x, \) and \(y\) replaced by \(\sqrt{q}, \sqrt{z}, \) and \(\sqrt{z}, \) respectively, and by (4.1.2), we see that, for \(|q| < |z| < 1,\)
\[
2\varphi(-q)B(z) = z^{3/2}2z^{1/2}\varphi(-q)\psi(q)(qz^2, q/z^2, q^2; q^2)\infty = \sum_{r, s = -\infty \atop \text{sg}(r) = \pm \text{sg}(s) \atop r \not\equiv s \pmod{2}} \text{sg}(r)q^{rs/2} z^{(r+s+3)/2} (-1)^t q^t z^{-t}. \quad (6.5.8)
\]
We note that the coefficients of \(z^0\) and \(z^1\) in (6.5.8) are \(-2\varphi(-q)\rho_6(q)\) and \(-2\varphi(-q)\sigma_6(q), \) respectively, by (6.5.5) and (6.5.6). In complete analogy with Theorem 6.4.2, we may similarly prove that
We now combine (6.5.13) with (6.4.26) (in which \(n \mapsto n - 3\)), with \(122^6\) Sixth Order Mock Theta Functions

we obtain the alternative representation

\[
\frac{z^4(q^6; q^6)^3}{\varphi(-q)(z, q^6/z^3, q^6; q^6)_{\infty}}
\]

we obtain the alternative representation

\[
B(z) = \frac{(q^6; q^6)_{\infty}(q^3, q^2; q^2)_{\infty}}{\varphi(-q)(z, q^6/z^3, q^6; q^6)_{\infty}}
\]

\[
\times \left( \frac{z^4(q^3, q^2; q^2)_{\infty}}{(q^3, q^6/z^3, q^6; q^6)_{\infty}} + \frac{z^3\varphi(-q^6)}{(q^3z^3, 1/z^3, q^3; q^6)_{\infty}} + \frac{z^2(q^3, q^6; q^6)_{\infty}}{(q^6z^3, q^3, q^6; q^6)_{\infty}} \right).
\]

We now proceed to compare (6.5.9) with (6.5.10) (in analogy with the analysis done to prove Theorem 6.4.4). Hence,

\[
q(z, q^6/z, q^6; q^6)_{\infty}\rho_6(q) = -\frac{z(q^6; q^6)^3_{\infty}}{\varphi(-q)(z, q^6/z^3, q^6; q^6)_{\infty}}.
\]

\[
\times \left( \frac{z^4(q^3, q^2; q^2)_{\infty}}{(q^3, q^6/z^3, q^6; q^6)_{\infty}} + \frac{z^3\varphi(-q^6)}{(q^3z^3, 1/z^3, q^3; q^6)_{\infty}} + \frac{z^2(q^3, q^6; q^6)_{\infty}}{(q^6z^3, q^3, q^6; q^6)_{\infty}} \right).
\]

We now combine (6.5.13) with (6.4.26) (in which \(q \mapsto q^2\)) to obtain

\[
q^{-1}\psi_6(q^2) + \rho_6(q) = \frac{(q^3; q^3)^3\varphi(q)}{\varphi(-q)(-q, -q^2, q^3; q^3)_{\infty}(q^2, q^4; q^6; q^6)_{\infty}} = \frac{\varphi(q)(-q, -q^5, q^6; q^6)_{\infty}}{(q^2; q^2)_{\infty}}.
\]
which is equivalent to (6.5.1).

By a similar argument, combining (6.5.12) with \( z = 1 \) and (6.4.20), in which \( q \) has been replaced by \( q^3 \), we find that

\[
\phi_6(q^2) + 2\sigma_6(q) = \frac{(-q, -q, q^2; q^2)_\infty(-q^3, -q^3, q^6; q^6)_\infty}{(q^2, q^2)_\infty},
\]

which is equivalent to (6.5.2).

To prove (6.5.3), we need a different representation for \( \Phi_6(q) \), namely,

\[
\Phi_6(q) = \sum_{n=0}^{\infty} \frac{(-q; q)_{2n}q^{n+1}}{(q; q^2)_{n+1}}
= \frac{q}{(1-q)^2} \sum_{n=0}^{\infty} \frac{(q^2, -q, q^3; q^3)_nq^n}{(q^2, q^4, q^4; q^4)_n}
= \frac{1}{2\varphi(-q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^nq^{(n+1)^2}}{1-q^{2n+1}}, \tag{6.5.15}
\]

where the last line follows from [20, p. 81, equation (4.1.3)] with \( q \) replaced by \( q^2 \), and then with \( \alpha = q^2, \beta = \gamma = q, \delta = -q, \epsilon = -q^2 \), and lastly with \( N \to \infty \). Hence, (6.5.3) is equivalent to the assertion that

\[
\rho_6(q) = \frac{1}{q\varphi(-q^3)} \sum_{n=-\infty}^{\infty} \frac{(-1)^nq^{3n^2+6n+3}}{1-q^{6n+3}} + \frac{(-q, -q^3, q^4; q^4)_\infty}{\varphi(-q^3)}, \tag{6.5.16}
\]

and this is equivalent to (6.5.11) with \( z = q^3 \). Thus, Entry 6.5.1 has been proved. \( \square \)

Next, we consider \( \lambda_6(q) \) and \( \mu_6(q) \).

**Entry 6.5.2 (p. 13, equations 2b, 1b).** We have

\[
2\phi_6(q^2) - 2\mu_6(-q) = (-q; q^2)_{\infty}(-q^3, -q^3, q^6; q^6)_{\infty}, \tag{6.5.17}
2q^{-1}\psi_6(q^2) + \lambda_6(-q) = (-q; q^2)_{\infty}(-q, -q^5, q^6; q^6)_{\infty}. \tag{6.5.18}
\]

**Proof.** From equations (6.3.24) and (6.3.25), we see that

\[
2(-q, -q^3, q^4; q^4)_{\infty}\lambda_6(q) = \sum_{\substack{r,s=\infty \\ sg(r)=sg(s) \\ s \equiv s \pmod{2}}} \left( r^{r+s+(r+s+4)/2} \right)^{-1} \tag{6.5.19}
\]
and

\[
2(-q, -q^3, q^4; q^4)_{\infty}\mu_6(q) = \sum_{\substack{r,s=\infty \\ sg(r)=sg(s) \\ r \equiv s \pmod{2}}} \left( r^{r+s+(r+s+2)/2} \right). \tag{6.5.20}
\]
We now define

\[ C(z) := \frac{z\varphi(-q)(-z, -q/z, q; q)_{\infty}}{(z, q^{2}/z, q^{2}; q^{2})_{\infty}}. \]  

(6.5.21)

By Lemma 5.3.6 in Chapter 5, with \( x = -\sqrt{z} \) and \( y = \sqrt{z} \), we find that

\[ (-q, -q^{3}, q^{4}; q^{4})_{\infty}C(z) \]

\[ = \frac{z\varphi(-q)\varphi(-q^{2})(-qz, -q/z, q^{2}; q^{2})_{\infty}(z^{2}, q^{4}/z^{2}, q^{4}; q^{4})_{\infty}(-z, -q/z, q; q)_{\infty}}{(z, q^{4}/z, q^{4}; q^{2})_{\infty}} \]

\[ = z \sum_{r,s=-\infty}^{\infty} \text{sg}(r)(-1)^{r} q^{r} z^{(r+s)/2} \sum_{t=-\infty}^{\infty} q^{t(t+1)/2} z^{-t}, \]  

(6.5.22)

for \(|q^{2}| < |z| < 1\). Now, by inspection, we find that the coefficients of \( z^{0} \) and \( z^{1} \) are, respectively, \( 2q(-q, -q^{3}, q^{4}; q^{4})_{\infty}\lambda_{6}(q) \) and \( 2(-q, -q^{3}, q^{4}; q^{4})_{\infty}\mu_{6}(q) \).

Again, an analysis similar to that for \( A(z) \) in the proof of Theorem 6.4.2 yields

\[ C(z) = 2\mu_{6}(q) \left( z(q^{2}z^{3}, q^{4}/z^{3}, q^{6}; q^{6})_{\infty} - z^{2}(q^{4}z^{3}, q^{2}/z^{3}, q^{6}; q^{6})_{\infty} \right) \]

\[ + 2q\lambda_{6}(q)(z^{3}, q^{6}/z^{3}, q^{6}; q^{6})_{\infty} + 4 \sum_{r=-\infty}^{\infty} \frac{(-1)^{r} q^{r(3r+1)} z^{3r+2}}{1 - zq^{2r}}. \]  

(6.5.23)

Our next step, as before, is to dissect \( C(z) \) into three parts in order to isolate alternative representations of \( \mu_{6}(q) \) and \( \lambda_{6}(q) \). This task turns out to be more challenging than in the cases of \( A(z) \) and \( B(z) \). We proceed by relating \( C(z) \) to \( A(z) \). In particular, we apply Lemma 5.3.3 of Chapter 5 with \( x = -1/z \) and \( y = z \), and after multiplying both sides of this new version of (5.3.5) by

\[ \frac{-z^{2}\varphi(-q)}{(z, q^{2}/z, q^{2}; q^{2})_{\infty}}, \]

we deduce that

\[ C(z) = 2A(z, q^{2}) - \frac{z\varphi(-q)(z, q/z, q; q)_{\infty}(qz, q/z, q^{2}; q^{2})_{\infty}}{(-q, -q^{3}, q^{4}; q^{4})_{\infty}}. \]  

(6.5.24)

Now we are able to subdivide each term on the right side into three parts, by using Lemma 6.2.4 on the second term on the right-hand side, and by applying (6.4.19) to \( A(z, q^{2}) \). This allows us first to identify the portion involving \( z^{3} \) and its powers. Equating the relevant powers in (6.5.23) and (6.5.24) and then replacing \( z^{3} \) by \( z \), we find that

\[ q(z, q^{6}/z, q^{6}; q^{6})_{\infty}\lambda_{6}(q) \]

\[ = \frac{2z(q^{2}; q^{2})_{\infty}(q^{2}, q^{10}, q^{12}; q^{12})_{\infty}}{(q^{4}; q^{4})_{\infty}} \frac{(-q^{2}z, -q^{2}/z, -q^{4}z, -q^{4}/z, q^{6}, q^{6}; q^{6})_{\infty}}{(z, q^{6}/z, q^{6}; q^{6})_{\infty}} \]
we deduce that
\[ a \text{ similar argument, combining (6.4.24) (with } q) \]

Now isolating the portion with exponents on \( z \) congruent to 1 modulo 3, we find that
\[ 2(z, q^6/z, q^6; q^6)_\infty \mu_6(q) \]

Combining (6.4.23) (with \( q \) replaced by \( q^2 \)) and (6.5.25), we find that
\[ q \lambda_6(q) - 2 \psi_6(q^2) = \frac{q \phi(-q)(-q^3, -q^3, q^{12}; q^{12})_\infty}{(-q, q^3, q^4; q^4)_\infty}. \]

Replacing \( q \) by \(-q\) in (6.5.22), we obtain a result equivalent to (6.5.18). By a similar argument, combining (6.4.24) (with \( q \) replaced by \( q^2 \)) and (6.27), we deduce that
\[ 2 \phi_6(q^2) - 2 \mu_6(q) = \frac{\phi(-q)(-q, -q^2, q^3; q^3)_\infty}{(-q, q^3, q^4; q^4)_\infty}. \]

We thus obtain a result equivalent to (6.5.17).

**Entry 6.5.3 (p. 17, equation 1).** Recall that \( \gamma_6(q) \) and \( \sigma_6(q) \) are defined in (6.1.7) and (6.1.4), respectively. Then
\[ 2 \gamma_6(q) = 3 \phi_6(q) - \frac{\phi^2(-q)}{(-q, q^2, q^3, q^3)_\infty}. \]

**Proof.** By two applications of Lemma 5.3.4, with \( \omega = e^{2 \pi i / 3} \),
\[ \sum_{r=-\infty}^{\infty} 3 z^r \left( \frac{1}{1 + q^r + q^{2r}} \right) = \sum_{r=-\infty}^{\infty} \frac{3 z^r}{(1 - \omega q^r)(1 - \omega^2 q^r)}, \]

\[ = (1 - \omega^2) \sum_{r=-\infty}^{\infty} \frac{z^r}{1 - \omega q^r} + (1 - \omega) \sum_{r=-\infty}^{\infty} \frac{z^r}{1 - \omega^2 q^r} \]

\[ = (1 - \omega^2)(q; q)_\infty^3 (\omega z, q^2 z, q; q)_\infty + (1 - \omega)(q; q)_\infty^3 (\omega^2 z, q^2 z, q; q)_\infty + \omega (q^2 z, q^2 z, q; q)_\infty. \]
Therefore, by (6.3.26), \((q; q)\gamma_6(q)\) is the coefficient of \(z^0\) in
\[
\sum_{r=\infty}^{\infty} \frac{3z^r}{1 + q^r + q^{2r}} \sum_{s=\infty}^{\infty} (-1)^s q^{s(3s+1)/2} z^{-s} = -\frac{(q; q)_\infty^2 (qz, q^2/z, q^3; q^3)_\infty}{(q^3, q^3)_\infty (z, q^3/z, q; q)_\infty} \left( \omega^2 (z, q \omega^2/z, q; q)_\infty + \omega (\omega^2 z, q \omega / z, q; q)_\infty \right) =: D(z). \tag{6.5.31}
\]

For future use, we reduce the expression in parentheses above by using the Jacobi triple product identity (4.1.2) to deduce that
\[
\omega^2 (z, q \omega^2/z, q; q)_\infty + \omega (\omega^2 z, q \omega / z, q; q)_\infty = \sum_{n=\infty}^{\infty} (-1)^n (\omega^{n+2} + \omega^{2n+1}) q^{n(n-1)/2} z^n = 3 \sum_{n=\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n - \sum_{n=\infty}^{\infty} (-1)^s q^{s(s-1)/2} z^s = -3z(q^6 z^3, q^3/z^3, q^3; q^3)_\infty - (z, q/z, q; q)_\infty. \tag{6.5.32}
\]

As in the previous analysis of \(A(z)\) (as well as \(B(z)\) and \(C(z)\)), we now determine from (6.5.30) and (6.5.31) that
\[
D(z) = \gamma_6(q)(qz, q^2/z, q^3; q^3)_\infty + 3 \sum_{r=\infty}^{\infty} \frac{(-1)^r q^{r(3r+5)/2} z^{r+1}}{1 - q^{3r+1} z}. \tag{6.5.33}
\]

Now replace \(z\) by \(q^2/z\) in (6.5.33) to arrive at
\[
(z, q^3/z, q^3; q^3)_\infty \gamma_6(q) = -\frac{(q; q)_\infty^2 (z, q^3/z, q^3; q^3)_\infty}{(q^3, q^3)_\infty (z, q^3/z, q; q)_\infty} \left( \omega^2 (z, q \omega^2/z, q; q)_\infty + \omega (\omega^2 z, q \omega / z, q; q)_\infty \right) - 3 \sum_{r=\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)/2} z^{r+1}}{1 - q^{3r+1} z}. \tag{6.5.34}
\]

In (6.5.34), we set \(z = -1\) and invoke (6.5.32) to deduce that
\[
(-1, -q^3, q^3; q^3)_\infty (2 \gamma_6(q) - 3 \phi_6(q)) = \frac{(q; q)_\infty^2 (-1, -q^3, q^3; q^3)_\infty}{(q^3, q^3)_\infty (-1, -q^3, q; q)_\infty} \left( \omega^2 (-\omega, -q \omega^2, q; q)_\infty + \omega (-\omega^2, -q \omega, q; q)_\infty \right).
\]

By Lemma 6.2.1, the last equality is equivalent to (6.5.29). \(\square\)
6.6 Two Further Identities

Entry 6.6.1 (p. 4, equation 1b). If $\psi_6(q)$ is defined by (6.1.2), then

$$
\sum_{n=0}^{\infty} \frac{(-1)^n(q^3; q^6)_n}{(-q^3; q^3)_n^2} - 2\psi_6(q) = \frac{\varphi^2(-q)}{2\psi(q^3)}.
$$

(6.6.1)

Our proof follows closely that in [32, Section 3].

Proof. We must first establish a meaning for the divergent series on the left side of (6.6.1). Thus, using [19, p. 266, equation (12.3.6)] in the second line below, we assert that

$$
\sum_{n=0}^{\infty} \frac{(-1)^n(q^3; q^6)_n}{(-q^3; q^3)_n^2} := \lim_{\alpha \to 1} \sum_{n=0}^{\infty} \frac{(\alpha q^3; q^3)_n(q^3; q^6)_n(-\alpha)^n}{(q^3; q^3)_n(-\alpha q^3; q^3)_n^2} = 2(q^3; q^6)_\infty \sum_{n=-\infty}^{\infty} q^{3n(n+1)/2} (1 + q^{3n})^2
$$

$$
= \frac{2}{\psi(q^3)} \left\{ \frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^{3n(n+1)/2}}{1 + q^{3n}} \right\}
$$

$$
= \frac{1}{\psi(q^3)} \sum_{n=-\infty}^{\infty} q^{3n(n+1)/2} \frac{1}{1 + q^{3n}},
$$

(6.6.2)

upon using (1.3.11) in the penultimate line. Hence, we shall assume that (6.6.1) is equivalent to the identity

$$
\sum_{n=-\infty}^{\infty} \frac{q^{3n(n+1)/2}}{1 + q^{3n}} - 2\psi(q^3)\psi_6(q) = \frac{1}{2}\varphi^2(-q).
$$

(6.6.3)

The identity (6.6.3) follows from (6.4.23) by setting $z = -1$ therein. \qed

Entry 6.6.2 (p. 16, equation 1b). If $\phi_6(q)$ is defined by (6.1.1), then

$$
\phi_6(q) + 2 \sum_{n=1}^{\infty} \frac{(-q; q)_{2n-1}q^n}{(q; q^2)_n} = \frac{1}{(q; q)_\infty} \left( 1 + 6 \sum_{n=1}^{\infty} \left( \frac{n}{3} \right) \frac{q^{2n}}{1 - q^{2n}} \right),
$$

(6.6.4)

where $\left( \frac{n}{3} \right)$ denotes the Legendre symbol.

The second term on the left side of (6.6.4) was studied extensively in [32], and we shall utilize one of their identities to establish this result.

Proof. We define

$$
\phi_{-}(q) := \sum_{n=1}^{\infty} \frac{(-q; q)_{2n-1}q^n}{(q; q^2)_n},
$$

\[\text{RAW TEXT CONTINUES HERE}\]
and
\[
\psi_-(q) := \sum_{n=1}^{\infty} \frac{\theta(q)_{2n} - 2q^n}{(q^2)^n}.
\]

Our first objective is to prove that
\[
\sigma_6(q) - \phi_-(q^2) = q(-q^2; q^2)_\infty (-q^6, -q^6, q^6)_\infty.
\]

To accomplish this, we proceed as before. We define
\[
D(z) := z^2(\zeta_2 q, z^2 / q^3, q^3 ; q^3)_\infty.
\]

Now \(D(z)\) is quite similar to \(A(z)\), studied in detail in the proof of Theorem 6.4.2. Consequently, we omit the details of the proof of the essential decomposition of \(D(z)\), namely \(32, \text{Theorem 2.3}\),
\[
D(z) = \sum_{n=\infty}^{\infty} \frac{(-1)^n q^n(3n+1)/2 \cdot 3n+2}{1 - q^n z} - 2\psi_-(q)(z^3, q^3 / z^3, q^3 ; q^3)_\infty
\]
\[
- \phi_-(q) \{z(q^3 / z^3, q^3 / z^3, q^3 ; q^3)_\infty - z^2 (q^3 / z^3, q^3 / z^3, q^3 ; q^3)_\infty\}.
\]

As noted above, the proof of this identity is parallel to the proof of Theorem 6.4.2. One requires representations for \(\phi_-(q)\) and \(\psi_-(q)\) using Theorem 6.3.1. Then replace \(q\) by \(q^2\), and set \(a = q^4\), \(b = q\), and \(c = 0\). Insert the resulting Bailey pair in Lemma 6.3.2 with \(q\) replaced by \(q^2\), \(\rho_1 = -q^2\), \(\rho_2 = -q^3\), and \(a = q^4\). After simplification, we find that
\[
-\varphi(-q)\phi_-(q) = \sum_{r, s = 0}^{\infty} (-1)^{r+s+1/2} q^{(2r+1)s + ((r+s+1)/2)^2}
\]
\[
- \sum_{r, s = 1}^{\infty} (-1)^{-(r-s+1)/2} q^{(2r-1)s + ((r+s-1)/2)^2}.
\]

Apply Theorem 6.3.1 with \(q\) replaced by \(q^2\), \(a = q^2\), \(b = q\), and \(c = 0\). Insert the resulting Bailey pair in Lemma 6.3.2 with \(q\) replaced by \(q^2\), \(\rho_1 = -q^2\), and \(\rho_2 = -q^2\) to find that, after simplification,
\[
-2\varphi(-q)\psi_-(q) = \sum_{r, s = 0}^{\infty} (-1)^{(r+s+2)/2} q^{(2r+1)s + ((r+s+2)/2)^2}
\]
\[
- \sum_{r, s = 1}^{\infty} (-1)^{-(r-s+2)/2} q^{(2r-1)s + ((r+s-2)/2)^2}.
\]
We wish to isolate that portion of (6.6.7) where the exponents of \( z \) are congruent to 1 modulo 3. To do this, we apply Lemma 6.2.2 with \( n = 3 \) to each curly bracketed factor in

\[
D(z) = \frac{z^2(q^6; q^6)_\infty^4 (z^3, q/z^3; q)_\infty}{(q; q)_\infty^2 (z^3, q^3/z^3; q^3)_\infty} \left\{ \frac{(q^3 z^2, q/z^2; q^3)_\infty (z^3, q^6/z^3; q^6)_\infty (q^2; q^2)_\infty^2}{(z, q^2/z, q^3 z, q/z^3; q^2)_\infty (q^6; q^6)_\infty} \right\}
\]

\[
\times \left\{ \frac{(q^2 z^2, q/z^2; q^3)_\infty (q^3 z^3, q^3/z^3, q^6)_\infty (q^2; q^2)_\infty^2}{(q, z, q^2/z, z^3, q^2/z^3, q^2)_\infty (q^6; q^6)_\infty^2} \right\}
\]

\[
= \frac{z^2(q^6; q^6)_\infty^4 (z^3, q/z^3; q)_\infty}{(q; q)_\infty^2 (z^3, q^3/z^3; q^3)_\infty} \sum_{k=0}^{2} \frac{z^k (q^{2k+1}, q^5-2k; q^6)_\infty}{(q^{2k+1}/z^3, q^5-2k z^3, q^3)_\infty} \times \sum_{j=0}^{2} \frac{(q/z)^j (q^{2j+3}, q^{3-2j}; q^6)_\infty}{(q^{2j+3}, q^{6-2j}/z^3, q^6)_\infty}.
\] (6.6.8)

We now consider \( D(q^{2/3}) \) in both (6.6.7) and (6.6.8), and identifying the two expressions, we obtain

\[
\frac{(q^6; q^6)_\infty^2 (q^3; q^3)_\infty (q^5; q^6)_\infty}{(q; q)_\infty} \sum_{k=0}^{2} \frac{q^{2k/3}(q^{2k+1}, q^5-2k; q^6)_\infty}{(q^{2k-1}, q^7-2k; q^6)_\infty}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)/2}(q^{4/3} + q^{n+2} + q^{2n+2+2/3})}{1 - q^{3n+2}}
\]

\[
- 2 \psi_-(q; q)_\infty - \phi_-(q; q)^{1/3}(q; q)_\infty.
\]

Now isolating those terms of the sort \( q^{n+1/3} \) in the identity above, we find that

\[
\phi_-(q; q)_\infty = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)/2+1}}{1 - q^{3n+2}} - \frac{q(q; q)_\infty (q^6; q^6)_\infty}{(q^2; q^2)_\infty (q^3; q^3)_\infty}. \] (6.6.9)

Next, setting \( z = q^2 \) in (6.5.12) gives

\[
\sigma_6(q)(q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)+2}}{1 - q^{6n+4}} + \frac{q(q; q)_\infty (q^6; q^6)_\infty}{(q^2; q^2)_\infty (q^3; q^3)_\infty}. \] (6.6.10)

Finally, if we replace \( q \) by \( q^2 \) in (6.6.9), subtract the result from (6.6.10), and divide both sides by \((q^2; q^2)_\infty \), we arrive at, after simplification,

\[
\sigma_6(q) - \phi_-(q^2) = q(q^2; q^2)_\infty (-q^6, -q^6, q^6; q^6)_\infty,
\] (6.6.11)

which is (6.6.4).

Now multiply (6.6.1) by 2 and subtract it from (6.5.2) to find that

\[
\phi_0(q^2) + 2 \phi_-(q^2) = (-q; q^2)_\infty (-q^3, -q^3, q^6; q^6)_\infty
\]

\[
+ 2 q(-q^2; q^2)_\infty (-q^6, -q^6, q^6; q^6)_\infty.
\] (6.6.12)
Noting that the left side of (6.6.12) is an even function of $q$, we see that, upon applying (1.3.1),

$$\phi_6(q^2) + 2\phi_-(q^2) = \text{even part of } (-q; q^2)_{\infty}(-q^3; q^6)_{\infty}$$

$$= \text{even part of } \frac{1}{(q^2; q^2)_{\infty}}(-q, -q, q^2; q^2)_{\infty}(-q^3, -q^3, q^6; q^6)_{\infty}$$

$$= \text{even part of } \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=\infty} q^{n^2} \sum_{m=\infty} q^{3m^2}$$

$$= \frac{1}{(q^2; q^2)_{\infty}} \left\{ \sum_{n=-\infty}^{\infty} q^{4n^2} \sum_{m=-\infty}^{\infty} q^{12m^2} + q^4 \sum_{n=-\infty}^{\infty} q^{4n^2+4n} \sum_{m=-\infty}^{\infty} q^{12m^2+12m} \right\}$$

$$= (q^2; q^2)_{\infty} c\Phi_3(q^2)$$

$$= \frac{1}{(q^2; q^2)_{\infty}} \left( 1 + 6 \sum_{n=0}^{\infty} \frac{q^{6n+2}}{1 - q^{6n+2}} - \frac{q^{6n+4}}{1 - q^{6n+4}} \right), \quad (6.6.13)$$

where $c\Phi_3(q)$ is the generating function for three-color generalized Frobenius partitions. In the last two equalities, we applied, respectively, [12, p. 14, equation (5.20)] and [12, p. 26, equation (9.4)]. The final identity in (6.6.13) is simply (6.6.4) with $q$ replaced by $q^2$. \hfill \Box

### 6.7 Further Work

Several papers have been written on sixth order mock theta functions following the appearance of [24]. We should begin by mentioning very general studies such as the work by Hickerson and E. Mortenson [88] and K. Bringmann, J. Lovejoy, and K. Mahlburg [38]. These works produce a general framework for many instances of mock theta functions, including those of the sixth order.

The paper [32] by Berndt and Chan contains two further sixth order mock theta functions, and we followed much of their exposition in Section 6.6.

Finally, papers by R. Antony [25], Y.–S. Choi [54], Choi and B. Kim [60], R. McIntosh [107], and S. Saba [134] are explicitly devoted to particular aspects of the sixth order mock theta functions.
Transformation Formulas for Third Order
Mock Theta Functions

7.1 Introduction

On page 9 in his lost notebook [128], Ramanujan offers eight identities for tenth order mock theta functions. Two of the tenth order mock theta functions are defined by

\[ \phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \quad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}. \] (7.1.1)

The seventh and eighth identities, which we prove in Chapter 8, are transformation formulas involving \( \phi(q) \), \( \psi(q) \), and what we now call a Mordell integral. It is curious that these are the only two transformation formulas involving mock theta functions in the lost notebook. On the other hand, the first published transformation formulas for mock theta functions are those by G.N. Watson [142] for third order mock theta functions. Is it possible that Watson saw these transformation formulas for tenth order mock theta functions and so surmised that possibly transformation formulas existed for third order mock theta functions? In view of the transformation formulas for tenth order mock theta functions, it is inconceivable that Ramanujan would not have discovered transformation formulas for third order mock theta functions. We conjecture that he indeed did discover transformation formulas for third order mock theta functions and that they likely were on pages originally with the lost notebook as we know it now, but that, sadly, these pages have been lost. Indeed, we think that Watson first discovered these transformation formulas while examining the lost notebook.

7.2 A Transformation Formula Involving \( f_3(q) \) and \( \omega_3(q) \)

We state and prove a transformation formula for the third order mock theta functions \( f_3(q) \) and \( \omega_3(q) \) first established in print by G.N. Watson, but likely due to Ramanujan.
Theorem 7.2.1. Let $\alpha$ and $\beta$ be positive numbers such that $\alpha \beta = \pi^2$. Set $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$. Then

$$q^{-1/24} f_3(q) = 2 \sqrt{\frac{2\pi}{\alpha}} q^{1/3} \omega_3(q^2) + 4 \sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-\frac{3}{2} \alpha x^2} \frac{\sinh(\alpha x)}{\sinh\left(\frac{1}{2} \alpha x\right)} \, dx. \quad (7.2.1)$$

Proof. Let $N$ denote a positive integer, and consider the positively oriented rectangular contour $C_N$ with vertical sides passing through the points $\pm (N + \frac{1}{2})$ and horizontal sides passing through the points $\pm ic$, where $0 < c < \pi/\alpha$. Consider

$$\frac{1}{2\pi i} \int_{C_N} \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2} \alpha z^2} \, dz.$$ 

We apply the residue theorem. By our choice of $c$, there are no zeros of $\cosh(\frac{1}{2} \alpha z)$ on the interior of $C_N$. Thus, the only poles are at $z = n$ for every integer $n$ arising from $\sin(\pi z)$. If $R_n$ denotes the residue of the integrand at $z = n$, then

$$R_n = \frac{(-1)^n e^{-\frac{3}{2} \alpha n^2}}{\cosh(\frac{1}{2} \alpha n)} = \begin{cases} 1, & \text{if } n = 0, \\ 2(-1)^n q^{n(3n+1)/2} & \frac{1}{1 + q^n}, & \text{if } n > 0. \end{cases} \quad (7.2.2)$$

We now let $N \to \infty$. By (7.2.2), (1.3.22), and the fact that $R_n = R_{-n}$, we conclude that

$$\sum_{n=-\infty}^{\infty} R_n = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 + q^n} = (q; q)_\infty f_3(q). \quad (7.2.3)$$

If $L_1$ and $L_2$ denote the left- and right-most vertical sides of $C_N$, it is clear that, for $j = 1, 2$,

$$\frac{1}{2\pi i} \int_{L_j} \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2} \alpha z^2} \, dz = o(1), \quad (7.2.4)$$

as $N \to \infty$. Hence, by (7.2.3), (7.2.4), and the residue theorem,

$$(q; q)_\infty f_3(q) = \frac{1}{2\pi i} \left( \int_{-\infty - ic}^{\infty - ic} + \int_{\infty + ic}^{\infty + ic} \right) \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2} \alpha z^2} \, dz. \quad (7.2.5)$$

Now, for $\text{Im } z > 0$,

$$\frac{1}{\sin(\pi z)} = \frac{2i}{e^{-i\pi z} (e^{2\pi iz} - 1)} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi iz}. \quad (7.2.6)$$

Hence,
7.2 A Transformation Formula Involving \( f_3(q) \) and \( \omega_3(q) \)

\[
\frac{1}{2\pi i} \int_{-\infty + ic}^{\infty + ic} \frac{e^{-\frac{3}{2} \alpha z^2}}{\sin(\pi z) \cosh(\frac{1}{2} \alpha z)} \, dz = 2 \sum_{n=0}^{\infty} \int_{-\infty + ic}^{\infty + ic} \frac{e^{(2n+1) \pi i z - \frac{3}{2} \alpha z^2}}{e^{\frac{1}{2} \alpha z} + e^{-\frac{1}{2} \alpha z}} \, dz
\]

\[
= 2 \sum_{n=0}^{\infty} \int_{-\infty + ic}^{\infty + ic} e^{(2n+1) \pi i z - \frac{3}{2} \alpha z^2} \left( e^{\frac{1}{2} \alpha z} + e^{-\frac{1}{2} \alpha z} - 1 \right) \, dz
\]

\[
= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{-\infty + ic}^{\infty + ic} F_n(z) \, dz, \tag{7.2.7}
\]

where

\[
F_n(z) := 4\pi i e^{(2n+1) \pi iz - \frac{3}{2} \alpha z^2} \frac{\left( e^{\frac{1}{2} \alpha z} + e^{-\frac{1}{2} \alpha z} - 1 \right)}{e^{\frac{1}{2} \alpha z} + e^{-\frac{1}{2} \alpha z}} \tag{7.2.8}
\]

and where we used the elementary identity

\[
(x^2 + x^{-2} - 1)(x + x^{-1}) = x^3 + x^{-3}.
\]

To evaluate the integral on the far right side of (7.2.7), we integrate around a positively oriented rectangle \( \gamma_n \) with vertical sides having ordinates \( \pm(N + \frac{1}{2}) \), where \( N \) is a positive integer, and horizontal sides having ordinates \( ic \) and \( z_n \), where

\[
z_n := \frac{(2n + 1) \pi i}{3\alpha}, \quad n \in \mathbb{Z}^+ \tag{7.2.9}
\]

However, the top horizontal side is indented at \( z_n \) by a small semi-circle of radius \( \epsilon > 0 \) with center \( z_n \) and lying above the line \( \text{Im} z = (2n + 1) \pi / (3\alpha) \). We shall apply the residue theorem to \( F_n(z) \), integrated over this indented rectangle \( \gamma_n \). On the interior of \( \gamma_n \), by our choice of \( c \), there exist (at most) simple poles at \( z = z_m, 0 \leq m \leq n \). The residue at \( z_m \) is given by

\[
R_{z_m} = 4\pi i \frac{2}{3\alpha} \frac{(-1)^m}{2i} e^{(2n+1) \pi iz_m - \frac{3}{2} \alpha z_m^2} (2 \cosh(\alpha z_m) - 1) =: \lambda_{n,m}. \tag{7.2.10}
\]

In analogy with (7.2.4), we readily see that if \( L_j, j = 1, 2 \), denote the vertical sides of \( \gamma_n \), then

\[
\frac{1}{2\pi i} \int_{L_j} F_n(z) \, dz = o(1), \tag{7.2.11}
\]

as \( N \) tends to infinity. Thus, employing the residue theorem, (7.2.10), and (7.2.11), letting \( \epsilon \to 0 \), and letting \( N \to \infty \), we find that

\[
\frac{1}{2\pi i} \left( \int_{-\infty + ic}^{\infty + ic} - PV \int_{-\infty + z_n}^{\infty + z_n} \right) F_n(z) \, dz = \lambda_{n,0} + \lambda_{n,1} + \cdots + \lambda_{n,n-1} + \frac{1}{2} \lambda_{n,n}. \tag{7.2.12}
\]

Summing on \( n \), \( 0 \leq n < \infty \), inverting the order of summation, and noting that

\[
\lambda_{n+1,m} = e^{2\pi iz_m} \lambda_{n,m}, \tag{7.2.13}
\]

we arrive at
We now observe that

\[
2 \cos \left( \frac{(2m+1)\pi}{3} \right) - 1 = \begin{cases} 
0, & \text{if } m \equiv -1, 0 \pmod{3}, \\
-3, & \text{if } m \equiv 1 \pmod{3}.
\end{cases}
\]

Thus, setting \( m = 3r + 1 \), we find that (7.2.14) yields

\[
S = \frac{2\pi}{3\alpha} \sum_{m=0}^{\infty} (-1)^m e^{-(2m+1)^2 \pi^2 / (9\alpha) + (2m+1)^2 \pi^2 / (6\alpha)} \\
\times \left\{ 2 \cos \left( \frac{(2m+1)\pi}{3} \right) - 1 \right\} 1 + e^{-2(2m+1)\pi^2 / (3\alpha)} 1 - e^{-2(2m+1)\pi^2 / (3\alpha)}
\]

\[
= \frac{2\pi}{3\alpha} \sum_{m=0}^{\infty} (-1)^m q_1^{(2m+1)^2/6} \left\{ 2 \cos \left( \frac{(2m+1)\pi}{3} \right) - 1 \right\} 1 + q_1^{2(2m+1)/3} 1 - q_1^{2(2m+1)/3}.
\]

Returning to (7.2.14) and the definition of \( F_n(z) \) in (7.2.8), we see that we must next examine

\[
P V \int_{-\infty}^{\infty} F_n(z)dz = PV \int_{-\infty}^{\infty} F_n(z_n + x)dx = 4\pi i
\]

\[
\times PV \int_{-\infty}^{\infty} e^{-(2n+1)^2 \pi^2 / (3\alpha) + (2n+1)\pi ix - \frac{3}{2} \alpha (- (2n+1)^2 \pi^2 / (9\alpha^2) + 2(2n+1)\pi ix / (3\alpha) + x^2)}
\]
7.2 A Transformation Formula Involving \( f_3(q) \) and \( \omega_3(q) \)

\[
\times \left( \frac{2 \cosh \alpha \left( \frac{(2n+1)\pi i}{3\alpha} + x \right)}{2 \cosh \frac{3\alpha}{2} \left( \frac{(2n+1)\pi i}{3\alpha} + x \right)} \right) \, dx
\]

\[
= 4\pi i PV \int_{-\infty}^{\infty} e^{-(2n+1)^2 \pi^2/(6\alpha) - \frac{3}{2} \alpha x^2} \frac{\cosh \left( \frac{\alpha x + (2n+1)\pi i}{3} \right) - \frac{1}{2}}{(-1)^n \sinh \frac{3\alpha x}{2}} \, dx,
\]

(7.2.16)

upon using the addition formula for \( \cosh z \). Furthermore,

\[
cosh \left( \frac{\alpha x + (2n+1)\pi i}{3} \right) - \frac{1}{2} = \cosh(\alpha x) \cosh \left( \frac{(2n+1)\pi i}{3} \right) + \sinh(\alpha x) \sinh \left( \frac{(2n+1)\pi i}{3} \right) - \frac{1}{2}.
\]

We shall show that the contribution of

\[
cosh(\alpha x) \cosh \left( \frac{(2n+1)\pi i}{3} \right) - \frac{1}{2}
\]

to the integral on the right side of (7.2.16) is equal to 0, because this portion of the integrand is odd. However, because the integral is a principal value, we shall provide a detailed argument. Set

\[
f(x) := 4\pi (-1)^n e^{-(2n+1)^2 \pi^2/(6\alpha) - \frac{3}{2} \alpha x^2} \left( \cosh(\alpha x) \cosh \left( \frac{(2n+1)\pi i}{3} \right) - \frac{1}{2} \right),
\]

which is even. Then

\[
PV \int_{-\infty}^{\infty} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx = \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\infty} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx + \int_{-\infty}^{-\epsilon} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx \right)
\]

\[
= \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\infty} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx + \int_{\epsilon}^{\infty} \frac{f(-x)}{\sinh \left( \frac{-3\alpha x}{2} \right)} \, dx \right)
\]

\[
= \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\infty} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx - \int_{\epsilon}^{\infty} \frac{f(x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx \right) = 0,
\]

as we had wished to show. In conclusion, from (7.2.16) and our calculations above, we have so far demonstrated that

\[
PV \int_{-\infty}^{\infty} F_n(z) \, dz
\]

\[
= 4\pi i (-1)^n \sin \left( \frac{(2n+1)\pi}{3} \right) q_1^{(2n+1)^2/6} \int_{-\infty}^{\infty} e^{-\frac{3}{2} \alpha x^2} \frac{\sinh(\alpha x)}{\sinh \left( \frac{3\alpha x}{2} \right)} \, dx.
\]

(7.2.17)
The integral along the lower side of the rectangular contour $C_N$ can be examined in the same manner as that on the upper side. In fact, all we need to do is to replace $i$ by $-i$. Hence, from (7.2.5), (7.2.14), (7.2.15), and (7.2.17), we conclude that

$$
(q; q)_{\infty} f_3(q) = 8 \sum_{n=0}^{\infty} (-1)^n \sin \left( \frac{(2n+1)\pi}{3} \right) q_1^{(2n+1)^2/6} \int_0^\infty e^{-\frac{3}{2} \alpha x^2} \frac{\sinh(\alpha x)}{\sinh(\frac{3}{2} \alpha x)} \, dx
$$

$$+ \frac{4\pi}{\alpha} \sum_{r=0}^{\infty} (-1)^r q_1^{3(2r+1)^2/2} \frac{1 + q_1^{2(2r+1)}}{1 - q_1^{2(2r+1)}}.
$$

(7.2.18)

Now recall (1.3.14), namely,

$$
(q^2; q^2)_{\infty} \omega_3(q) = \sum_{r=0}^{\infty} (-1)^r q^{3r(r+1)} \frac{1 + q^{2r+1}}{1 - q^{2r+1}}.
$$

Thus, we can recast (7.2.18) in the form

$$
(q; q)_{\infty} f_3(q) = 8 \sum_{n=0}^{\infty} (-1)^n \sin \left( \frac{(2n+1)\pi}{3} \right) q_1^{(2n+1)^2/6} \int_0^\infty e^{-\frac{3}{2} \alpha x^2} \frac{\sinh(\alpha x)}{\sinh(\frac{3}{2} \alpha x)} \, dx
$$

$$+ \frac{4\pi}{\alpha} \omega_3(q_1^2)(q_1^2; q_1^2)_{\infty} q_1^{3/2}.
$$

(7.2.19)

Replacing $n$ by $-n-1$ in the second portion of the series below and eventually using the Jacobi triple product identity (4.1.2), we find that

$$
2 \sum_{n=0}^{\infty} (-1)^n \sin \left( \frac{(2n+1)\pi}{3} \right) q_1^{(2n+1)^2/6}
$$

$$= \frac{1}{i} \sum_{n=0}^{\infty} (-1)^n \left( e^{(2n+1)\pi i/3} - e^{-(2n+1)\pi i/3} \right) q_1^{(2n+1)^2/6}
$$

$$= \frac{1}{i} \sum_{n=0}^{\infty} (-1)^n e^{(2n+1)\pi i/3} q_1^{(2n+1)^2/6} + \frac{1}{i} \sum_{n=-1}^{-\infty} (-1)^n e^{(2n+1)\pi i/3} q_1^{(2n+1)^2/6}
$$

$$= \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n e^{(2n+1)\pi i/3} q_1^{(2n+1)^2/6}
$$

$$= \frac{1}{i} e^{\pi i/3} q_1^{1/6} \sum_{n=-\infty}^{\infty} (-1)^n e^{2\pi i n/3} q_1^{2n(n+1)/3}
$$

$$= \frac{1}{i} e^{\pi i/3} q_1^{1/6} f(-e^{2\pi i/3} q_1^{1/6}, -e^{-2\pi i/3})
$$

$$= \frac{1}{i} e^{\pi i/3} q_1^{1/6} \left( q_1^{4/3}, q_1^{4/3} \right)_\infty (e^{2\pi i/3} q_1^{4/3}, q_1^{4/3})_\infty (1 - e^{-2\pi i/3}) (e^{-2\pi i/3} q_1^{4/3}, q_1^{4/3})_\infty
$$

$$= \sqrt{3} q_1^{1/6} (q_1^{4/3}, q_1^{4/3})_\infty (e^{2\pi i/3} q_1^{4/3}, q_1^{4/3})_\infty (e^{-2\pi i/3} q_1^{4/3}, q_1^{4/3})_\infty
$$
\( f_3(q) = \sqrt{3}q_1^{1/6}(q_1^4; q_1^4)_\infty \).  \hspace{1cm} (7.2.20)

Substituting (7.2.20) into (7.2.19), we arrive at

\[
(q; q)_\infty f_3(q) = 4\sqrt{3}q_1^{1/6}(q_1^4; q_1^4)_\infty \int_0^\infty e^{-\frac{\alpha x^2}{2}} \sinh(\alpha x) \sinh\left(\frac{3}{2}\alpha x\right) dx \\
+ 4\frac{\pi}{\alpha^{3/2}} q_1^{3/2} \omega_3(q_1^2)(q_1^4; q_1^4)_\infty.  \hspace{1cm} (7.2.21)
\]

In order to simplify (7.2.21), we recall the transformation formula for the Dedekind eta function [28, p. 43, Entry 27(iii)]. If

\[
e^{-\alpha/24}(\alpha/2)^{1/4}f(-e^{-\alpha}) = e^{-\beta/6}(2\beta)^{1/4}f(-e^{-4\beta}),  \hspace{1cm} (7.2.22)
\]

or

\[
q^{-1/24}\frac{f(-e^{-\beta})}{f(-e^{-\alpha})} = e^{\beta/6}\sqrt{\frac{\alpha}{2\pi}}.
\]

Thus,

\[
4\frac{\pi}{\alpha} q_1^{3/2} q^{-1/24}(q_1^4; q_1^4)_\infty = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{4/3}
\]

and

\[
4\sqrt{3} q_1^{1/6} q^{-1/24}(q_1^4; q_1^4)_\infty = 4\sqrt{\frac{3\alpha}{2\pi}}.
\]

Using the last two identities in (7.2.21), we conclude that

\[
q^{-1/24} f_3(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{4/3} \omega_3(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \sinh(\alpha x) \sinh\left(\frac{3}{2}\alpha x\right) dx.  \hspace{1cm} (7.2.23)
\]

The identity (7.2.23) is the same as (7.2.1), and so our proof of Theorem 7.2.1 is complete. \( \Box \)

We now show that the integral in Theorem 7.2.1, i.e.,

\[
J_1(\alpha) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh(\alpha x)}{\sinh\left(\frac{3}{2}\alpha x\right)} dx
\]

satisfies a beautiful transformation law. Also, let

\[
J_2(\alpha) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(\alpha x)}{\cosh(3\alpha x)} dx.
\]

**Theorem 7.2.2.** With \( J_1(\alpha) \) and \( J_2(\alpha) \) given above,

\[
J_1(\alpha) = \sqrt{\frac{2\pi^3}{\alpha^3}} J_2(\beta).  \hspace{1cm} (7.2.24)
\]
We first observe that, for \( x > 0 \),
\[
\int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \cos(3\pi xy) dy = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{3}{2} \beta y^2 + 3\pi i x y} dy
\]
\[
= \frac{1}{2} e^{-\frac{3}{2} \alpha x^2} \int_{-\infty}^{\infty} e^{-\frac{3}{2} \beta (y-i\alpha x/\pi)^2} dy
\]
\[
= \frac{1}{2} e^{-\frac{3}{2} \alpha x^2} \int_{-\infty}^{\infty} e^{-\frac{3}{2} \beta y^2} dy
\]
\[
= \frac{\sqrt{\pi}}{2 \sqrt{3\beta}} \int_{-\infty}^{\infty} e^{-x^2} dx
\]
\[
= \frac{\sqrt{\pi}}{2 \sqrt{3\beta}} e^{-\frac{3}{2} \alpha x^2}, \quad (7.2.25)
\]
where we applied Cauchy’s theorem upon integrating over a rectangle with horizontal sides passing through the origin and \(-i\alpha x/\pi\). Hence, using (7.2.25) and an integral evaluation from [81, p. 537, equation 3.981, no. 5], we find that
\[
J_1(\alpha) = \sqrt{\frac{6\beta}{\pi}} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \cos(3\pi xy) \sinh(\alpha x) \sinh(3\beta y) dx dy
\]
\[
= \sqrt{\frac{6\beta}{\pi}} \int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \frac{\sin(2\pi/3)}{3\alpha \cosh(2\beta y) + \cos(2\pi/3)} dy
\]
\[
= \sqrt{\frac{6\beta \pi}{3\alpha}} \int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \frac{\sqrt{3/2}}{\cosh(2\beta y) - 1} dy
\]
\[
= \sqrt{\frac{2\beta \pi}{\alpha^2}} \int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \frac{dy}{2 \cosh(2\beta y) - 1} dy. \quad (7.2.26)
\]
It is easily checked that
\[
\frac{1}{2 \cosh(2\beta y) - 1} = \frac{\cosh(\beta y)}{\cosh(3\beta y)}.
\]
Thus, we can write (7.2.26) in the form
\[
J_1(\alpha) = \sqrt{\frac{2\pi^3}{\alpha^2}} \int_{0}^{\infty} e^{-\frac{3}{2} \beta y^2} \frac{\cosh(\beta y)}{\cosh(3\beta y)} dy,
\]
which is equivalent to (7.2.24). \(\square\)

We now derive a second transformation formula, this one involving the third order mock theta functions \(\phi_3(q)\) and \(\psi_3(q)\). Before establishing this second transformation formula for third order mock theta functions, we establish a lemma providing representations for \(\phi_3(q)\) and \(\psi_3(q)\) that are analogous to those for \(f_3(q)\) and \(\omega_3(q)\) given in (1.3.22) and (1.3.14), respectively.
Lemma 7.2.1. We have

\[(q; q)\infty \phi_3(q) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n(3n+1)/2(1 + q^n)}{1 + q^{2n}} \quad (7.2.27)\]

and

\[(q^2; q^4)\infty \psi_3(q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n(n+1)+1}}{1 - q^{4n+1}}. \quad (7.2.28)\]

Proof. We begin by rewriting (1.3.13) in the equivalent form

\[
(1 - c) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - cq^n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(cq)_n(q/c)_n}. \quad (7.2.29)
\]

Set \(c = i\) in (7.2.29) to find that

\[
S := (1 - i) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - iq^n} = (q; q)\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n} = (q; q)\infty \phi_3(q). \quad (7.2.30)
\]

We need to rewrite the left side of (7.2.30) so that it does not involve non-real numbers. To that end,

\[
(1 - i) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - iq^n} = (1 - i) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + iq^n)}{1 + q^{2n}}. \quad (7.2.31)
\]

Let us assume, without loss of generality, that \(q\) is real. The real part of the right-hand side of (7.2.31) is thus equal to

\[
S = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{1 + q^{2n}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}q^n}{1 + q^{2n}}.
\]

In that part of the series over \(-1 \geq n > -\infty\), we replace \(n\) by \(-n\) to deduce that

\[
S = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{1 + q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}(1 + q^{-n})}{1 + q^{-2n}}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{1 + q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{1 + q^{2n}}
\]

\[
= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{1 + q^{2n}}.
\]
Putting the far right side of the equation above into (7.2.30), we complete the proof of (7.2.27).

Lastly, we turn to the proof of (7.2.28), which will be somewhat more difficult. Set \( c = q^{1/4} \) in (7.2.29) and then replace \( q \) by \( q^4 \) to deduce that

\[
1 - q^{1/4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(3n+1)}}{1 - q^{4n+1}} = \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^5; q^4)_n(q^3; q^4)_n} =: R. \tag{7.2.32}
\]

We rewrite the right-hand side of (7.2.32), and then at the third equality below, we set \( m = 2n \) in the first series and \( m = 2n + 1 \) in the second series. To that end,

\[
R = 1 + \sum_{n=1}^{\infty} \frac{q^{4n^2}}{(1 - q^3)(1 - q^{3}) \cdots (1 - q^{4n+1})} \\
= 1 + \frac{q^{4n^2}}{(1 - q^3)(1 - q^{3}) \cdots (1 - q^{4n-1})} \\
+ \sum_{n=1}^{\infty} \frac{q^{(2n+1)^2}}{(1 - q^3)(1 - q^{3}) \cdots (1 - q^{4n+1})} \\
= 1 + \sum_{m=2}^{\infty} \frac{q^{m^2}}{(1 - q^3)(1 - q^{3}) \cdots (1 - q^{2m+1})} \\
= (1 - q) \left\{ \frac{1}{1 - q} - \frac{q}{1 - q} + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q q^2)_m} \right\} \\
= (1 - q) \left\{ 1 + \psi_3(q) \right\}. \tag{7.2.33}
\]

Inserting (7.2.33) in (7.2.32), we arrive at

\[
\frac{1}{(q^4; q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(3n+1)}}{1 - q^{4n+1}} = 1 + \psi_3(q),
\]

or, by the pentagonal number theorem (1.3.27),
7.3 A Transformation Formula Involving $\phi_3(q)$ and $\psi_3(q)$

Theorem 7.3.1. Let $\alpha$ and $\beta$ be positive numbers such that $\alpha \beta = \pi^2$. Set $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$. Then

$$q^{-1/24} \phi_3(q) = 2 \sqrt{\frac{\pi}{\alpha}} q_1^{-1/24} \psi_3(q_1)$$

\[+ \sqrt{\frac{6\alpha}{\pi}} \int_0^\infty e^{-\frac{3}{2} \alpha x^2} \frac{\cosh(\frac{1}{2} \alpha x) + \cosh(\frac{1}{2} \alpha x)}{\cosh(3 \alpha x)} \, dx.\]  

(7.3.1)

Proof. Consider

$$\frac{1}{2\pi i} \int_{C_N} \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2} \alpha z^2} \frac{\cosh(\frac{1}{2} \alpha z)}{\cosh(\alpha z)} \, dz,$$

(7.3.2)

where $C_N$ denotes the same contour as in the proof of Theorem 7.2.1, except that now we require that $c < \pi/(2\alpha)$. We apply the residue theorem. The integrand has simple poles at each integer $n$ with residue

$$R_n = (-1)^n e^{-\frac{3}{2} \alpha n^2} \frac{\cosh(\frac{1}{2} \alpha n)}{\cosh(\alpha n)} = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n(3n+1)/2}(1+q^n), & \text{if } n \neq 0. \end{cases}$$

(7.3.3)

It is readily shown that, if $L_1$ and $L_2$ are the vertical sides of $C_N$, as before,

$$\frac{1}{2\pi i} \int_{L_j} \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2} \alpha z^2} \frac{\cosh(\frac{1}{2} \alpha z)}{\cosh(\alpha z)} \, dz = o(1), \quad j = 1, 2,$$

(7.3.4)

as $N \to \infty$. Thus, upon letting $N \to \infty$, we find that, by (7.2.27),

$$\sum_{n=-\infty}^{\infty} R_n = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{2n}} = (q; q)_{\infty} \phi_3(q),$$

(7.3.5)

and so (7.2.28) has also been established. \[\square\]
and by (7.3.2), (7.3.4), and (7.3.5),

\[
(q;q)_{\infty}\phi_3(q) = \frac{1}{2\pi i} \left( \int_{-\infty-i\infty}^{\infty-i\infty} + \int_{\infty+i\infty}^{\infty-i\infty} \right) \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2}z^2} \frac{\cosh(\frac{1}{2}z)}{\cosh(2z)} \, dz.
\]

(7.3.6)

Now we evaluate the line integral above that lies in the upper half-plane. Using (7.2.6) and the easily verified elementary identity

\[
\cosh(3\alpha z) = 2 \cosh(2\alpha z) - 1,
\]

we find that

\[
\frac{1}{2\pi i} \int_{-\infty-i\infty}^{\infty-i\infty} \frac{\pi}{\sin(\pi z)} e^{-\frac{3}{2}z^2} \frac{\cosh(\frac{1}{2}z)}{\cosh(\alpha z)} \, dz = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{-\infty+i\infty}^{\infty+i\infty} F_n(z),
\]

(7.3.7)

where

\[
F_n(z) := 2\pi i e^{(2n+1)\pi i z - \frac{3}{2}z^2} \cosh(\frac{1}{2}z) \left\{ 2 \cosh(\alpha z) - 1 \right\} \frac{\cosh(2\alpha z)}{\cosh(3\alpha z)}.
\]

(7.3.8)

In order to evaluate the integrals above, we integrate over the same contour \(\gamma_n\) as we did in the proof of Theorem 7.2.1. We note that the integrand, i.e., \(F_n(z)\), has simple poles, for each integer \(n\), at

\[
\frac{(4m+1)\pi i}{6\alpha} = \zeta_m, \quad \text{and} \quad \frac{(4m-1)\pi i}{6\alpha} = \eta_m, \quad m \in \mathbb{Z},
\]

where we have subdivided the original set of poles into two classes. By straightforward calculations the residues at \(\zeta_m\) and \(\eta_m\), are, respectively,

\[
R_{\zeta_m} = \frac{2\pi}{3\alpha} e^{(2n+1)\pi i \zeta_m - \frac{3}{2}\alpha z^2} \cos \left( \frac{4m+1}{12} \pi \right) \left\{ 2 \cos \left( \frac{(4m+1)\pi}{3} \right) - 1 \right\} =: \mu_{n,m}
\]

(7.3.9)

and

\[
R_{\eta_m} = -\frac{2\pi}{3\alpha} e^{(2n+1)\pi i \eta_m - \frac{3}{2}\alpha \eta_m^2} \cos \left( \frac{4m-1}{12} \pi \right) \left\{ 2 \cos \left( \frac{(4m-1)\pi}{3} \right) - 1 \right\} =: \nu_{n,m}.
\]

(7.3.10)

As in the proof of Theorem 7.2.1, the integrals over the vertical sides of \(\gamma_n\) tend to 0 as \(N \to \infty\). Thus, by the residue theorem,

\[
\frac{1}{2\pi i} \left\{ \int_{-\infty+i\infty}^{\infty+i\infty} - \int_{-\infty+i\infty}^{-\infty+z_n} \right\} F_n(z) \, dz = \mu_{n,0} + \mu_{n,1} + \cdots + \mu_{n,n} + \nu_{n,1} + \nu_{n,2} + \cdots + \nu_{n,n}.
\]

(7.3.11)

Proceeding again as in the proof of Theorem 7.2.1, we sum both sides of (7.3.11) on \(n\), \(0 \leq n < \infty\). Clearly, analogues of (7.2.13) hold for \(\mu_{n,m}\) and
Thus, inverting the order of summation and eventually using the definitions of $\mu_{m,m}$ from (7.3.9) and $\nu_{m,m}$ from (7.3.10), we find that

$$S := \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left\{ \int_{-\infty+i\epsilon}^{\infty+i\epsilon} - \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \right\} F_n(z)dz$$

(7.3.12)

$$= \sum_{n=0}^{\infty} \{ \mu_{n,0} + \mu_{n,1} + \cdots + \mu_{n,n} \} + \sum_{n=0}^{\infty} \{ \nu_{n,1} + \nu_{n,2} + \cdots + \nu_{n,n} \}$$

$$= \sum_{m=0}^{\infty} \{ \mu_{m,m} + \mu_{m+1,m} + \mu_{m+2,m} + \cdots \}$$

$$+ \sum_{m=1}^{\infty} \{ \nu_{m,m} + \nu_{m+1,m} + \nu_{m+2,m} + \cdots \}$$

$$= \sum_{m=0}^{\infty} \{ \mu_{m,m} + e^{2\pi i\zeta_m} \mu_{m,m} + e^{4\pi i\zeta_m} \mu_{m,m} \cdots \}$$

$$+ \sum_{m=1}^{\infty} \{ \nu_{m,m} + e^{2\pi i\eta_m} \nu_{m,m} + e^{4\pi i\eta_m} \nu_{m,m} \cdots \}$$

$$= \sum_{m=0}^{\infty} \frac{\mu_{m,m}}{1 - e^{2\pi i\zeta_m}} + \sum_{m=1}^{\infty} \frac{\nu_{m,m}}{1 - e^{2\pi i\eta_m}}$$

$$= \frac{2\pi}{3\alpha} \left\{ \sum_{m=0}^{\infty} \frac{q_1^{(4m+1)(4m+3)/24}}{1 - q_1^{(4m+1)/3}} \cos \frac{(4m + 1)\pi}{12} \left\{ 2\cos \frac{(4m + 1)\pi}{3} - 1 \right\} \right\} - \sum_{m=1}^{\infty} \frac{q_1^{(4m-1)(4m+5)/24}}{1 - q_1^{(4m-1)/3}} \cos \frac{(4m - 1)\pi}{12} \left\{ 2\cos \frac{(4m - 1)\pi}{3} - 1 \right\} \right\}.$$

Now

$$2 \cos \frac{(4m + 1)\pi}{3} - 1 = \begin{cases} 0, & \text{if } m \equiv 0, 1 \pmod{3}, \\ -3, & \text{if } m \equiv 2 \pmod{3}. \end{cases}$$

and

$$2 \cos \frac{(4m - 1)\pi}{3} - 1 = \begin{cases} 0, & \text{if } m \equiv 0, 2 \pmod{3}, \\ -3, & \text{if } m \equiv 1 \pmod{3}. \end{cases}$$

Thus, in the first sum on the far right side of (7.3.12), we set $m = 3n + 2$, while in the second sum on the extreme right side of (7.3.12), we set $m = 3n + 1$. We note that

$$\cos \frac{4(3n + 2) + 1}{12} \pi = \frac{(-1)^{n+1}}{\sqrt{2}}$$

and

$$\cos \frac{4(3n + 1) - 1}{12} \pi = \frac{(-1)^n}{\sqrt{2}}.$$

Employing the trigonometric evaluations above, we find that the equation on the far right-hand side of (7.3.12) reduces to the form
Transformation Formulas for Third Order Mock Theta Functions

\[
S = \frac{\pi \sqrt{2}}{\alpha q_1^{g/8}} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^nq_1^{2n(3n+5)+3}}{1-q_1^{4n+3}} + \sum_{n=0}^{\infty} \frac{(-1)^nq_1^{6n(n+1)}}{1-q_1^{4n+1}} \right\}.
\]  

(7.3.13)

From (7.3.12) and (7.3.13), it is clear that our next task is to evaluate

\[
\int_{-\infty}^{\infty} F_n(z)dz = \int_{-\infty}^{\infty} F_n(x+z_n)dx
\]

(7.3.14)

\[
= 2\pi i \int_{-\infty}^{\infty} e^{-(2n+1)^2\pi^2/(6\alpha)} \frac{\frac{3}{2}\alpha x^2}{\cosh(\frac{1}{2}\alpha z)} \frac{2\cosh(2\alpha z) - 1}{\cosh(3\alpha z)} dx
\]

\[
= 2\pi i \int_{-\infty}^{\infty} e^{-(2n+1)^2\pi^2/(6\alpha)} \frac{\frac{3}{2}\alpha x^2}{\cosh(\frac{5}{2}\alpha x) + \cosh(\frac{3}{2}\alpha x) - \cosh(\frac{1}{2}\alpha x)} \cosh(3\alpha x) dx,
\]

where we have temporarily set

\[
z = x + \frac{(2n+1)\pi i}{3\alpha}
\]

and used the elementary identity

\[
\cosh(\frac{1}{2}x)\{2\cosh(2x) - 1\} = \cosh(\frac{5}{2}x) + \cosh(\frac{3}{2}x) - \cosh(\frac{1}{2}x).
\]

We apply the addition formula for \(\cosh x\) three times in the integrand on the right-hand side of (7.3.14). Noting that contributions from terms involving \(\sinh x\) will be 0, we find, after simplification, that

\[
\int_{-\infty}^{\infty} F_n(z)dz = 2\pi i \int_{-\infty}^{\infty} e^{-(2n+1)^2\pi^2/(6\alpha)} \frac{\frac{3}{2}\alpha x^2}{\cosh(3\alpha x)} dx
\]

(7.3.15)

\[
= -2\pi i q_1^{(2n+1)^2/6} \cos \frac{(2n+1)\pi}{6} \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(\frac{5}{2}\alpha x) + \cosh(\frac{3}{2}\alpha x) - \cosh(\frac{1}{2}\alpha x)}{\cosh(3\alpha x)} dx.
\]

We now sum (7.3.15), \(0 \leq n < \infty\), and put the result, along with (7.3.13), into (7.3.6). We also observe that the integral along the lower side of \(C_N\) may be evaluated by simply replacing \(i\) by \(-i\) in the analysis above. In summary, so far we have shown that

\[
(q;q)_{\infty} \phi_3(q) = \frac{2\pi \sqrt{2}}{\alpha} q_1^{g/8} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^nq_1^{2n(3n+5)+3}}{1-q_1^{4n+3}} + \sum_{n=0}^{\infty} \frac{(-1)^nq_1^{6n(n+1)}}{1-q_1^{4n+1}} \right\}
\]

(7.3.16)

\[
+ 2\sum_{n=0}^{\infty} q_1^{(2n+1)^2/6} \cos \frac{(2n+1)\pi}{6} \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(\frac{5}{2}\alpha x) + \cosh(\frac{3}{2}\alpha x) - \cosh(\frac{1}{2}\alpha x)}{\cosh(3\alpha x)} dx.
\]

Next, replacing \(n\) by \(-n - 1\) in the second sum on the right-hand side below, we find that
From (7.2.28), we have

\[ 2 \sum_{n=0}^{\infty} q_1^{(2n+1)^2/6} \cos \left( \frac{(2n+1)\pi}{6} \right) = q_1^{1/6} \sum_{n=0}^{\infty} q_1^{2n(n+1)/3} (e^{(2n+1)\pi i/6} + e^{-(2n+1)\pi i/6}) \]

Now, by the Jacobi triple product identity (4.1.2),

\[ e^{\pi i/6} f(q_1^{4/3}, e^{\pi i/3}, e^{-\pi i/3}) \]

\[ = e^{\pi i/6} (-q_1^{4/3}, e^{\pi i/3}; q_1^{4/3})_{\infty} (-e^{-\pi i/3}, q_1^{4/3})_{\infty} \]

\[ = 2 \cos(\pi/6) (-q_1^{4/3}, e^{\pi i/3}; q_1^{4/3})_{\infty} (-q_1^{4/3}, e^{-\pi i/3}; q_1^{4/3})_{\infty} \]

\[ = \sqrt{3}(q_1^{1/6} f(q_1^{4/3}, e^{\pi i/3}, e^{-\pi i/3}) \quad (7.3.17) \]

Next, if \( \alpha, \beta > 0, \alpha \beta = \pi^2 \), and \( f(q) = (q; q)_{\infty} \), then \([28, \text{p. 43, Entry 27(iii)}]\]

\[ f(-e^{-\alpha}) = q^{-1/24} e^{-\beta/6} \sqrt{\frac{2\pi}{\alpha}} f(-e^{-\beta}) , \]

or

\[ q^{1/24} f(-q) = \sqrt{\frac{2\pi}{\alpha}} q_1^{1/6} f(-q_1^{4}). \quad (7.3.19) \]

Hence, combining (7.3.18) and (7.3.19) yields

\[ \frac{q_1^{1/6} e^{\pi i/6} f(q_1^{4/3}, e^{\pi i/3}, e^{-\pi i/3})}{q^{1/24} f(-q)} = \sqrt{\frac{3\alpha}{2\pi}}. \quad (7.3.20) \]

Now return to (7.3.16) and replace \( n \) by \( -n - 1 \) in the first sum on the left side below to obtain

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q_1^{2n(3n+5)+4}}{1 - q_1^{4n+3}} + \sum_{n=0}^{\infty} \frac{(-1)^n q_1^{6n(n+1)+1}}{1 - q_1^{4n+1}} \]

\[ = \sum_{n=-\infty}^{-1} \frac{(-1)^n q_1^{6n(n+1)+1}}{1 - q_1^{4n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q_1^{6n(n+1)+1}}{1 - q_1^{4n+1}} \]

\[ = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q_1^{6n(n+1)+1}}{1 - q_1^{4n+1}}. \quad (7.3.21) \]

From (7.2.28), we have

\[ (q^4; q^4)_{\infty} \psi_3(q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q_1^{6n(n+1)+1}}{1 - q_1^{4n+1}}. \quad (7.3.22) \]
Dividing both sides of (7.3.16) by $q^{1/24}(q; q)_\infty$, employing (7.3.22) in (7.3.21), and using (7.3.19), we are led to the simplification of

$$\frac{2\pi\sqrt{2}}{\alpha} \frac{1}{q^{1/2} q^{4/15} q^{1/18}} \psi_3(q_1) - 2\sqrt{\frac{\pi}{\alpha}} q_1^{-1/24} \psi_3(q_1).$$

(7.3.23)

Substituting (7.3.20) and (7.3.23) into (7.3.16) after having divided it by $q^{1/24}(q; q)_\infty$, we finally deduce that

$$q^{-1/24} \phi_3(q) = 2\sqrt{\frac{\pi}{\alpha}} q_1^{-1/24} \psi_3(q_1) + \sqrt{\frac{6\alpha}{\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\left(\frac{5}{2} \alpha x\right) + \cosh\left(\frac{1}{2} \alpha x\right)}{\cosh(3\alpha x)} dx.$$

This identity is the same as (7.3.1), and so the proof of Theorem 7.3.1 is complete.

We now demonstrate that the integral

$$J(\alpha) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\left(\frac{5}{2} \alpha x\right) + \cosh\left(\frac{1}{2} \alpha x\right)}{\cosh(3\alpha x)} dx$$

(7.3.24)

satisfies the following beautiful transformation formula.

**Theorem 7.3.2.** If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$J(\alpha) = \sqrt{\frac{\pi^3}{\alpha^3}} J(\beta).$$

(7.3.25)

**Proof.** We appeal once again to (7.2.25) and then use the formula [81, p. 538, formula 3.981, no. 10] to deduce that

$$J(\alpha) = \sqrt{\frac{6\beta}{\alpha^3}} \int_0^\infty \int_0^\infty e^{-\frac{3}{2}\beta y^2} \cos(3\pi xy) \frac{\cosh\left(\frac{5}{2} \alpha x\right) + \cosh\left(\frac{1}{2} \alpha x\right)}{\cosh(3\alpha x)} dx dy$$

(7.3.26)

$$= \sqrt{\frac{6\beta}{3\alpha}} \int_0^\infty e^{-\frac{3}{2}\beta y^2} \left\{ \cos\left(\frac{5}{12} \pi\right) \frac{\cosh(\frac{1}{2} \beta y)}{\cosh(\beta y) + \cos(\frac{5}{6} \pi)} + \cos\left(\frac{1}{12} \pi\right) \frac{\cosh(\frac{1}{2} \beta y)}{\cosh(\beta y) + \cos(\frac{1}{6} \pi)} \right\} dy.$$

Now,

$$C := \frac{\cos\left(\frac{5}{12} \pi\right) \cosh(\frac{1}{2} \beta y)}{\cosh(\beta y) + \cos(\frac{5}{6} \pi)} + \frac{\cos\left(\frac{1}{12} \pi\right) \cosh(\frac{1}{2} \beta y)}{\cosh(\beta y) + \cos(\frac{1}{6} \pi)}$$

$$= \cosh(\frac{1}{2} \beta y) \left\{ \cos\left(\frac{5}{12} \pi\right) \cosh(\beta y) + \frac{\sqrt{3}}{2} \right\} + \cos\left(\frac{1}{12} \pi\right) \left\{ \cosh(\beta y) - \frac{1}{2} \sqrt{3} \right\} \left(\cosh(\beta y) + \frac{1}{2} \sqrt{3} \right)$$

$$= \frac{\cosh(\frac{1}{2} \beta y) \left\{ \cos\left(\frac{5}{12} \pi\right) \cosh(\beta y) + \frac{\sqrt{3}}{2} \right\} + \cos\left(\frac{1}{12} \pi\right) \left\{ \cosh(\beta y) - \frac{1}{2} \sqrt{3} \right\} \left(\cosh(\beta y) + \frac{1}{2} \sqrt{3} \right)}{\left(\cosh(\beta y) - \frac{1}{2} \sqrt{3}\right)\left(\cosh(\beta y) + \frac{1}{2} \sqrt{3}\right)}.$$
We now note that
\[
\left( \sqrt{\frac{1 - \frac{1}{2} \sqrt{3}}{2}} + \sqrt{\frac{1 + \frac{1}{2} \sqrt{3}}{2}} \right)^2 = \frac{3}{2} \quad \text{and} \quad \left( \sqrt{\frac{1 - \frac{1}{2} \sqrt{3}}{2}} - \sqrt{\frac{1 + \frac{1}{2} \sqrt{3}}{2}} \right)^2 = \frac{1}{2}.
\]

Hence,
\[
C = \frac{\cosh\left(\frac{1}{2} \beta y\right)}{\cosh^2(\beta y) - \frac{3}{4}} \left\{ \sqrt{\frac{3}{2}} \cosh(\beta y) - \frac{1}{2} \sqrt{3} \right\}
= \sqrt{\frac{3}{2}} \left\{ \frac{\cosh\left(\frac{1}{2} \beta y\right)}{\cosh^2(\beta y) - \frac{3}{4}} \left( \cosh(\beta y) - 1 \right) \right\}
= \sqrt{\frac{3}{2}} \frac{\cosh\left(\frac{1}{2} \beta y\right) \cosh(\beta y)}{\cosh(3\beta y)}
= \sqrt{\frac{3}{2}} \frac{\cosh\left(\frac{1}{2} \beta y\right) + \cosh\left(\frac{1}{2} \beta y\right)}{\cosh(3\beta y)}. \tag{7.3.27}
\]

Putting (7.3.27) in (7.3.26), we conclude that
\[
J(\alpha) = \sqrt{\frac{6\beta}{\pi}} \frac{\pi}{3\alpha} \sqrt{\frac{3}{2}} J(\beta) = \sqrt{\frac{\pi^3}{\alpha^3}} J(\beta),
\]
which completes the proof of (7.3.25).

\[\square\]

### 7.4 A Brief Survey of Further Transformation Formulas for Mock Theta Functions

The proofs in this chapter are due to Watson [142]. Following his proofs of the two transformation formulas in Theorems 7.2.1 and 7.3.1, he provides six
additional transformation formulas for third order mock theta functions given by

\[ q^{-\frac{1}{24}} f(q) - 2 \sqrt{\frac{2\pi}{\alpha}} q^{\frac{4}{3}} \omega(q^{\frac{1}{2}}) = 2 \sqrt{\left(\frac{6\alpha}{\pi}\right)} J_1(\alpha) = \frac{4\beta\sqrt{3}}{\pi} J_2(\beta), \]

\[ q^{-\frac{1}{24}} f(-q) + \sqrt{\left(\frac{\pi}{\alpha}\right)} q^{\frac{1}{24}} f(-q) = 2 \sqrt{\left(\frac{6\alpha}{\pi}\right)} J(\alpha) = \frac{2\beta\sqrt{6}}{\pi} J(\beta), \]

\[ q^{-\frac{1}{24}} \phi(q) - 2 \sqrt{\left(\frac{\pi}{\alpha}\right)} q^{\frac{1}{24}} \psi(q) = \sqrt{\left(\frac{6\alpha}{\pi}\right)} J(\alpha) = \frac{\beta\sqrt{6}}{\pi} J(\beta), \]

\[ q^{-\frac{1}{24}} \phi(-q) - 2 \sqrt{\left(\frac{2\pi}{\alpha}\right)} q^{\frac{1}{12}} \psi(-q) = \sqrt{\left(\frac{6\alpha}{\pi}\right)} J_1(\alpha) = \frac{2\beta\sqrt{3}}{\pi} J_2(\beta), \]

\[ q^{-\frac{1}{24}} \psi(-q) - \sqrt{\left(\frac{\pi}{2\alpha}\right)} q^{\frac{1}{12}} \psi(q) = - \sqrt{\left(\frac{3\alpha}{2\pi}\right)} J_1(\alpha) = - \frac{\beta\sqrt{3}}{\pi} J_2(\beta), \]

\[ q^{\frac{1}{3}} \omega(q) - \sqrt{\left(\frac{\pi}{4\alpha}\right)} q^{\frac{1}{12}} f(q^{\frac{1}{2}}) = - \sqrt{\left(\frac{3\alpha}{2\pi}\right)} J_2(\frac{1}{2}\alpha) = - \frac{2\beta\sqrt{3}}{\pi} J_1(2\beta). \]

Watson also gives asymptotic expansions for the Mordell integrals appearing in the transformation formulas. For example, as \( \alpha \to 0 \),

\[ J_2(\alpha) = \sqrt{\frac{\pi}{6\alpha}} \left\{ 1 - \frac{4}{3} \alpha + \frac{44}{9} \alpha^2 - \cdots \right\}. \]  

(7.4.1)

He does not provide any details on how he derived these asymptotic expansions. However, they can be readily derived from what is now commonly called Watson’s Lemma [62, p. 50].

**Theorem 7.4.1 (Watson’s Lemma).** Let \( \phi(t) \) be an analytic function of \( t \), holomorphic in a neighborhood of the origin, and let

\[ \phi(t) = \sum_{m=0}^{\infty} a_m t^m \]

be its Taylor series valid in some neighborhood of the origin. Suppose that for certain positive constants \( K, b, \) and \( r, \) independent of \( t, \)

\[ |\phi(t)| \leq Ke^{bt}, \quad t > 0. \]

Then

\[ \int_0^{\infty} e^{-zt} \phi(t) dt \sim \sum_{m=0}^{\infty} \frac{a_m}{r^m} \Gamma \left( \frac{m+1}{r} \right) z^{-(m+1)r}, \]

as \( |z| \to \infty \) in the sector \( |\arg(z)| \leq \frac{1}{2} \pi - \epsilon < \frac{1}{2} \pi. \)
Thus, to apply Watson’s Lemma to $J_2(\alpha)$, as $\alpha \to 0$ or as $\beta \to \infty$, first, let $\alpha x = u$ to obtain

$$J_2(\alpha) = \frac{1}{\alpha} \int_0^\infty e^{-\frac{3}{2\alpha} u^2} \frac{\cosh u}{\cosh 3u} du,$$

and second, set $z = \frac{3}{2\alpha}$ and $r = 2$.

In their excellent, highly recommended survey [78, pp. 114–115], B. Gordon and R. McIntosh provide a list of 18 transformation formulas for third order mock theta functions, ten of which can be found in Watson’s paper [142]. Four different Mordell integrals appear in the transformation formulas. We have observed three of them in our proofs above; the fourth is given by

$$\int_0^\infty e^{-3\alpha x^2} \frac{\sinh(\alpha x)}{\sinh(3\alpha x)} dx.$$

They observe that these transformation formulas, as well as those for fifth and seventh order mock theta functions, can be obtained from a universal transformation formula involving

$$g_3(x, q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x; q)_{n+1}(q/x; q)_{n+1}} \tag{7.4.2}$$

and

$$h_3(x, q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq; q)_n(q/x; q)_n}. \tag{7.4.3}$$

Note that, by (5.4.11), $g_3(x, q) = g(x, q)$. If $r$ is any real number, then their transformation law is given by [77], [78, p. 110, equation (4.2)]

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}} g_3(q^r, q) = \sqrt{\frac{\pi}{2\alpha}} \csc(\pi r) q^\frac{1}{8} \left( e^{2\pi ir} q^\frac{1}{4} \right) - \sqrt{\frac{3\alpha}{2\pi}} L(r, \alpha),$$

where

$$L(r, \alpha) := \int_0^\infty e^{-\frac{3}{2} \alpha x^2 \cosh \{(3r - 1)\alpha x\} + \cosh \{(3r - 2)\alpha x\}} \frac{\cosh \{(3\alpha x)\}}{\cosh \{(3\alpha x)\}} dx. \tag{7.4.4}$$

In their paper [77] and survey paper [78], Gordon and McIntosh record 20 transformation formulas for fifth order mock theta functions. We offer here just one example, namely,

$$q^{-\frac{1}{60}} f_0(q) = \sqrt{\frac{2\pi(5 - \sqrt{5})}{5\alpha} q_1^{-\frac{1}{60}} (F_0(q^2) - 1)} + \sqrt{\frac{2\pi(5 + \sqrt{5})}{5\alpha} q_1^{\frac{71}{60}} F_1(q^2)} + \sqrt{\frac{60\alpha}{\pi}} L \left( \frac{1}{5}, 10\alpha \right),$$
where \( L(r, \alpha) \) is defined in (7.4.4).

Gordon and McIntosh \[77\], \[78\] also derived transformation formulas for seventh order mock theta functions. We cite only one of the six that they list in \[78, p. 119\], namely,

\[
q^{-\frac{1}{168}} (F_0(q) - 2) = \sqrt{\frac{8\pi}{7\alpha}} \sin \left( \frac{\pi}{7} \right) q_1^{-\frac{1}{42}} F_0(q_1^4) + \sqrt{\frac{8\pi}{7\alpha}} \sin \left( \frac{2\pi}{7} \right) q_1^{-\frac{25}{42}} F_1(q_1^4) + \sqrt{\frac{8\pi}{7\alpha}} \sin \left( \frac{3\pi}{7} \right) q_1^{-\frac{47}{42}} F_2(q_1^4) - \sqrt{\frac{42\alpha}{\pi}} L \left( 1, 7\alpha \right).
\]

In analogy with the mock theta functions of odd order being related to the function \( g_3(x, q) \), defined in (7.4.2), the mock theta functions of even order are related to the function

\[
g_2(x, q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(x; q)_{n+1}(q/x; q)_{n+1}}. \tag{7.4.5}
\]

The mock theta functions of order 2 are defined by \[80\], \[78, p. 120, equation (5.1)]

\[
A(q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+1)} (q; q^2)_{n+1}}{(q; q^2)_{n+1}}, \\
B(q) = \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n(n+1)} (q; q^2)_{n+1}}{(q; q^2)_{n+1}}, \\
\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(-q^2; q^2)_{n+1}}.
\]

McIntosh \[106\], \[78, p. 120, equation (5.3)] established six transformation formulas featuring \( A(q) \), \( B(q) \), and \( \mu(q) \), of which one is given by

\[
q^{-\frac{1}{8}} A(q) = \sqrt{\frac{\pi}{16\alpha}} q_1^{-\frac{1}{8}} \mu(-q_1) - \sqrt{\frac{\alpha}{2\pi}} K(\alpha),
\]

where \( K(\alpha) \) is one of the two Mordell integrals appearing in the transformation formulas for second order mock theta functions; these integrals are defined by

\[
J(\alpha) := \int_0^\infty \frac{e^{-\alpha x^2}}{\cosh(\alpha x)} \, dx \quad \text{and} \quad K(\alpha) := \int_0^\infty e^{-\frac{1}{2} \alpha x^2} \frac{\cosh(\frac{1}{2} \alpha x)}{\cosh(\alpha x)} \, dx.
\]

Gordon and McIntosh \[78, pp. 122–123\] record 20 transformation formulas for the sixth order mock theta functions that we studied in a previous chapter. We cite one example. If \( \beta(q) \) and \( \gamma(q) \) are given by

\[
\beta(q) := \sum_{n=0}^{\infty} \frac{q^{3n^2 + 3n + 1}}{(q; q^3)_{n+1}(q^2; q^3)_{n+1}} = qg_4(q, q^3)
\]
and
\[\gamma(q) := \sum_{n=0}^{\infty} \frac{(q; q)_n q^{n^2}}{(q^3; q^3)_n} = h_3(e^{2\pi i/3}, q),\]
then
\[q^{-\frac{1}{8}} \beta(q) = \sqrt{\frac{2\pi}{9\alpha}} q^{-\frac{1}{18}} \gamma(q_{1/4}) - \sqrt{\frac{81\alpha}{8\pi}} \left\{ J\left(\frac{9\alpha}{2}\right) + \frac{1}{9} J\left(\frac{\alpha}{2}\right) \right\},\]
where \(J(\alpha)\) is defined, as before, by (7.3.24).

Gordon and McIntosh [76], [78] also studied eighth order mock theta functions, which are defined by

\[S_0(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(-q^4; q^4)_n}, \quad S_1(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(-q^2; q^2)_n},\]
\[T_0(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{(n+1)(n+2)}}{(-q; q^2)_{n+1}}, \quad T_1(q) := \sum_{n=0}^{\infty} \frac{(-q^3; q^3)_n q^{n(n+1)}}{(-q; q^2)_{n+1}}.\]

Appearing in the transformation laws of these eighth order mock theta functions are the functions

\[U_0(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(-q^4; q^4)_n} = S_0(q^2) + q S_1(q^2),\]
\[U_1(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(-q^2; q^2)_{n+1}} = T_0(q^2) + q T_1(q^2),\]
\[V_0(q) := -1 + 2 \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q; q^2)_n},\]
\[V_1(q) := \sum_{n=0}^{\infty} \frac{(-q^3; q^3)_n q^{n(n+1)^2}}{(q; q^2)_{n+1}}.\]

One of the 16 transformation formulas found by McIntosh and Gordon is given by [78, p. 125]
\[q^{-\frac{1}{16}} S_0(q) = \sqrt{\frac{\pi}{4\alpha}} V_0(q_1) + \sqrt{\frac{2\pi}{\alpha}} q^{-\frac{1}{4}} V_1(q_1) + \sqrt{\frac{4\alpha}{\pi}} K_3(\alpha),\]
where \(K_3(\alpha)\) is a Mordell integral given by
\[K_3(\alpha) := \int_0^{\infty} e^{-\alpha x^2} \frac{\sinh(\frac{3}{2} \alpha x)}{\sinh(2\alpha x)} \, dx.\]

Two of the four mock theta functions of order 10 are given by (7.1.1). The remaining two tenth order mock theta functions are defined by
\[ X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(-q; q)_{2n}}, \quad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}}{(-q; q)_{2n+1}}. \]

Gordon and McIntosh [78, p. 129] provide eight transformation laws for tenth order mock theta functions, one of which is given by

\[ \frac{1}{q^{\frac{5}{10}}} \phi(q) = \sqrt{\frac{(5 + \sqrt{5})\pi}{10\alpha}} q^{\frac{1}{20}} X(q^{\frac{1}{10}}) - \sqrt{\frac{(5 - \sqrt{5})\pi}{10\alpha}} q^{\frac{9}{20}} \chi(q^{\frac{1}{10}}) - \sqrt{\frac{20\alpha}{\pi}} J_4(\alpha), \]

where \( J_4(\alpha) \) is a Mordell integral defined by

\[ J_4(\alpha) := \int_{0}^{\infty} e^{-5\alpha x^2} \frac{\cosh(\alpha x)}{\cosh(5\alpha x)} \, dx. \]
Transformation Formulas: 10th Order Mock Theta Functions

8.1 Introduction

On page 9 in his lost notebook [128], Ramanujan offers eight identities for tenth order mock theta functions. Let us recall the four tenth order mock theta functions:

\[ \phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}}, \quad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}}, \quad (8.1.1) \]

\[ X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(-q; q)_{2n}}, \quad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q; q)_{2n+1}}. \quad (8.1.2) \]

The seventh and eighth identities are transformation formulas involving \( \phi(q) \), \( \psi(q) \), and what we now call a Mordell integral. The goal of this chapter is to prove these two transformation formulas.

**Entry 8.1.1 (p. 9).** If \( \phi(q) \) and \( \psi(q) \) are defined by (8.1.1), then, for \( n > 0 \),

\[ \int_{0}^{\infty} \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1 + \sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi(-e^{-\pi/n}) \]

\[ = \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \phi(-e^{-\pi n}) - \sqrt{5} + 1 \frac{e^{-\pi/(5n)}}{2\sqrt{n}} \phi(-e^{-\pi/n}). \quad (8.1.3) \]

**Entry 8.1.2 (p. 9).** If \( \phi(q) \) and \( \psi(q) \) are defined by (8.1.1), then, for \( n > 0 \),

\[ \int_{0}^{\infty} \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1 - \sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi(-e^{-\pi/n}) \]

\[ = -\sqrt{\frac{5 - \sqrt{5}}{2}} e^{\pi n/5} \psi(-e^{-\pi n}) + \sqrt{5} - 1 \frac{e^{-\pi/(5n)}}{2\sqrt{n}} \phi(-e^{-\pi/n}). \quad (8.1.4) \]
In order to relate Choi’s proofs [53] of Entries 8.1.1 and 8.1.2, we review some notation and results, some of which were used in previous chapters. Recall that Ramanujan’s theta function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (8.1.5)$$

Recall the Jacobi triple product identity [28, p. 35]

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \quad (8.1.6)$$

We shall use the elementary properties [28, p. 34]

$$f(-1, a) = 0, \quad |a| < 1, \quad (8.1.7)$$

and, for any integer $n$,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \quad (8.1.8)$$

The identities

$$(a^2; q)_{\infty} = (a^2; q^2)_{\infty} (a^2 q; q^2)_{\infty} = (-a; q)_{\infty} (a; q)_{\infty} (a^2 q; q^2)_{\infty} \quad (8.1.9)$$

will be frequently used, usually without comment. Lastly, recall the pentagonal number theorem [28, p. 36]

$$f(-q, -q^2) = (q; q)_{\infty}. \quad (8.1.10)$$

We next recall some definitions and results from [51]. Let

$$L_1(-q, z) := 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n} z^{n+1}}{1 - q^{10n+3} z}, \quad (8.1.11)$$

$$L_2(-q, z) := 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n-1} z^{n+1}}{1 - q^{10n+1} z}, \quad (8.1.12)$$

$$A(z, x, q) := \frac{(q; q)_{\infty}^2 (x^2; q^2)_{\infty} f(-z x, -q/(z x))}{f(-q, -q) f(-x, -q/x) f(-z q, -q/z)}, \quad (8.1.13)$$

$$a_1(q) := -\frac{q(q^5; q^5)_{\infty} (q^{10}; q^{10})_{\infty} f(-q^2, -q^6)}{f(-q, -q^4) f(-q^4, -q^8)}, \quad (8.1.14)$$

and

$$a_2(q) := \frac{(q^5; q^5)_{\infty} (q^{10}; q^{10})_{\infty} f(-q^4, -q^6)}{f(-q^2, -q^3) f(-q^2, -q^8)}. \quad (8.1.15)$$

Then, with $\phi(q)$ and $\psi(q)$ defined by (8.1.1) [51],
\[ \psi(q)f(-z, -q^{10}/z) = L_1(-q, z) + a_1(q)f(-z, -q^{10}/z) + 2qA(z/q^2, q, q^5) \]

\begin{equation}
(8.1.16)
\end{equation}

and

\[ \phi(q)f(-z, -q^{10}/z) = L_2(-q, z) + a_2(q)f(-z, -q^{10}/z) + 2qA(z/q^4, q^2, q^5). \]

\begin{equation}
(8.1.17)
\end{equation}

The identities (8.1.16) and (8.1.17) show that \( \phi(q) \) and \( \psi(q) \) can be expressed in terms of generalized Lambert series and theta functions. D. Hickerson [86, 87] derived analogous results for fifth and seventh order mock theta functions, and the first author and Hickerson [24] established similar results for sixth order mock theta functions. Choi [52] also derived similar identities for the tenth order mock theta functions \( X(q) \) and \( \chi(q) \).

We shall establish Entries 8.1.1 and 8.1.2 with the aid of (8.1.16), (8.1.17), and a transformation formula for a certain Mordell integral that was derived by Mordell [112, p. 333]. More precisely, Mordell proved that

\[ \int_{\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi x t}}{e^{2\pi t} - e^{2\pi \theta}} dt = e^{-\pi i (\theta^2 \omega + 2\theta x + 2\theta)} \times \frac{F((x + \theta \omega)/\omega, -1/\omega) + i\omega F(x + \theta \omega, \omega)}{\omega \theta_{11}(x + \theta \omega, \omega)}, \quad q = e^{\pi i \omega}, \quad \text{Im}(\omega) > 0, \]

\begin{equation}
(8.1.18)
\end{equation}

where

\[ iF(x, \omega) := \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2 + m + 1/4} e^{(2m+1)\pi i x}}{1 + q^{2m+1}} \]

\begin{equation}
(8.1.19)
\end{equation}

and

\[ i\theta_{11}(x, \omega) := \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2 + m + 1/4} e^{(2m+1)\pi i x}}{1 + q^{2m+1}}. \]

\begin{equation}
(8.1.20)
\end{equation}

In Section 8.2, we establish nine identities for theta functions. Using (8.1.18), we demonstrate in Section 8.3 that the integral in (8.1.3) may be represented in terms of \( F(x, \omega) \) and \( \theta_{11}(x, \omega) \). Next, using (8.1.11), (8.1.12), (8.1.16), and (8.1.17) in the aforementioned integral identity, we show that the integral on the left-hand side of (8.1.3) can be represented in terms of the tenth order mock theta functions \( \phi(q), \psi(q) \), and theta functions. The identities in Section 8.2 enable us to simplify the representation and complete the proof of (8.1.3). In Section 8.4, we employ similar ideas to prove Entry 8.1.2.

8.2 Some Theta Function Identities

The first two theta function identities, found in Ramanujan’s second notebook [127], [28, p. 45, Entry 29], are extremely useful.
Lemma 8.2.1. If \( ab = cd \), then
\[
f(a, b) f(c, d) + f(-a, -b) f(-c, -d) = 2 f(ac, bd) f(ad, bc)
\] (8.2.1)

and
\[
f(a, b) f(c, d) - f(-a, -b) f(-c, -d) = 2 a f \left( \frac{b}{c}, \frac{c}{a} \frac{abcd}{b} \right) f \left( \frac{b}{c}, \frac{d}{a} \frac{abcd}{b} \right).
\] (8.2.2)

Lemma 8.2.2. If \( xy \neq 0 \), then
\[
f(-x, -q/x)f(-y, -q/y) = f(xy, q^2/(xy))f(yq/x, xq/y) - x f(xyq/qx, xyq)/y).
\]

Proof. In Lemma 8.2.1, set \( a = -x \), \( b = -q/x \), \( c = -y \), and \( d = -q/y \) in both (8.2.1) and (8.2.2). Add the resulting two identities and divide by 2 to achieve the desired result. \(\square\)

Lemma 8.2.3. If \( a, b, c, \) and \( d \) are any nonzero complex numbers, then
\[
af \left( -ab, -\frac{q}{ab} \right) f \left( \frac{b}{a}, -\frac{a}{b} q \right) f \left( -cd, -\frac{q}{cd} \right) f \left( \frac{c}{d}, -\frac{d}{c} q \right)
+ bf \left( -bc, -\frac{q}{bc} \right) f \left( \frac{c}{b}, -\frac{b}{c} q \right) f \left( -ad, -\frac{q}{ad} \right) f \left( \frac{a}{d}, -\frac{d}{a} q \right)
+ cf \left( -ac, -\frac{q}{ac} \right) f \left( \frac{a}{c}, -\frac{c}{a} q \right) f \left( -bd, -\frac{q}{bd} \right) f \left( \frac{b}{d}, -\frac{d}{b} q \right) = 0. \tag{8.2.3}
\]

Proof. Recall the definition of the classical theta function
\[
\vartheta_1(z) := \vartheta_1(z, q) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)+1/4} e^{(2n+1)iz}
= -i q^{1/4} e^{iz} f(-q^2 e^{2iz}, -e^{-2iz}), \quad \text{Im} \, z > 0, \tag{8.2.4}
\]
upon the use of Ramanujan’s notation (8.1.5). A fundamental identity discovered by K. Weierstrass, but expressed in terms of the Weierstrass \( \sigma \)-function, is given by [145, p. 451, Example 5], [98]
\[
\vartheta_1(u + u_1) \vartheta_1(u - u_1) \vartheta(u_2 + u_3) \vartheta(u_2 - u_3)
+ \vartheta_1(u + u_2) \vartheta_1(u - u_2) \vartheta(u_3 + u_1) \vartheta(u_3 - u_1)
+ \vartheta_1(u + u_3) \vartheta_1(u - u_3) \vartheta(u_1 + u_2) \vartheta(u_1 - u_2) = 0. \tag{8.2.5}
\]
With the use of (8.2.4), we translate (8.2.5) into an identity involving Ramanujan’s theta functions with the substitutions \( a = e^{2iu} \), \( b = e^{-2iu_1} \), \( c = e^{-2iu_2} \), and \( d = e^{-2iu_3} \). We also replace \( q^2 \) by \( q \) and divide both sides of the equation by \( \sqrt{q} \). Hence,
We apply Lemma 8.2.2 with Lemma 8.2.4. We have the resulting equality by \(abc\) total of 7 times to obtain all 12 theta functions appearing in (8.2.3). Multiply in order to establish (8.2.3), we see that we need to apply (8.1.8) with next, in Lemma 8.2.3, replace \(a\) by \(q\) and \(y = q^2\). Accordingly,

\[
f(q, q^4)f(-q^2, -q^3) = f(-q^3, -q^7)f(-q^4, -q^6) + qf(-q, -q^9)f(-q^2, -q^8).
\]

(8.2.8)

Next, in Lemma 8.2.3, replace \(q\) by \(q^{10}\) and set \(a = q^3, b = q^2, c = q^5\), and \(d = q\) to deduce that

\[
q^3f(-q^5, -q^9)f(-q^{-1}, -q^{11})f^2(-q^6, -q^4) \\
+ q^2f^2(-q^2, -q^3)f(-q^4, -q^6)f(-q^2, -q^8) \\
+ q^5f(-q^8, -q^2)f(-q^{-2}, -q^{12})f(-q^3, -q^7)f(-q, -q^9) = 0.
\]

(8.2.9)

Now apply (8.1.8) twice with \(n = 1\) in (8.2.9) and divide both sides by \(q^2\) to arrive at

\[
f(-q^2, -q^8)f^2(-q^3, -q^7)f(-q^4, -q^6) \\
= f(-q, -q^9)f^2(-q^4, -q^6)f(-q^5, -q^9) \\
+ qf(-q, -q^9)f^2(-q^2, -q^8)f(-q^3, -q^7).
\]

(8.2.10)

Multiplying both sides of (8.2.8) by \(f(-q^2, -q^8)f(-q^3, -q^7)\) and using (8.2.10), we find that

\[
f(-q^2, -q^8)f(-q^3, -q^7)f(q, q^4)f(-q^2, -q^3) \\
= f(-q, -q^9)f^2(-q^4, -q^6)f(-q^5, -q^9) \\
+ 2qf(-q, -q^9)f^2(-q^2, -q^8)f(-q^3, -q^7).
\]

(8.2.11)
Dividing both sides of (8.2.11) by
\[ f(-q, -q^3)f^2(-q^2, -q^8)f(-q^3, -q^7)f(-q^4, -q^6)f(-q^5, -q^5)/(q^{10}; q^{10})_\infty^3 \]
using (8.1.9), and replacing \( q \) by \(-q\), we complete the proof of Lemma 8.2.4.

\[ \square \]

**Lemma 8.2.5.** We have
\[
\frac{(-q^5; q^5)_\infty (q^{10}; q^{10})_\infty f(-q^2, -q^3)}{f(q, -q^4)f(-q^4, -q^6)} + 2 \frac{(-q^5; q^5)_\infty (q^{10}; q^{10})_\infty}{f^2(q^3, q^3)f(-q^2, -q^8)}
\]
\[ = \frac{f(-q^3, -q^7)f(q, q^3)f(q^2, q^8)}{f(-q^2, -q^8)f(q^3, q^4)}. \tag{8.2.12} \]

**Proof.** Replacing \( q, x, \) and \( y \) by \( q^5, q, \) and \(-q^3\), respectively, we find that
\[
f(-q, -q^4)f(q^2, q^3) = f(-q^3, -q^7)f(-q^4, -q^6) - qf(-q, -q^9)f(-q^2, -q^8). \tag{8.2.13} \]

Replacing \( q \) by \( q^{10} \), setting \( a = q^4, b = q, c = q^5, \) and \( d = q^3 \) in Lemma 8.2.3, applying (8.1.8) three times with \( n = 1 \) in each instance, and dividing both sides by \( q \), we find that
\[
qf^2(-q, -q^3)f(-q^2, -q^8)f(-q^4, -q^6)
\]
\[ = f^2(-q^2, -q^8)f(-q^3, -q^7)f(-q^5, -q^5)
\]
\[ - f(-q, -q^9)f(-q^3, -q^7)f^2(-q^4, -q^6). \tag{8.2.14} \]

Multiplying both sides of (8.2.13) by \( f(-q, -q^9)f(-q^4, -q^6) \) and using (8.2.14), we deduce that
\[
f(-q, -q^9)f(-q^4, -q^6)f(-q, -q^4)f(q^2, q^3)
\]
\[ = 2f(-q, -q^9)f(-q^3, -q^7)f^2(-q^4, -q^6)
\]
\[ - f^2(-q^2, -q^8)f(-q^3, -q^7)f(-q^5, -q^5). \tag{8.2.15} \]

Dividing both sides of (8.2.15) by
\[
f(-q, -q^9)f(-q^2, -q^8)f(-q^3, -q^7)f^2(-q^4, -q^6)f(-q^5, -q^5)/(q^{10}; q^{10})_\infty^3,
\]
using (8.1.9), and replacing \( q \) by \(-q\), we finish the proof of Lemma 8.2.5. \( \square \)

**Lemma 8.2.6.** We have
\[
f(-q^{15}, -q^{35})f(q^{25}, q^{25}) + q^3f(-q^5, -q^{45})f(q^{25}, q^{25})
\]
\[ + f(-q^{25}, -q^{35})f(q^{15}, q^{35}) - q^4f(-q^5, -q^{45})f(q^{15}, q^{35})
\]
\[ - q^3f(-q^{25}, -q^{25})f(q^5, q^{45}) - q^4f(-q^{15}, -q^{35})f(q^5, q^{45})
\]
\[ = 2f(-q^4, -q^{16})f(-q^8, -q^{12}). \tag{8.2.16} \]
Proof. Let \(a = -q^{15}, b = -q^{35}, \) and \(c = d = q^{25}\) in (8.2.1), so that \(ab = cd\). Therefore,
\[
f(-q^{15}, -q^{35})f(q^{25}, q^{25}) + f(-q^{25}, -q^{25})f(q^{15}, q^{35}) = 2f^2(-q^{40}, -q^{60}).
\]
Next, set \(a = -q^5, b = -q^{45}, \) and \(c = d = q^{25}\) in (8.2.2), so that once again \(ab = cd\). Hence,
\[
f(-q^5, -q^{45})f(q^{25}, q^{25}) - f(-q^{25}, -q^{25})f(q^5, q^{45}) = -2q^5f^2(-q^{20}, -q^{80}).
\]
Setting \(a = -q^5, b = -q^{45}, \) and \(c = q^{15}, \) and \(d = q^{35}\) in (8.2.1), so that of course \(ab = cd\), we find that
\[
f(-q^5, -q^{45})f(q^{15}, q^{35}) + f(-q^{15}, -q^{35})f(q^5, q^{45}) = 2f(-q^{40}, -q^{60}).
\]
If we let \(L(q)\) denote the left-hand side of (8.2.16) and use (8.2.17)—(8.2.19), we find that
\[
L(q) = 2f^2(-q^{40}, -q^{60}) - 2q^8f^2(-q^{20}, -q^{80}) - 2q^4f(-q^{20}, -q^{80})f(-q^{40}, -q^{60}).
\]
From [29, p. 191], we recall that
\[
f(xA, x/A)f(xB, x/B) = f(x^5A^2B, x^5A^{-2}B^{-1})f(x^5A^{-1}B^2, x^5AB^{-2})
\]
\[
+ xBf(x^7A^2B, x^3A^{-2}B^{-1})f(x^9A^{-1}B^2, xAB^{-2})
\]
\[
+ xB^{-1}f(x^3A^2B, x^7A^{-2}B^{-1})f(xA^{-1}B^2, x^9AB^{-2})
\]
\[
+ xAf(x^9A^2B, xA^{-2}B^{-1})f(x^3A^{-1}B^2, x^7AB^{-2})
\]
\[
+ xA^{-1}f(xA^2B, x^9A^{-2}B^{-1})f(x^7A^{-1}B^2, x^3AB^{-2}).
\]
Setting \(x = -q^{10}, A = q^6, \) and \(B = q^{-2}\) in (8.2.21), and then multiplying both sides by 2, we find that
\[
2f(-q^5, -q^{45})f(-q^8, -q^{12}) = 2f^2(-q^{40}, -q^{60}) - 2q^8f^2(-q^{20}, -q^{80}) - 2q^4f(-q^{20}, -q^{80})f(-q^{40}, -q^{60}).
\]
Substituting (8.2.22) into (8.2.20), we see that we have completed the proof of (8.2.16).

The next six theta function identities that we need are proved using the theory of modular forms. We follow the exposition in Choi’s paper [53]. The method is also described in the second author’s book [28, pp. 326–345]. We assume that the reader is familiar with basic facts of modular forms; if not, we recommend the book [95] by M. Knopp for a lucid and quick introduction.
Let $\mathcal{H} = \{ z : \text{Im}(z) > 0 \}$. Recall the definition of the Dedekind eta function $\eta(z)$, which is a modular form of weight $\frac{1}{2}$ on the full modular group. If $n$ is a positive integer, from its definition, we note that

$$\eta_n := \eta(nz) = q^{n/24}(q^n; q^n) = q^{n/24}f(-q^n, -q^{2n}), \quad q = e^{2\pi iz}, \quad z \in \mathcal{H}.$$  \hspace{1cm} (8.2.23)

We need to also define the generalized Dedekind eta function.

**Definition 8.2.1.** For $z \in \mathcal{H}$, the generalized Dedekind eta function is defined by

$$\eta_{n,m}(z) := e^{\pi i P_2(m/n)nz} \prod_{k=1}^{\infty} (1 - e^{2\pi ikz}) \prod_{k \equiv m \pmod{n}}^{\infty} (1 - e^{2\pi ikz})$$

$$= e^{\frac{1}{2} P_2(m/n)n} f(-q^m, -q^{n-m}) \left( q^n; q^n \right)_\infty,$$ \hspace{1cm} (8.2.24)

where $P_2(t) = \{ t \}^2 - \{ t \} + \frac{1}{6}$ is the second Bernoulli function, and $\{ t \} = t - [t]$ is the fractional part of $t$.

The full modular group of linear fractional transformations $(az + b)/(cz + d)$, $a, b, c, d \in \mathbb{Z}$, is denoted by $\Gamma(1)$. Let $\Gamma_1(N)$ denote the subgroup of $\Gamma(1)$ where $a \equiv d \equiv 1 \pmod{N}$ and $c \equiv 0 \pmod{N}$. Let $\{ \Gamma, r, v \}$ denote the space of modular forms of weight $r$ and multiplier system $v$ on a subgroup $\Gamma$ of finite index in $\Gamma(1)$. Let $\text{ord}(f; z)$ denote the invariant order of a modular form $f$ at $z$, and let $\text{Order}(f; z)$ denote the order of $f$ with respect to $\Gamma$, defined by

$$\text{Ord}_{\Gamma}(f; z) := \frac{1}{\ell} \text{ord}(f; z),$$

where $\ell$ is the order of $f$ at $z$ as a fixed point of $\Gamma$.

**Lemma 8.2.7.** [95, p. 51] Let $\zeta_{24}$ be a primitive $24$th root of unity, and $M = \left(\begin{array}{cc}a & b \\ c & d \end{array}\right) \in \Gamma(1)$. The multiplier system $v_\eta$ for the modular form $\eta(z)$ is given by

$$v_\eta(M) = \begin{cases} \left(\frac{d}{|c|}\right) \zeta_{24}^{bd(1-c^3)+c(a+d)-3c}, & \text{if } c \text{ is odd,} \\ \left(\frac{c}{|d|}\right) \zeta_{24}^{ac(1-d^3)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd and either} \\ & \quad c \geq 0 \text{ or } d \geq 0, \\ \left(\frac{c}{|d|}\right) \zeta_{24}^{ac(1-d^3)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd, } c < 0, \text{ and } d \geq 0. \end{cases}$$

**Lemma 8.2.8.** (The valence formula) [131, Theorem 4.1.4] If $f \in \{ \Gamma, r, v \}$ and $f$ is not identically $0$, then

$$\sum_{z \in \mathcal{R}} \text{Ord}_{\Gamma}(f; z) = \mu r,$$
where $R$ is any fundamental region for $\Gamma$ and

$$\mu := \frac{1}{12}[\Gamma(1):\Gamma].$$

**Lemma 8.2.9.** [53, Lemma 2.2.4] If $n \in \mathbb{Z}^+$, $m_1, m_2, \ldots, m_{2n}$ are positive integers, $N$ is a positive even integer, and $[m_1, m_2, \ldots, m_{2n}]|N$, then, for $z \in \mathcal{H}$,

$$\eta(m_1z)\eta(m_2z)\cdots\eta(m_{2n}z) \in \{\Gamma_1(N), n, v\},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$. Furthermore, if $\zeta_{24}$ is a primitive 24th root of unity, then

$$v(A) = 2n \prod_{j=1}^{2n} \left( \frac{c/m_j}{|d|} \right) \zeta_{24}^{ac(1-d^2)/m_j+4d(m_jb-c/m_j)+3(d-1)}.$$ 

**Lemma 8.2.10.** [58, Lemma 2.6.5] For each positive integer $n$,

$$[\Gamma(1):\Gamma_1(n)] = n^2 \prod_{p|n} \left( 1 - \frac{1}{p^2} \right),$$

where the product is over all primes $p$ dividing $n$.

**Lemma 8.2.11.** [133, p. 126, Theorem 3] For $z \in \mathcal{H}$, let

$$f(z) = \prod_{\substack{n|N \ 0\leq m<n \ 0\leq r_{n,m}}} \eta^{r_{n,m}}(z),$$

where $r_{n,m}$ are integers. If

$$\sum_{\substack{n|N \ 0\leq m<n \ \text{mod} \ 2}} nP_2 \left( \frac{m}{n} \right) r_{n,m} \equiv 0 \pmod{2} \quad \text{and} \quad \sum_{\substack{n|N \ 0\leq m<n \ \text{mod} \ 2}} \frac{N}{n} P_2(0) r_{n,m} \equiv 0 \pmod{2},$$

then $f(z) \in \{\Gamma_1(N), 0, I\}$, where for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ and $I(M) = 1$.

**Lemma 8.2.12.** For $z \in \mathcal{H}$,

$$\eta_{50}^{18} \eta_{10,2}^{4} \eta_{10,3}^{2} \eta_{10,20}^{2} \eta_{20,50}^{2} \eta_{100,20}^{4} \eta_{100,50}^{4} \times \left( -\eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,25} \eta_{100,40} + \eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,40} - \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,20} + 2\eta_{50,5} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,40} - 2\eta_{50,5} \eta_{50,20} \eta_{50,25} \eta_{100,30} \eta_{100,40} + \eta_{50,5} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,40} \eta_{100,50} \right)$$

$$= \eta_{10}^{2} \eta_{100}^{16} \eta_{10,1}^{2} \eta_{10,2}^{2} \eta_{10,3}^{2} \eta_{10,4}^{2} \eta_{10,5}^{2} \eta_{10,6}^{2} \eta_{10,7}^{2} \eta_{10,8}^{2} \eta_{50}^{2} \eta_{50,10}^{2} \eta_{50,15}^{2} \eta_{50,20}^{2} \eta_{50,25}^{2} \eta_{100,50}^{2}. \quad (8.2.25)
Proof. For $1 \leq j \leq 7$, let $f_j^1$ denote the product of eta functions in each of the 7 products in (8.2.25), and let $g_j^1$ denote the product of the generalized eta functions in each of the seven products in (8.2.25). Each $f_j^1$ is the product of 18 eta functions, and by Lemma 8.2.9 and a straightforward calculation, each $f_j^1$ is a modular form of weight 9 on $\Gamma_1(300)$ with the multiplier system $v_1$, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(300),$

$$v_1(A) = \frac{1}{\zeta_{24}} e^{(a-c d^2-d)+12bd+3(d-1)}.$$

By Lemma 8.2.11 and a straightforward calculation, each $g_j^1$ is a modular form of weight 0 on $\Gamma_1(300)$ with the multiplier system $v_1$. By Lemma 8.2.10, $[\Gamma(1) : \Gamma_1(300)] = 57600$. Let $F_1$ denote the difference of the left- and right-hand sides of (8.2.25). Applying Lemma 8.2.8 for a fundamental region $R$ for $\Gamma_1(300)$, we deduce that

$$\sum_{z \in R} \text{Ord}_{\Gamma_1(300)}(F_1; z) = \frac{9 \cdot 57600}{12} = 43200 \geq \text{ord}(F_1; \infty), \quad (8.2.26)$$

since both sides of (8.2.25) are analytic on $R$. Using Mathematica, we calculate the Taylor series of $F_1$ about $q = 0$ (or about the cusp $z = \infty$) and find that $F_1 = O(q^{4221})$. Unless $F_1$ is a constant, we have a contradiction to (8.2.26). Our proof of Lemma 8.2.12 is therefore completed. 

$$\square$$

Lemma 8.2.13. For $z \in \mathcal{H},$

$$\eta_{50}^{18} \eta_{10,1}^{16} \eta_{10,2}^{16} \eta_{20,6}^{20} \eta_{20,8}^{20} \eta_{100,50}^{20} \eta_{50,15}^{20} \eta_{50,20}^{20} \eta_{50,25}^{20} \eta_{100,20}^{20} + \eta_{50,5} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,20}^{20} - \eta_{50,5}^{2} \eta_{50,10}^{2} \eta_{50,15} \eta_{50,25} \eta_{100,20}^{20} - 2 \eta_{50,5} \eta_{50,10}^{2} \eta_{50,15} \eta_{50,25} \eta_{100,20}^{20} + \eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,25} \eta_{100,20}^{9}$$

$$= \eta_{10}^{16} \eta_{100}^{16} \eta_{10,2}^{16} \eta_{10,3}^{20} \eta_{50,5}^{20} \eta_{50,10}^{20} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{100,50}^{9}. \quad (8.2.27)$$

Proof. For $1 \leq j \leq 7$, let $f_j^2$ denote the product of eta functions in each of the 7 products in (8.2.27), and let $g_j^2$ denote the product of the generalized eta functions in each of the seven products in (8.2.27). Each $f_j^2$ is the product of 18 eta functions, and by Lemma 8.2.9 and a straightforward calculation, each $f_j^2$ is a modular form of weight 9 on $\Gamma_1(300)$ with the multiplier system $v_2$, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(300),$

$$v_2(A) = \frac{1}{\zeta_{24}} e^{(a-c d^2-d)+12bd+3(d-1)}.$$
modular form of weight 9 on $\Gamma_1(300)$ with multiplier system $v_2$. Let $F_2$ denote the difference of the left- and right-hand sides of (8.2.27). Applying Lemma 8.2.8 for a fundamental region $R$ for $\Gamma_1(300)$, we deduce that

$$\sum_{z \in R} \text{Ord}_{\Gamma_1(300)}(F_2; z) = \frac{9 \cdot 57600}{12} = 43200 \geq \text{ord}(F_2; \infty), \quad (8.2.28)$$

since both sides of (8.2.27) are analytic on $R$. Using Mathematica, we calculate the Taylor series of $F_2$ about $q = 0$ (or about the cusp $z = i\infty$) and find that $F_2 = O(q^{43201})$. Unless $F_2$ is a constant, we have a contradiction to (8.2.28). Our proof of Lemma 8.2.13 is thus finished.

Lemma 8.2.14. For $z \in \mathcal{H}$,

$$\eta_5^2 \eta_{100}^2 \eta_{15}^2 \eta_{20}^2 \eta_{25}^2 \cdot \eta_{50}^3 \eta_{100}^3 \eta_{100,40} \eta_{100,50} + 3 \eta_{50,10} \eta_{50,20} \eta_{100,20} \eta_{100,50} - 3 \eta_{50,10} \eta_{50,20} \eta_{100,25}^2$$

$$= \eta_{20}^2 \eta_{25}^2 \eta_{1}^2 \eta_{6}^2 \eta_{50}^2 \eta_{50,20} \eta_{100}^2 \eta_{100,20} \eta_{100,25}.$$

(8.2.29)

Proof. For $1 \leq j \leq 4$, let $f^3_j$ denote the product of eta functions in each of the 4 products in (8.2.29), and let $g^3_j$ denote the product of the generalized eta functions in each of the 4 products in (8.2.29). Each $f^3_j$ is the product of 14 eta functions, and by Lemma 8.2.9 and a straightforward calculation, each $f^3_j$ is a modular form of weight 2 on $\Gamma_1(300)$ with the multiplier system $v_3$, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(300)$,

$$v_3(A) = \frac{1}{24} \zeta_{24}^{\frac{4}{3}(a-d^2-d)+18bd+3(d-1)}.$$ 

By Lemma 8.2.11 and a straightforward calculation, each $g^3_j$ is a modular form of weight 0 on $\Gamma_1(300)$ with the multiplier system $I$. Hence, each $f^3_j g^3_j$ is a modular form of weight 2 on $\Gamma_1(300)$ with multiplier system $v_3$. Let $F_3$ denote the difference of the left- and right-hand sides of (8.2.29). Applying Lemma 8.2.8 for a fundamental region $R$ for $\Gamma_1(300)$, we deduce that

$$\sum_{z \in R} \text{Ord}_{\Gamma_1(300)}(F_3; z) = \frac{2 \cdot 57600}{12} = 9600 \geq \text{ord}(F_3; \infty), \quad (8.2.30)$$

since both sides of (8.2.29) are analytic on $R$. Using Mathematica, we calculate the Taylor series of $F_3$ about $q = 0$ (or about the cusp $z = i\infty$) and find that $F_3 = O(q^{9601})$. Unless $F_3$ is a constant, we have a contradiction to (8.2.30). Our proof of Lemma 8.2.14 has therefore been completed.

Lemma 8.2.15. For $z \in \mathcal{H}$,
\[
\eta_{10}^2 \eta_{50}^2 \eta_{10,1}^2 \eta_{10,2}^2 \eta_{10,3}^3 \\
\times (\eta_{50,5}^5 \eta_{50,15} \eta_{50,25} \eta_{100,50} + \eta_{50,5} \eta_{50,15} \eta_{50,25} \eta_{100,30} \\
- \eta_{50,5} \eta_{50,15} \eta_{50,25} \eta_{100,10} - \eta_{50,5} \eta_{50,15} \eta_{50,25} \eta_{100,30} \\
+ 2 \eta_{50,5} \eta_{50,25} \eta_{100,30} - \eta_{50,15} \eta_{50,25} \eta_{100,10})) \\
= \eta_{10}^2 \eta_{20}^2 \eta_{10,1}^3 \eta_{10,2}^3 \eta_{20,4}^3 \eta_{20,8}^3 \eta_{50,5}^3 \eta_{50,15} \eta_{50,25}. \\
\tag{8.2.31}
\]

**Proof.** For \(1 \leq j \leq 8\), let \(f_j^4\) denote the product of \(\eta\) functions in each of the 8 products in (8.2.31), and let \(g_j^4\) denote the product of the generalized \(\eta\) functions in each of the 8 products in (8.2.31). Each \(f_j^4\) is the product of 4 \(\eta\) functions, and by Lemma 8.2.9 and a straightforward calculation, each \(f_j^4\) is a modular form of weight 2 on \(\Gamma_1(300)\) with the multiplier system \(v_4\), where for \(A = (a\ b\ c\ d) \in \Gamma_1(300)\),

\[
v_4(A) = \frac{2^d}{\phi(\Delta) (a-d^2-d) + 12bd + 3(d-1)}.
\]

By Lemma 8.2.11 and a straightforward calculation, each \(g_j^4\) is a modular form of weight 0 on \(\Gamma_1(300)\) with the multiplier system \(I\). Hence, each \(f_j^4 g_j^4\) is a modular form of weight 2 on \(\Gamma_1(300)\) with multiplier system \(v_4\). Let \(F_4\) denote the difference of the left- and right-hand sides of (8.2.31). Applying Lemma 8.2.8 for a fundamental region \(R\) for \(\Gamma_1(300)\), we deduce that

\[
\sum_{z \in R} \text{Ord}_{\Gamma_1(300)}(F_4; z) = \frac{2 \cdot 57600}{12} = 9600 \geq \text{ord}(F_4; \infty), \tag{8.2.32}
\]

since both sides of (8.2.31) are analytic on \(R\). Using Mathematica, we calculate the Taylor series of \(F_4\) about \(q = 0\) (or about the cusp \(z = i\infty\)) and find that \(F_4 = O(q^{3001})\). Unless \(F_4\) is a constant, we have a contradiction to (8.2.32). Our proof of Lemma 8.2.15 has thus been accomplished. \(\square\)

**Lemma 8.2.16.** For \(z \in \mathcal{H}\),

\[
\eta_{10}^3 \eta_{20}^3 \eta_{50,5} \eta_{50,10}^3 \eta_{10,4}^3 \eta_{20,2}^3 \eta_{20,8}^3 \eta_{50,5} \eta_{50,15} \eta_{50,25} \eta_{200,50} \eta_{200,100} \\
\times (\eta_{50,5} \eta_{20,50} \eta_{10,1} \eta_{10,4} \eta_{20,2} \eta_{20,8} \eta_{50,10} \eta_{10,10} \eta_{10,40} - 4 \eta_{10,1} \eta_{20,2} \eta_{20,40} + \eta_{10,1} \eta_{20,4} \eta_{20,2} \eta_{20,10}) \\
+ 4 \eta_{10,1} \eta_{10,1} \eta_{10,4} \eta_{20,4} \eta_{20,8} \eta_{10,10} \eta_{10,40} \eta_{10,10} \eta_{10,40} + 6 \eta_{10,1} \eta_{10,4} \eta_{20,4} \eta_{20,2} \eta_{20,10} \\
- 4 \eta_{10,1} \eta_{20,4} \eta_{20,8} \eta_{10,10} \eta_{10,40} \eta_{10,10} \eta_{10,40} + \eta_{10,1} \eta_{20,10} \eta_{10,1} \eta_{10,4} \eta_{20,2} \eta_{20,3} \eta_{20,10}^2) \\
\times (\eta_{50,5} \eta_{20,50} \eta_{10,1} \eta_{10,4} \eta_{20,2} \eta_{20,8} \eta_{50,50} \eta_{50,15} \eta_{50,25} \eta_{200,50} \eta_{200,100} \\
= \eta_{10}^8 \eta_{20}^3 \eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,20} \eta_{200,50} \eta_{200,100} \\
\times (\eta_{50,5} \eta_{20,50} \eta_{10,1} \eta_{10,4} \eta_{20,2} \eta_{20,8} - 4 \eta_{10,1} \eta_{20,10} \eta_{10,3} \eta_{10,4} \eta_{20,2} \eta_{20,5} \eta_{20,8} \\
- \eta_{10,1} \eta_{20,4} \eta_{20,10} \eta_{20,8} \eta_{20,10} \eta_{20,4} \eta_{20,10} \\
+ \eta_{10,1} \eta_{20,50} \eta_{10,50} \eta_{10,50} \eta_{10,50} \eta_{200,50} \eta_{200,100} \eta_{200,100} \\
\times (2 \eta_{10,1} \eta_{20,8} \eta_{20,10} + 2 \eta_{10,1} \eta_{10,3} \eta_{10,4} \eta_{20,2} \eta_{20,5} \eta_{20,8}
\]

\[
\tag{8.2.33}
\]
\[
+ 3\eta_{10,3}^3\eta_{20,2}\eta_{20,4}\eta_{20,10}
- 8\eta_{10,2}\eta_{50,10}\eta_{50,15}\eta_{50,20}\eta_{20,100,10}\eta_{200,50}\eta_{200,100}
\times (3\eta_{10,1}\eta_{20,6}\eta_{20,8}\eta_{20,10}^2 - 2\eta_{10,1}\eta_{10,3}\eta_{70,4}\eta_{20,2}\eta_{20,5}\eta_{20,8}
+ 2\eta_{10,3}\eta_{20,2}\eta_{20,4}\eta_{20,10}).
\] (8.2.33)

**Proof.** For \(1 \leq j \leq 15\), let \(f_j^5\) denote the product of eta functions in each of the 15 products in (8.2.33), and let \(g_j^5\) denote the product of the generalized eta functions in each of the 15 products in (8.2.33). Each \(f_j^5\) is the product of 12 eta functions, and by Lemma 8.2.9 and a straightforward calculation, each \(f_j^5\) is a modular form of weight 6 on \(\Gamma_1(200)\) with the multiplier system \(v_5\), where for \(A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_1(200)\),

\[
v_5(A) = \left(\frac{10}{|d|}\right) \frac{1}{\zeta_{24}^{50}} c(a - ad^2 - d) + 22b + 3(d - 1).
\]

By Lemma 8.2.11 and a straightforward calculation, each \(g_j^5\) is a modular form of weight 0 on \(\Gamma_1(200)\) with the multiplier system \(I\). Hence, each \(f_j^5\) is a modular form of weight 6 on \(\Gamma_1(200)\) with multiplier system \(v_5\). By Lemma 8.2.10, \([\Gamma(1): \Gamma_1(200)] = 28800\). Let \(F_5\) denote the difference of the left and right sides of (8.2.33). Applying Lemma 8.2.8 for a fundamental region \(R\) for \(\Gamma_1(200)\), we deduce that

\[
\sum_{z \in R} \text{Ord}_{\Gamma_1(200)}(F_5; z) = \frac{6 \cdot 28800}{12} = 14400 \geq \text{ord}(F_5; \infty),
\] (8.2.34)

since both sides of (8.2.33) are analytic on \(R\). Using Mathematica, we calculate the Taylor series of \(F_5\) about \(q = 0\) (or about the cusp \(z = i\infty\)) and find that \(F_5 = O(q^{14401})\). Unless \(F_5\) is a constant, we have a contradiction to (8.2.34).

Our proof of Lemma 8.2.16 is therefore finished. \(\square\)

**Lemma 8.2.17.** For \(z \in \mathcal{H}\),

\[
\eta_{10,1}^8\eta_{20,5}^3\eta_{10,1}^3\eta_{20,4}\eta_{20,6}\eta_{20,8}\eta_{50,5,15}\eta_{50,25}\eta_{200,50}\eta_{200,100}
\times (\eta_{10,1}\eta_{20,10}\eta_{50,10}\eta_{100,40} - \eta_{10,1}\eta_{20,10}\eta_{50,20}\eta_{100,20}
- 2\eta_{10,3}\eta_{20,2}\eta_{20,50,20}\eta_{100,20} + 4\eta_{10,1}\eta_{20,5}\eta_{50,100}\eta_{10,1}\eta_{10,3}\eta_{20,2}\eta_{20,4}\eta_{20,6}
\times \eta_{20,8}\eta_{20,10}\eta_{40,10}\eta_{50,5,15}\eta_{50,20}\eta_{100,25}\eta_{200,100}
\times \eta_{10,1}^3\eta_{20,5}^3\eta_{50,10}\eta_{50,20}\eta_{200,50}\eta_{200,100}
\times (\eta_{10,1}\eta_{20,6}\eta_{20,8}\eta_{20,10}\eta_{50,5,15}\eta_{100,50}
+ \eta_{10,3}\eta_{20,2}\eta_{20,4}\eta_{20,5}\eta_{20,10}\eta_{50,5,15}\eta_{100,50}
- 2\eta_{10,1}\eta_{10,3}\eta_{20,2}\eta_{20,4}\eta_{20,5}\eta_{20,6}\eta_{20,8}\eta_{50,5,15}\eta_{100,30}
- \eta_{10,3}\eta_{20,2}\eta_{20,4}\eta_{20,10}\eta_{50,5,25}\eta_{100,30}
- \eta_{10,1}\eta_{20,6}\eta_{20,8}\eta_{20,10}\eta_{50,15}\eta_{50,25}\eta_{100,10}
\]
\begin{equation}
+ 2\eta_{10,1}\eta_{10,2}\eta_{20,2}\eta_{20,4}\eta_{20,5}\eta_{20,6}\eta_{20,8}\eta_{50,15}\eta_{50,25}\eta_{100,10}.
\end{equation}

(8.2.35)

**Proof.** For \(1 \leq j \leq 10\), let \(f_j\) denote the product of \(\eta\) functions in each of the 10 products in (8.2.35), and let \(g_j\) denote the product of the \(\eta\) functions in each of the 10 products in (8.2.35). Each \(f_j\) is the product of 14 \(\eta\) functions, and by Lemma 8.2.9 and a straightforward calculation, each \(f_j\) is a modular form of weight 7 on \(\Gamma_1(200)\) with the multiplier system \(v_6\), where for \(A = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(200)\),

\[
v_6(A) = \left( \frac{10}{|d|} \right) \zeta_{20}^{c(a-ad^2-d)+2bd+3(d-1)}.
\]

By Lemma 8.2.11 and a straightforward calculation, each \(g_j\) is a modular form of weight 0 on \(\Gamma_1(200)\) with the multiplier system \(I\). Hence, each \(f_j^6g_j^6\) is a modular form of weight 7 on \(\Gamma_1(200)\) with multiplier system \(v_6\). Let \(F_6\) denote the difference of the left and right sides of (8.2.35). Applying Lemma 8.2.8 for a fundamental region \(R\) for \(\Gamma_1(200)\), we deduce that

\[
\sum_{z \in R} \text{Ord}_{\Gamma_1(200)}(F_6; z) = \frac{7 \cdot 28800}{12} = 16800 \geq \text{ord}(F_6; \infty),
\]

since both sides of (8.2.35) are analytic on \(R\). Using *Mathematica*, we calculate the Taylor series of \(F_6\) about \(q = 0\) (or about the cusp \(z = \infty\)) and find that \(F_6 = O(q^{16801})\). Unless \(F_6\) is a constant, we have a contradiction to (8.2.36). Our proof of Lemma 8.2.17 is thus finished.

Our next task is to take each of the six \(\eta\) function identities from Lemmas 8.2.12–8.2.17 and rewrite them in terms of Ramanujan’s \(\eta\) functions \(f(a, b)\).

**Lemma 8.2.18.** If \(f(a, b)\) is defined by (8.1.5), then

\[
\begin{align*}
\frac{f(-q^5, -q^{15})f(-q^{40}, -q^{60})f(-q^{50}, -q^{50})}{f^2(-q^{20}, -q^{40})} & \times (q^6f(-q^{10}, -q^{40}) - q^4f(-q^{20}, -q^{30})) \\
& - \frac{f(-q^{50}, -q^{50})}{f(-q^{10}, -q^{40})} \\
& \times (q^3f(-q^{15}, -q^{35})f(-q^{20}, -q^{80}) - 2q^2f(-q^5, -q^{15})f(-q^{40}, -q^{60})) \\
& - 2q^2f^2(-q^5, -q^{45})f(-q^{40}, -q^{60})(q^{50}; q^{50})_\infty(q^{100}; q^{100})_\infty \\
& \frac{f(-q^{15}, -q^{35})f(-q^{10}, -q^{40})f(-q^{20}, -q^{80})}{f(-q^{25}, -q^{25})f(-q^{15}, -q^{35})f(-q^{20}, -q^{80})} \\
& + \frac{f^2(-q^5, -q^{45})f(-q^{40}, -q^{60})(q^{50}; q^{50})_\infty}{f(-q^{25}, -q^{25})f(-q^{15}, -q^{35})f(-q^{20}, -q^{80})f(-q^{10}, -q^{70})} \\
& = \frac{f^2(-q, -q^9)f(-q^8, -q^{12})(q^{10}; q^{10})_\infty(q^{20}; q^{20})_\infty}{f(-q^3, -q^3)f(-q^2, -q^8)f(-q^4, -q^{16})}.
\end{align*}
\]

(8.2.37)
\textit{Proof.} Simplify slightly the right-hand side of (8.2.25) by using the identity \[ q^{15/30} \eta_{10}^2 \eta_{10,1}^2 \eta_{10,2}^2 \eta_{10,3} \eta_{20,2} \eta_{10,10} \eta_{50,15} \eta_{50,20} \eta_{50,25}^2 / \eta_{100}^2. \]

Using (8.2.23) and (8.2.24) to convert the resulting identity, we deduce (8.2.37).

\textbf{Lemma 8.2.19.} We have
\[
\begin{align*}
&\frac{f(-q^5, -q^{15}) f(-q^{20}, -q^{30}) f(-q^{40}, -q^{50})}{f^2(-q^{10}, -q^{40})} \\
&\times \left( q^2 f(-q^{10}, -q^{40}) + f(-q^{20}, -q^{30}) \right) \\
&\times \left( q^2 f(-q^{5}, -q^{15}) f(-q^{20}, -q^{30}) \right) \\
&+ \frac{1}{f(-q^{10}, -q^{40})} f(-q^{10}, -q^{40}) f(-q^{20}, -q^{30}) f(-q^{40}, -q^{60}) \\
&= f^2(-q^{3}, -q^{7}) f(-q^{4}, -q^{14}) (q^{10}, q^{10}) \infty (q^{20}, q^{20}) \infty.
\end{align*}
\] (8.2.38)

\textit{Proof.} Simplify the right-hand side of (8.2.27) with the identity \[ q^{589/30} \eta_{10,1}^2 \eta_{10,2}^2 \eta_{10,3} \eta_{20,2} \eta_{10,10} \eta_{50,15} \eta_{50,20} \eta_{50,25}^2 / \eta_{100}^2. \]

Using (8.2.23) and (8.2.24) to convert the resulting identity to a \(q\)-series identity, we deduce (8.2.38).

\textbf{Lemma 8.2.20.} We have
\[
\begin{align*}
q^5 f(-q^5, -q^{15}) f(-q^{10}, -q^{40}) f(-q^{20}, -q^{50}) f(-q^{40}, -q^{60}) &\\
&f^2(-q^{10}, -q^{40}) \\
&+ f(-q^{15}, -q^{35}) f(-q^{20}, -q^{30}) f(-q^{40}, -q^{50}) f(-q^{40}, -q^{60}) f(-q^{40}, -q^{50}) &\\
&f^2(-q^{10}, -q^{40}) \\
&- 3q^5 f^2(-q^{25}, -q^{75}) = f^2(-q^{5}, -q^{15}).
\end{align*}
\] (8.2.39)

\textit{Proof.} Employ the identity \[ \eta_{25,5} \eta_{25,10} \eta_{25} = \eta_{5} \] on the right-hand side of (8.2.29) in \textbf{Lemma 8.2.14}, divide both sides of (8.2.29) by \[ q^{25/4} \eta_{5} \eta_{5}^{10} \eta_{5}^{2} \eta_{50,10} \eta_{50,20}. \]

apply (8.2.23) and (8.2.24) to express the identity in terms of theta functions, and finally deduce (8.2.39). \(\square\)
**Lemma 8.2.21.** We have

\[
f(q^{25}, q^{25}) f(-q^{25}, -q^{25}) + q^{2} f(q^{15}, q^{35}) f(-q^{15}, -q^{35}) \\
- q^{8} f(q^{5}, q^{45}) f(-q^{5}, -q^{45}) - q^{4} f(q^{25}, q^{25}) f(-q^{5}, -q^{45}) \\
- q f(q^{15}, q^{35}) f(-q^{25}, -q^{25}) + 2q^{5} f(q^{15}, q^{35}) f(-q^{5}, -q^{45}) \\
- q^{5} f(q^{5}, q^{45}) f(-q^{5}, -q^{35}) = \frac{f(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18})}. \quad (8.2.40)
\]

**Proof.** Using the Jacobi triple product identity, we can readily show that, for each integer \( n \),

\[
\frac{(q^{50}; q^{50})_{\infty}^{2} f(-q^{2n}, -q^{100-2n})}{(q^{100}; q^{100})_{\infty} f(-q^{n}, -q^{50-n})} = f(q^{n}, q^{50-n}). \quad (8.2.41)
\]

Now use the identity \( \eta_{20}/\eta_{20,4}^{2} = \eta_{4} \) on the right side of (8.2.31), divide both sides of (8.2.31) by

\[
q^{329/30} \eta_{4}^{1/10,1} \eta_{10,2}^{1/10,3} \eta_{20,2}^{1/30,5} \eta_{50,15}^{1/15,45} \eta_{50,25},
\]

apply (8.2.23), (8.2.24), and (8.2.41) with \( n = 5, 15, \) and \( 25 \), respectively, and thus complete the proof of Lemma 8.2.21. \( \Box \)

**Lemma 8.2.22.** We have

\[
f(q^{20}, q^{30}) \left( \frac{f(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}^{2}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18})} - 4qf(-q^{4}, -q^{16}) f(-q^{8}, -q^{12}) \right) \\
+ q^{2} f(q^{10}, q^{40}) \left( \frac{f(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}^{2}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18})} \\
+ 6qf(-q^{4}, -q^{16}) f(-q^{8}, -q^{12}) \right) \\
- 4q^{6} f(q^{50}, q^{150}) \left( \frac{f(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}^{2}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18})} \\
+ qf(-q^{4}, -q^{16}) f(-q^{8}, -q^{12}) \right) \]

\[
= f(q^{25}, q^{25}) \left( \frac{f^{2}(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}(q^{20}; q^{20})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18}) f(-q^{4}, -q^{16})} \\
- 4qf^{2}(-q^{5}, -q^{15}) - q^{2} \frac{f^{2}(-q^{3}, -q^{7}) f(-q^{4}, -q^{16})(q^{10}; q^{10})_{\infty}(q^{20}; q^{20})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18}) f(-q^{4}, -q^{16})} \right) \\
+ qf(q^{15}, q^{35}) \left( 2 \frac{f^{2}(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}(q^{20}; q^{20})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18}) f(-q^{4}, -q^{16})} + 2q^{2} \frac{f^{2}(-q^{3}, -q^{7}) f(-q^{4}, -q^{16})(q^{10}; q^{10})_{\infty}(q^{20}; q^{20})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18}) f(-q^{4}, -q^{16})} \right) \\
- q^{4} f(q^{25}, q^{25}) \left( 3 \frac{f^{2}(-q, -q^{9})(q^{4}; q^{4})_{\infty}(q^{10}; q^{10})_{\infty}(q^{20}; q^{20})_{\infty}}{f(-q^{3}, -q^{7}) f(-q^{2}, -q^{18}) f(-q^{4}, -q^{16})} \right).
\]
Lemma 8.2.23. We have
\[-2q f^2(-q^5, -q^{15}) + 2q^2 f^2(-q^3, -q^7) f(-q^4, -q^{16})(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty \frac{f(-q, -q^9) f(-q^6, -q^{14}) f(-q^8, -q^{12})}{f(-q, -q^9) f(-q^6, -q^{14}) f(-q^8, -q^{12})} \]
(8.2.42)

Proof. First, employ the identities
\[
\begin{align*}
\eta_{20}^6 \eta_{20,10} &= \eta_{10}^2, \\
\eta_{20}^9 \eta_{40,10} &= \eta_{10}, \\
\eta_{20} \eta_{20,4} \eta_{20,8} &= \eta_4, \\
\eta_5 \eta_{50,10} \eta_{50,20} &= \eta_1
\end{align*}
\]
in (8.2.33). Second, divide both sides of (8.2.33) by
\[
q^{233/6} \eta_{10}^2 \eta_{20} \eta_{10,1} \eta_{10,3} \eta_{20,2} \eta_{20,4} \eta_{20,6} \eta_{20,8} \\
\times \eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{200,12} \eta_{200,100}
\]
Third, use the identities (8.2.23), (8.2.24), and (8.2.41) with \(n\) replaced by 5, 10, 15, 20, and 25, respectively, and with \(q\) replaced by \(q^4\) and \(q^{10}\), respectively. The identity (8.2.33) then assumes the form (8.2.42). \(\square\)

Lemma 8.2.23. We have
\[
f(q^{20}, q^{30}) \frac{f(-q, -q^9)(q^4; q^4)_\infty (q^{10}; q^{10})^2}{f(-q^3, -q^7) f(-q^2, -q^{18})} \\
- q^2 f(q^{10}, q^{40}) \left( \frac{f(-q, -q^9)(q^4; q^4)_\infty (q^{10}; q^{10})^2}{f(-q^3, -q^7) f(-q^2, -q^{18})} \right) \\
+ 2q f(-q^4, -q^{16}) f(-q^8, -q^{12}) \\
+ 4q^7 f(q^{50}, q^{150}) f(-q^4, -q^{16}) f(-q^8, -q^{12}) \\
eq f(q^{25}, q^{25}) \left( \frac{f^2(-q, -q^9) f(-q^8, -q^{12})(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty}{f(-q^3, -q^7) f(-q^2, -q^{18}) f(-q^4, -q^{16})} \right) \\
+ q \frac{f^2(-q^3, -q^7) f(-q^4, -q^{16})(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty}{f(-q, -q^9) f(-q^6, -q^{14}) f(-q^8, -q^{12})} \\
- q^2 f(q^{15}, q^{35}) \left( 2f^2(-q^5, -q^{15}) \right) \\
+ 3q \frac{f^2(-q^3, -q^7) f(-q^4, -q^{16})(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty}{f(-q, -q^9) f(-q^6, -q^{14}) f(-q^8, -q^{12})} \\
- q^4 f(q^5, q^{45}) \left( \frac{f^2(-q, -q^9) f(-q^8, -q^{12})(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty}{f(-q^3, -q^7) f(-q^2, -q^{18}) f(-q^4, -q^{16})} \right) \\
- 2q f^2(-q^5, -q^{15}). \quad (8.2.43)
\]
Proof. First, simplify (8.2.35) by using the identities
\[ \eta_{20}\eta_{10} = \eta_{10}, \]
\[ \eta_{20}\eta_{40} = \eta_{40}, \]
\[ \eta_{20}^2 = \eta_{20}, \]
\[ \eta_{20}^2 \eta_{40} = \eta_{40}, \]
\[ \eta_{20}^2 \eta_{80} = \eta_{80}, \]
\[ \eta_{50}\eta_{100,25} = \eta_{25}. \]

Second, divide both sides of (8.2.35) by
\[ q^{233/6} \eta_{10}^2 \eta_{50}^2 \eta_{10,1} \eta_{10,3} \eta_{20,2} \eta_{20,6} \eta_{20,8} \]
\[ \times \eta_{50,5} \eta_{50,10} \eta_{50,15} \eta_{50,20} \eta_{50,25} \eta_{200,50} \eta_{200,100}. \]

Third, apply (8.2.23), (8.2.24), and (8.2.41) with \( n \) replaced by 5, 10, 15, 20, and 25, respectively, and with \( q \) replaced by \( q^4 \) and \( q^{50} \), respectively. Then (8.2.43) follows.

8.3 Proof of Entry 8.1.1

Entry 8.3.1 (p. 9). Recall that the tenth order mock theta functions \( \phi(q) \) and \( \psi(q) \) are defined in (8.1.1). If \( n > 0 \), then
\[
\int_0^\infty \frac{e^{-\pi nx^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + 1 + \frac{\sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi(-e^{-\pi/n})
\]
\[ = \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \phi(-e^{-\pi/n}) - \frac{\sqrt{5} + 1}{2\sqrt{n}} e^{-\pi/(5n)} \phi(-e^{-\pi/n}). \]  

(8.3.1)

Proof. Replacing \( \omega, x, t, \) and \( \theta \) by \( 5in, -5n, 1+z/\sqrt{5}, \) and \( \theta'-i \), respectively, on the left-hand side of (8.1.18), we find that
\[
\int_{-\infty}^{\infty} \frac{e^{\pi n x^2} - 2\pi x}{e^{2\pi x} - e^{2\pi i \theta}} dt = \frac{e^{5\pi n} - 2\pi}{\sqrt{5}} \int_{-\infty}^{\infty} e^{-\pi n z^2} dz. \]  

(8.3.2)

Setting \( \theta' = \frac{2}{5} \) in the right side of (8.3.2) and using (8.1.18), we deduce that
\[ \int_{-\infty}^{\infty} e^{-\pi n z^2} dz = \sqrt{5} e^{4\pi n/5 - 4\pi i/5} \frac{F(\frac{1}{5}, \frac{1}{5})}{5i \theta_{11}(2in, 5in)}. \]  

(8.3.3)

and setting \( \theta' = \frac{3}{5} \) in the right side of (8.3.2) and using (8.1.18), we further deduce that
\[ \int_{-\infty}^{\infty} e^{-\pi n z^2} dz = \sqrt{5} e^{9\pi n/5 - 6\pi i/5} \frac{F(\frac{3}{5}, \frac{1}{5})}{5i \theta_{11}(3in, 5in)}. \]  

(8.3.4)
Employing (8.3.3) and (8.3.4), we arrive at

\[
\int_0^\infty \frac{e^{-\pi nx^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1}{4}} \, dx = \frac{1}{2} \int_0^\infty \frac{e^{-\pi nx^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1}{4}} \, dx
\]

\[
= -\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-\pi nx^2}}{i \sin(\pi/5)} \left( \frac{1}{1 + e^{2\pi x/\sqrt{5} + \pi i/5}} - \frac{1}{1 + e^{2\pi x/\sqrt{5} - \pi i/5}} \right) \, dx
\]

\[
= -\frac{1}{2i \sin(\pi/5)} \left( e^{-\pi i/5} \int_{-\infty}^{\infty} \frac{e^{-\pi nx^2}}{e^{4\pi x/\sqrt{5}} + e^{-\pi i/5}} \, dx - e^{\pi i/5} \int_{-\infty}^{\infty} \frac{e^{-\pi nx^2}}{e^{4\pi x/\sqrt{5}} + e^{\pi i/5}} \, dx \right)
\]

\[
= -\frac{\sqrt{5 + \sqrt{5}}}{5n\sqrt{2}} \left( e^{-\pi n/5 + \pi n} \frac{F(\frac{3}{5} + \frac{i}{5})}{\theta_{11}(2in, 5in)} - 5nF(2in, 5in) \right) - e^{-\pi n/5 + 2\pi n} \frac{F(\frac{3}{5} + \frac{i}{5})}{\theta_{11}(3in, 5in)} \right).
\](8.3.5)

To continue the evaluation from (8.3.5), we note that, from (8.1.19) and (8.1.20), respectively,

\[
F(x, \tau) = -F(-x + \tau, \tau)
\](8.3.6)

and

\[
\theta_{11}(x, \tau) = -\theta_{11}(-x, \tau) = e^{\pi i(-2x + \tau)} \theta_{11}(-x + \tau, \tau).
\](8.3.7)

Using (8.3.6) and (8.3.7), we see that

\[
F(3in, 5in) = -F(2in, 5in) \quad \text{and} \quad \theta_{11}(3in, 5in) = e^{\pi n} \theta_{11}(2in, 5in).
\](8.3.8)

Therefore, from (8.3.8),

\[
e^{\pi n} \frac{F(3in, 5in)}{\theta_{11}(3in, 5in)} = -\frac{F(2in, 5in)}{\theta_{11}(2in, 5in)}.
\](8.3.9)

Let \( q = e^{-\pi n} \). Hence, using (8.3.9), (8.1.19), (8.1.20), (8.1.5), (8.1.12) with \( q \) and \( z \) replaced by \(-q\) and \( q^4\), respectively, (8.1.17), (8.1.13) with \( q, z, \) and \( x \) replaced by \(-q^5, 1, \) and \( q^2 \), respectively, (8.1.15) with \( q \) replaced by \(-q, \) and Theorem 8.2.4, we find that

\[
\sqrt{\frac{5 + \sqrt{5}}{2}} \left( e^{-\pi n/5 + \pi n} \frac{F(2in, 5in)}{\theta_{11}(2in, 5in)} - e^{-\pi n/5 + 2\pi n} \frac{F(3in, 5in)}{\theta_{11}(3in, 5in)} \right)
\](8.3.10)
\[\begin{align*}
&= -\sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \left( \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{5m^2 + 9m + 3}}{1 + q^{10m + 5}} f(-q^4, -q^6) \right) + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{5m^2 + 9m + 3}}{1 + q^{10m + 5}} f(-q^4, -q^6) \\
&= \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \frac{L_2(q, q^4)}{f(-q^4, -q^6)} \\
&= \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \left( \phi(-q) - \frac{(-q^5; -q^5)_\infty (q^{10}; q^{10})_\infty f(-q^4, -q^6)}{f(-q^2, -q^8)} \right. \\
&\quad \left. + 2 q \frac{(-q^5; -q^5)_\infty (q^{10}; q^{10})^2_{\infty}}{f^2(q^4, q^4) f(-q^4, -q^6)} \right) \\
&= \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \phi(-q) - \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} f(-q, -q^3) f(q^3, q^7) f(q^4, q^6) \\
&\quad \cdot f(-q^4, -q^6) f(q^4, q^6). 
\end{align*}\]

To further continue the evaluation in (8.3.5), we need the transformation formula [112, p. 330]

\[\theta_{11} \left( \frac{x}{\omega}, -\frac{1}{\omega} \right) = -i \sqrt{-2\omega} e^{\pi i x^2} \theta_{11}(x, \omega). \quad (8.3.11)\]

Putting \(x = 2in\) and \(\omega = 5in\) in (8.3.11), we find that

\[\theta_{11} \left( \frac{2}{5}, \frac{i}{5n} \right) = -i \sqrt{5n e^{-4n \pi/5}} \theta_{11}(2in, 5in), \quad (8.3.12)\]

and putting \(x = 3in\) and \(\omega = 5in\) in (8.3.11), we further find that

\[\theta_{11} \left( \frac{3}{5}, \frac{i}{5n} \right) = -i \sqrt{5n e^{-9n \pi/5}} \theta_{11}(3in, 5in). \quad (8.3.13)\]

Let \(q_1 = e^{-\pi/(5n)}\), \(\alpha = e^{2\pi i/5}\), and \(\beta = e^{3\pi i/5} = -1/\alpha\). Using the definitions of \(F\) in (8.1.19) and \(\theta_{11}\) in (8.1.20), replacing \(m\) by \(-m\), and then replacing \(m\) by \(m+1\), we see that

\[\begin{align*}
F \left( \frac{3}{5}, \frac{i}{5n} \right) &= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m} \alpha^{-2m}}{1 + q_1^{2m+1}} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 - m} \alpha^{2m}}{1 + q_1^{2m+1}} \\
\theta_{11} \left( \frac{3}{5}, \frac{i}{5n} \right) &= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m} \alpha^{2m}}{1 + q_1^{2m-1}} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m} \alpha^{2m}}{1 + q_1^{2m+1}} \\
&= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m} \alpha^{2m}}{1 + q_1^{2m-1}} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m} \alpha^{2m}}{1 + q_1^{2m+1}}. 
\end{align*}\]

(8.3.14)
By a similar, but less involved, calculation,

\[
F \left( \frac{2}{5}, \frac{i}{5n} \right) = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m \alpha^2 m}}{1 + q_1^{2n+1}}. \tag{8.3.15}
\]

Thus, using (8.3.12), (8.3.13), (8.3.14), and (8.3.15), we find that

\[
\frac{i}{\sqrt{5n}} \left( e^{4\pi n/5} F \left( \frac{2}{5}, \frac{i}{5n} \right) - e^{9\pi n/5} F \left( \frac{2}{5}, \frac{1}{5n} \right) \right)
= \frac{F \left( \frac{2}{5}, \frac{i}{5n} \right)}{\theta_1 \left( \frac{2}{5}, \frac{1}{5n} \right)} - \frac{F \left( \frac{2}{5}, \frac{1}{5n} \right)}{\theta_1 \left( \frac{3}{5}, \frac{1}{5n} \right)}
= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m \alpha^2 m} \theta_1 \left( \frac{3}{5}, \frac{1}{5n} \right)}{1 + q_1^{2n+1}} \tag{8.3.16}
\]

Using the definition of \( f(a, b) \) in (4.1.1), (4.1.3), and (4.1.4), we find that the denominator on the right side of (8.3.16) can be written in the form

\[
\sum_{m=-\infty}^{\infty} (-1)^m q_1^{m^2 + m \alpha^2 m} = \sum_{j=0}^{4} \sum_{m=-\infty}^{\infty} (-1)^{m+j} q_1^{25m^2 + 10mj + j^2 + 5m + j}
= (1 - \alpha^8) f(-q_1^{20}, -q_1^{30}) - q_1^2 (\alpha^2 - \alpha^6) f(-q_1^{10}, -q_1^{40}). \tag{8.3.17}
\]

We now multiply the numerator and denominator on the right-hand side of (8.3.16) by \( 1 - q_1^{2m+1} + q_1^{4m+2} - q_1^{6m+3} + q_1^{8m+4} \). It is easily seen that the new numerator is

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2 + m \alpha^2 m}}{1 + q_1^{10m+5}} \left\{ 1 - 2(q_1^{2m+1} - q_1^{4m+2} + q_1^{6m+3} - q_1^{8m+4} - q_1^{10m+5}) \right\}
= \sum_{j=0}^{4} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+j} q_1^{(5m+j)^2 + (5m+j)j}}{1 + q_1^{50m+10j+5}} \tag{8.3.18}
\]

We next transform each of the series above into series \( L_1 \) and \( L_2 \) and products of theta functions. We need Bailey’s summation of the \( \theta_6 \) given by [75, p. 140]
Using (8.1.5) and (8.1.6), we find that the right-hand side of (8.3.19) becomes

\[ (8.3.22). \]

Hence, using again the Jacobi triple product identity (8.1.6) several times, we have

\[
\sum_{m=-\infty}^{\infty} \frac{(qa^{1/2}; q)_n(-qa^{1/2}; q)_n(b; q)_n(c; q)_n(d; q)_n(e; q)_n}{(a^{1/2}; q)_n(-a^{1/2}; q)_n(aq/b; q)_n(aq/c; q)_n(aq/d; q)_n(aq/e; q)_n} \left( \frac{qa^2}{bcde} \right)^n
\]

provided that \(|qa^2/bcde| < 1.\)

Let \(q = -q_1^{5/2}, a = q_1^5, b = q_1^{5/2} i, c = -q_1^{5/2} i,\) and let \(d, e \rightarrow \infty\) in (8.3.19). Then the left-hand side of (8.3.19) becomes

\[
\sum_{m=-\infty}^{\infty} \frac{(q_1^{55}; q_1^{50})_m(-q_1^5; q_1^{50})_m(-q_1^{25}; q_1^{50})_{m(1-1)}}{(q_1^5; q_1^{50})_m(-q_1^{55}; q_1^{50})_m} = \frac{1 + q_1^5}{1 - q_1^5} \sum_{m=-\infty}^{\infty} 1 - q_1^{50m+5}(1-1)^m q_1^{25m^2+5m}. \quad (8.3.20)
\]

Using (8.1.5) and (8.1.6), we find that the right-hand side of (8.3.19) becomes

\[
\frac{(q_1^{5}; q_1^{5})_\infty(-q_1^{25}; q_1^{5})_\infty(-q_1^{25}; q_1^{25})_\infty(-q_1^{20}; q_1^{25})_\infty}{(q_1^{25+5/2}; q_1^{25+5/2})_\infty(-q_1^{25+5/2}; q_1^{25+5/2})_\infty(-q_1^{25-5/2}; q_1^{25-5/2})_\infty(-q_1^{25-5/2}; q_1^{25-5/2})_\infty}
\]

\[
= \frac{(q_1^{5}; q_1^{5})_\infty(-q_1^{25}; q_1^{25})_\infty(-q_1^{25}; q_1^{5})_\infty(-q_1^{20}; q_1^{5})_\infty}{(1-q_1^5)(q_1^5; q_1^{5})_\infty(-q_1^{45}; q_1^{45})_\infty}
\]

\[
= \frac{1 + q_1^5}{1 - q_1^5} f(-q_1^5; q_1^{20})(-q_1^{25}; q_1^{5})_\infty(q_1^{50}; q_1^{50})_\infty. \quad (8.3.21)
\]

Combining (8.3.20) and (8.3.21) and then multiplying both sides by \((1 - q_1^{50})(1 + q_1^5),\) we find that

\[
\sum_{m=-\infty}^{\infty} (-1)^m q_1^{25m^2+5m} \frac{1 - q_1^{50m+5}}{1 + q_1^{50m+5}} = \frac{f(-q_1^5; q_1^{20})(-q_1^{25}; q_1^{5})_\infty(q_1^{50}; q_1^{50})_\infty}{f(q_1^5; q_1^{15})}. \quad (8.3.22)
\]

We now want to obtain an alternative representation for the right side of (8.3.22). Hence, using again the Jacobi triple product identity (8.1.6) several times, we find that

\[
\frac{f(-q_1^5; q_1^{20})(-q_1^{25}; q_1^{5})_\infty(q_1^{50}; q_1^{50})_\infty}{f(q_1^5; q_1^{15})}
\]
Using (8.1.11), (8.1.12), (8.3.18), (8.3.22), (8.3.25), and (8.3.26), we deduce

Lastly, from [28, Entry 8(vii), p. 114], we readily see that

Substituting (8.3.23) into (8.3.22), we conclude that

Substituting (8.3.3) into (8.3.22), we conclude that

Next, let \( q = -q_1^{25}, a = q_1^{15}, b = q_1^{15/2}i, \) and \( c = -q_1^{15/2}i, \) and let \( d, e \to \infty \) in (8.3.19). We now follow the same paths as we did above in deriving (8.3.20)–(8.3.23). Omitting the details, since they are similar, we derive the analogues of (8.3.24) and (8.3.22), namely,

Lastly, from [28, Entry 8(vii), p. 114], we readily see that

Using (8.1.11), (8.1.12), (8.3.18), (8.3.22), (8.3.25), and (8.3.26), we deduce that
simplify this huge bevy of theta functions, we employ Theorems 8.2.18–8.2.20.

Our goal of writing the right side of (8.3.27) in terms of theta functions. In order to achieve this goal, we employ (8.1.13)–(8.1.15) to further complete our understanding of (8.3.27) so that we can begin to express the right side of (8.3.27) in terms of theta functions.

We now use (8.1.16) and (8.1.17) to replace $L_1$ and $L_2$ on the right-hand side of (8.3.27) so that we can begin to express the right side of (8.3.27) in terms of theta functions. We then employ (8.1.13)–(8.1.15) to further complete our goal of writing the right side of (8.3.27) in terms of theta functions. In order to simplify this huge bevy of theta functions, we employ Theorems 8.2.18–8.2.20.

To that end, using also (8.3.28), we find that

$$
\begin{align*}
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+m} \alpha^{2m} (1 - q_1^{2m+1})}{1 + q_1^{2m+1}} & \quad (8.3.27) \\
& \quad \begin{aligned}
&= (1 + \alpha^8) (q_1 L_1(q_1^5 \cdot q_1^{40}) - q_1^7 L_2(q_1^5, 1) - q_1^{-1} L_1(q_1^5 \cdot q_1^{30}) - q_1^3 L_2(q_1^5, q_1^{40}) \\
& \quad + (\alpha^2 + \alpha^6) (q_1^3 L_1(q_1^5, 1) + q_1^3 L_2(q_1^5, q_1^{10}) + q_1^{-1} L_1(q_1^5, q_1^{20}) + q_1 L_2(q_1^5, q_1^{10})) \\
& \quad - 2\alpha^4 (q_1 L_1(q_1^5, q_1^{10}) + q_1 L_2(q_1^5, q_1^{20})) \\
& \quad + (1 + \alpha^8) \frac{f(-q_1^5, q_1^{20})(-q_1^{25}; -q_1^{25}) = (q_1^{50}; q_1^{50}) = -2\alpha^4 q_1^2 f^2(-q_1^{25}, -q_1^{25}).
\end{aligned}
\end{align*}
\)

Recalling that

$$
2 \sin \frac{\pi}{5} = \sqrt{\frac{5 - \sqrt{5}}{2}} \quad \text{and} \quad 2 \sin \frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{2}},
$$

we can easily check the identities

$$
\begin{cases}
1 - 2\alpha^4 + \alpha^8 = i \sqrt{\frac{5 - \sqrt{5}}{2}} (-\alpha^2 + \alpha^6), \\
-1 + \alpha^2 + \alpha^6 - \alpha^8 = i \sqrt{\frac{5 - \sqrt{5}}{2}} (1 - \alpha^8) = i \sqrt{\frac{5 + \sqrt{5}}{2}} (-\alpha^2 + \alpha^6), \\
\alpha^2 - 2\alpha^4 + \alpha^6 = i \sqrt{\frac{5 + \sqrt{5}}{2}} (1 - \alpha^8).
\end{cases}
$$

(8.3.28)

We now use (8.1.16) and (8.1.17) to replace $L_1$ and $L_2$ on the right-hand side of (8.3.27) so that we can begin to express the right side of (8.3.27) in terms of theta functions. We then employ (8.1.13)–(8.1.15) to further complete our goal of writing the right side of (8.3.27) in terms of theta functions. In order to simplify this huge bevy of theta functions, we employ Theorems 8.2.18–8.2.20.
+ (\alpha^2 + \alpha^6) \left(2q_1^{10} A(q_1^{-10}, -q_1^5, -q_1^{25}) - q_1^{-5} a_2(-q_1^5)f(-q_1^{10}, -q_1^{40})
- q_1^{-1} a_1(-q_1^5)f(-q_1^{20}, -q_1^{30}) + 2q_1^{4} A(q_1^{-10}, -q_1^5, -q_1^{25}) - q_1 a_2(-q_1^5)f(-q_1^{20}, -q_1^{30})
+ 2q_1^{6} A(q_1^{-10}, q_1^5, -q_1^{25}) - q_1^2 f(-q_1^{-15}, q_1^{10})(-q_1^{25}; -q_1^{25}) \infty (q_1^{30}; q_1^{30}) \infty \right)
+ 2\alpha^4 q_1 \left(a_1(-q_1^5)f(-q_1^{10}, -q_1^{40}) + a_2(-q_1^5)f(-q_1^{20}, -q_1^{30})
- 2q_1^{5} f^2(-q_1^{25}, -q_1^{75}) - q_1^5 f^2(-q_1^{25}, -q_1^{75}) \right)
\sum_{m=-\infty}^{\infty} (-1)^m q_1^{m^2 + \alpha^2m} \left(5 - \sqrt{5}ight) \frac{q_1^{1} \psi(-q_1^5) + i \sqrt{5 + \sqrt{5}}}{2} q_1 \phi(-q_1^5) \\
+ (1 + \alpha^8) \frac{f^2(-q_1, -q_1^6)f(-q_1^8, -q_1^{12})(q_1^{10}; q_1^{10}) \infty}{f(-q_1^{15}, -q_1^5)f(-q_1^{16}, -q_1^{16})} \\
- (\alpha^2 + \alpha^6) q_1 \frac{f^2(-q_1^{15}, -q_1^5)f(-q_1^{14}, -q_1^{14})}{f(-q_1, -q_1^5)f(-q_1^6, -q_1^{16})} \\
+ 2\alpha^4 q_1 f^2(-q_1^{15}, -q_1^{15}). \quad (8.3.29)

Thus, by (8.3.16), (8.3.27), and (8.3.29), we deduce that

\frac{1}{\sqrt{n}} q_1^{-1} \psi(-q_1^5) - \frac{5 + 1}{2\sqrt{n}} q_1 \phi(-q_1^5) + i \sqrt{\frac{5 + \sqrt{5}}{\sqrt{10n}}} \sum_{m=-\infty}^{\infty} (-1)^m q_1^{m^2 + \alpha^2m} \\
\left(1 + \alpha^8 \right) \frac{f^2(-q_1, -q_1^6)f(-q_1^8, -q_1^{12})(q_1^{10}; q_1^{10}) \infty}{f(-q_1^{15}, -q_1^5)f(-q_1^{16}, -q_1^{16})} \\
- (\alpha^2 + \alpha^6) q_1 \frac{f^2(-q_1^{15}, -q_1^5)f(-q_1^{14}, -q_1^{14})}{f(-q_1, -q_1^5)f(-q_1^6, -q_1^{16})} \\
+ 2\alpha^4 q_1 f^2(-q_1^{15}, -q_1^{15}). \quad (8.3.30)

To complete our proof, we need several identities for theta functions. Using only the definition (8.1.5) and appropriate dissections, we can show that

\begin{align*}
\sum_{\gamma=\pm1} (1 - \gamma)f(-q_1^6, -q_1^{12}), & \quad \text{if } \gamma = e^{\pi i/3}, \\
(1 - \gamma^8)f(-q_1^{20}, -q_1^{30}) + (\gamma^2 - \gamma^6)q_2^2 f(-q_1^{10}, -q_1^{40}), & \quad \text{if } \gamma = e^{\pi i/5}, \\
(1 + \gamma^8)f(q_1^{30}, q_1^{30}) - (\gamma^2 + \gamma^6)q_2^2 f(q_1^{10}, q_1^{40}) \\
+ 2\gamma^4 q_6^6 f(q_1^{50}, q_1^{50}), & \quad \text{if } \gamma = e^{\pi i/10} \text{ or } e^{3\pi i/10},
\end{align*}

and
Transformation Formulas: 10th Order Mock Theta Functions

by

Utilizing (8.3.31), (8.3.32), Theorem 8.2.21, and Theorem 8.2.6 and replacing $\alpha$ throughout, we find that

\[
f(q^2, q) = \frac{f(-q^2, q) - (\gamma^2 + \gamma^8)qf(-q^1, q)}{\gamma + \gamma^6)qf(-q^1, q^5)}, \text{ if } \gamma = e^{\pi i/5} \text{ or } e^{2\pi i/5},
\]

\[
f(q^2, q^5) - (\gamma^2 - \gamma^8)qf(q^1, q^5),
\]

Employing the four transformation formulas above, we find that

\[
f(-q, -q^7) = -\frac{i}{\sqrt{5n}} e^{4\pi n/5-\pi/(20n)+\pi i/10} f(-q^1, q^1), \text{ if } \gamma_1 = e^{2\pi i/10},
\]

\[
f(q, q^9) = \frac{1}{\sqrt{5n}} e^{4\pi n/5} f(q^1, q^1),
\]

\[
f(q^3, q^7) = \frac{1}{\sqrt{5n}} e^{\pi n/5} f(q^1, q^1), \text{ if } \gamma_2 = e^{6\pi i/10},
\]

\[
f(q^4, q^6) = \frac{1}{\sqrt{5n}} e^{\pi n/20} f(q^1, q^1), \text{ if } \gamma_3 = e^{4\pi i/3}.
\]

Using the transformation formula (8.3.11), we can easily verify that

\[
\frac{f(-q, -q^9) f(q^3, q^7) f(q^4, q^6) f(q^4, q^4)}{f(q, q^9)} = \frac{1}{5n} e^{-\pi/(20n)+\pi n/5+\pi n/20} \alpha^4
\]

\[
\times \frac{f(-q^2, q^1) f(-q^1, q^1) f(-q^2, q^1) f(-q^1, q^1)}{f(-q^1, q^1)}.
\]

Employing the four transformation formulas above, we find that

\[
(1 + \alpha^2) f(q^{10}, q^{10}) + (\alpha^3 + \alpha^4) q^2 f(q^{10}, q^{10}) + 2\alpha q^8 f(q^{50}, q^{150})
\]

\[
\times (f(q^{25}, q^{25}) + (\alpha + \alpha^4) q^2 f(q^{50}, q^{150}) + (\alpha^2 + \alpha^3) q^2 f(q^{50}, q^{150}))
\]

\[
\times (f(q^{25}, q^{25}) - (\alpha^2 + \alpha^3) q^2 f(q^{25}, q^{25}) + (\alpha + \alpha^4) q^2 f(q^{50}, q^{150}))
\]

\[
= \alpha^4 (1 + \alpha^2) f(q^{20}, q^{30}) + (\alpha^3 + \alpha^4) q^2 f(q^{10}, q^{40}) + 2\alpha q^6 f(q^{50}, q^{150})
\]

\[
\times \left( \frac{f(-q^1, q^1) f(-q^1, q^1) f(-q^1, q^1) f(-q^1, q^1)}{f(-q^1, q^1) f(-q^1, q^1)} \right)
\]

\[
- 2(\alpha^2 + \alpha^3) q^2 f(-q^1, -q^1) f(-q^8, -q^1)
\]

\[
= (\alpha + \alpha^4) f(q^{20}, q^{30}) + (\alpha^3 + \alpha^4) q^2 f(q^{10}, q^{40}) + 2\alpha q^6 f(q^{50}, q^{150})
\]

\[
+ 2q^2 f(q^{20}, q^{30}) f(-q^1, -q^1) f(-q^8, -q^1)
\]

\[
+ (\alpha^2 + \alpha^3) q^2 f(q^{10}, q^{40}) f(-q^1, -q^1) f(-q^8, -q^1)
\]

\[
- 2(2 + \alpha + \alpha^4) q^3 f(q^{10}, q^{40}) f(-q^4, -q^1) f(-q^8, -q^1)
\]

\[
\frac{\gamma}{q}. 
\]

(8.3.32)
8.3 Proof of Entry 8.1.1 179

\[ + 2q_1^6 f(q_1, q_1^{150}) \frac{f(-q_1, -q_1^{9}(q_1^4); q_1^{10}q_1^{10})}{f(-q_1^4, -q_1^4)f(-q_1^8, -q_1^8)} \]

\[ - 4(\alpha^2 + \alpha^3)q_1^7 f(q_1, q_1^{150})f(-q_1^4, -q_1^{16})f(-q_1^8, -q_1^{12}). \] (8.3.34)

We need some elementary identities for \( \alpha = e^{2\pi i/5} \). It is easily checked that

\[
\begin{aligned}
\alpha + \alpha^4 &= -1 + \sqrt{5}, \\
\alpha^2 + \alpha^3 &= -1 - \sqrt{5}, \\
2 + \alpha + \alpha^4 &= \frac{3 + \sqrt{5}}{2}.
\end{aligned}
\] (8.3.35)

Let \( L_j(q_1) \) and \( R_j(q_1), \) \( j = 1, 2, \) denote, respectively, the left and right sides of (8.2.42) and (8.2.43) with \( q \) replaced by \( q_1 \). Furthermore, set

\[
LS_1(q) = -\frac{1}{2}L_1(q), \quad LS_2(q) = \frac{\sqrt{5}}{2}L_2(q),
\]

\[
RS_1(q) = -\frac{1}{2}R_1(q), \quad RS_2(q) = \frac{\sqrt{5}}{2}R_2(q).
\]

Note that the right-hand side of (8.3.34) is equal to \( LS_1(q) + LS_2(q) \), and so we find that the right-hand side of (8.3.34) is equal to \( RS_1(q) + RS_2(q) \). Using (8.3.32) and (8.3.35), we therefore find that

\[
RS_1(q) + RS_2(q)
\]

\[
= (\alpha + \alpha^4)f(q_1^{25}, q_1^{25})f^2(-q_1^4, -q_1^4)f(-q_1^8, -q_1^{12})(q_1^{10}; q_1^{10})_\infty(q_1^{20}; q_1^{20})_\infty
\]

\[
+ 2q_1^6 f(q_1^{25}, q_1^{25})f^2(-q_1^4, -q_1^{15})
\]

\[
- (\alpha^2 + \alpha^3)q_1^7 f(q_1^{25}, q_1^{25})f^2(-q_1^4, -q_1^4)f(-q_1^8, -q_1^{16})(q_1^{10}; q_1^{10})_\infty(q_1^{20}; q_1^{20})_\infty
\]

\[
- q_1^7 f(q_1^{15}, q_1^{15})f^2(-q_1^4, -q_1^4)f(-q_1^8, -q_1^{12})(q_1^{10}; q_1^{10})_\infty(q_1^{20}; q_1^{20})_\infty
\]

\[
+ 2(\alpha^2 + \alpha^3)q_1^7 f(q_1^{15}, q_1^{15})f^2(-q_1^4, -q_1^{15})
\]

\[
- (2 + \alpha + \alpha^4)q_1^7 f(q_1^{15}, q_1^{15})f^2(-q_1^4, -q_1^4)(q_1^{10}; q_1^{10})_\infty(q_1^{20}; q_1^{20})_\infty
\]

\[
+ 2(\alpha + \alpha^4)q_1^7 f(q_1^{15}, q_1^{15})f^2(-q_1^4, -q_1^{15})
\]

\[
+ q_1^7 f(q_1^{45}, q_1^{45})f^2(-q_1^4, -q_1^4)f(-q_1^8, -q_1^{12})
\]

\[
= \alpha (f(q_1^{25}, q_1^{25}) + (\alpha^2 + \alpha^3)q_1 f(q_1^{15}, q_1^{15}) + (\alpha + \alpha^4) f(q_1^{45}, q_1^{45}))
\]
we complete the proof of Entry 8.3.1.

In conclusion, putting (8.3.10) and (8.3.30) in (8.3.5) and utilizing (8.3.38),

\[
\begin{align*}
&= \alpha \sum_{m=-\infty}^{\infty} (-1)^m q_1^m e^{2m\pi i/10} \\
&\times \left( (1 + \alpha^8 \frac{f^2(-q, -q^3) f(-q^4, -q^6) f(-q^8, -q^9)}{f(-q_1, -q_1^2) f(-q_1^2, -q_1^4) f(-q_1^8, -q_1^9)} \\
&\quad - (\alpha^2 + \alpha^6) q_1 \frac{f^2(-q_1^3, -q_1^7) f(-q_1^4, -q_1^8) f(-q_1^6, -q_1^9)}{f(-q_1^4, -q_1^6) f(-q_1^5, -q_1^7)} \\
&\quad + 2\alpha^4 q_1 f^2(-q_1^5, -q_1^{15}) \right) .
\end{align*}
\]

(8.3.36)

From (8.1.5), (8.3.12), and (8.1.20),

\[
f(-q^4, -q^6) = \frac{-i\alpha}{\sqrt{5n}} e^{-\pi/(20n) + \pi n/20} f(-q_1^2 \alpha^2, -\alpha^{-2}).
\]

(8.3.37)

Hence, by (8.3.33), (8.3.34), (8.3.36), and (8.3.37), we conclude that

\[
- \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \frac{f(-q, -q^3) f(q^3, q^7) f(q^4, q^9)}{f(-q^4, -q^6) f(q, q^3)} \\
= - \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi/(20n) + \pi n/20} i \sqrt{5ne^{\pi/(20n)} - \pi n/20} \\
\times \alpha^4 f(-q_1^2 \gamma_3, -\gamma_1^{-1}) f(-q_1 \gamma_2, -q_1 \gamma_2^{-1}) f(-q_1 \gamma_3, -q_1 \gamma_3^{-1}) \\
\quad f(-q_1^2 \alpha^2, -\alpha^{-2}) f(-q_1 \gamma_1, -q_1 \gamma_1^{-1}) \\
= - \sqrt{\frac{5 + \sqrt{5}}{10n}} e^{-\pi/(20n) + \pi n/20} i \sqrt{5ne^{\pi/(20n)} - \pi n/20} \\
\times \left( (1 + \alpha^8 \frac{f^2(-q, -q^3) f(-q^4, -q^6) f(-q^8, -q^9)}{f(-q_1, -q_1^2) f(-q_1^2, -q_1^4) f(-q_1^8, -q_1^9)} \\
&\quad - (\alpha^2 + \alpha^6) q_1 \frac{f^2(-q_1^3, -q_1^7) f(-q_1^4, -q_1^8) f(-q_1^6, -q_1^9)}{f(-q_1^4, -q_1^6) f(-q_1^5, -q_1^7)} \\
&\quad + 2\alpha^4 q_1 f^2(-q_1^5, -q_1^{15}) \right) .
\]

(8.3.38)

In conclusion, putting (8.3.10) and (8.3.30) in (8.3.5) and utilizing (8.3.38),

we complete the proof of Entry 8.3.1. \(\square\)
8.4 Proof of Entry 8.1.2
Two Identities Involving a Mordell Integral and Appell–Lerch Sums

9.1 Introduction

On page 202 in his Lost Notebook, Ramanujan recorded without proofs two modular transformations involving a Mordell integral, \( q \)-hypergeometric series, and generalized Lambert series. These two formulas were first proved by Y.-S. Choi [59], and in this chapter we relate his proofs.

In [112] and [113], L.J. Mordell studied the integral

\[
\int_{-\infty}^{\infty} e^{at^2+bt} e^{ct+dt} \, dt, \quad \text{Re}(a) < 0,
\]

which first appeared in the work of L. Kronecker [99], [100] and B. Riemann (as related by C.L. Siegel) [138]. However, Mordell was the first to study its behavior under modular transformations, and consequently integrals of this type are commonly called Mordell integrals. In [112], Mordell derived the formula

\[
\int_{-\infty}^{\infty} e^{\pi i \tau^2 - 2\pi i t} \frac{e^{-\pi i (\theta^2 \tau + 2\theta x + 2\delta)}}{e^{2\pi i t} - e^{2\pi i \theta}} \, dt = e^{-\pi i (\theta^2 \tau + 2\theta x + 2\delta)} F[(x + \theta \tau)/\tau, -1/\tau] + i\tau F(x + \theta \tau, \tau)
\]

where \( q = e^{\pi i \tau}, \text{Im}(\tau) > 0 \),

\[
iF(x, \tau) := \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+m+1/4} e^{(2m+1)\pi i x}}{1 + q^{2m+1}}, \quad (9.1.2)
\]

and

\[
i\theta_{11}(x, \tau) := \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2+m+1/4} e^{(2m+1)\pi i x}.
\]

(9.1.3)
On page 9 in his lost notebook [128], Ramanujan recorded two modular transformations involving Mordell integrals and his tenth order mock theta functions $\phi(q)$ and $\psi(q)$, namely,

$$
\int_{0}^{\infty} e^{-\pi n x^2} \cosh(2\pi x/\sqrt{5}) \, dx + \frac{1}{\sqrt{n}} e^{\pi/(5n)} \psi(-e^{-\pi/n})
$$

$$
= \sqrt{\frac{5 + \sqrt{5}}{2}} e^{-\pi n/5} \phi(-e^{-\pi/n}) - \sqrt{\frac{5 - 1}{2\sqrt{n}}} e^{\pi/(5n)} \phi(-e^{-\pi/n}), \quad (9.1.4)
$$

$$
\int_{0}^{\infty} e^{-\pi n x^2} \cosh(2\pi x/\sqrt{5}) + \frac{1}{\sqrt{n}} \psi(-e^{-\pi/n})
$$

$$
= -\sqrt{\frac{5 - \sqrt{5}}{2}} e^{-\pi n/5} \phi(-e^{-\pi/n}) + \sqrt{\frac{5 - 1}{2\sqrt{n}}} e^{\pi/(5n)} \phi(-e^{-\pi/n}), \quad (9.1.5)
$$

where

$$
\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}}.
$$

The identities (9.1.4) and (9.1.5) were first proved by Choi [53].

On page 202 in his Lost Notebook, Ramanujan recorded without proofs two further identities including a Mordell integral.

**Entry 9.1.1 (p. 202).** If $q_1 = e^{-\pi/(3n)}$ and $q = e^{-3\pi n}$, then

$$
\frac{2}{\sqrt{3}} \int_{0}^{\infty} \frac{e^{-\pi n x^2/3} \cos \pi t x}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx
$$

$$
= q^{1/18} \sum_{m=1}^{\infty} \frac{q^{(2m-1)/2} / (q; q^2)_m}{(-e^{-\pi t/q^{1/3}}; q^{2/3})_m (-e^{-\pi t/q^{2/3}}; q^{2/3})_m}
$$

$$
+ \frac{e^{-3\pi t^2/(4n)} q^{1/2}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q^{3(2m-1)/2} / (q; q^6)_m}{(-e^{-\pi t/q^{1/3}}; q^{1/3})_m (-e^{-\pi t/q^{2/3}}; q^{1/3})_m}
$$

$$
= q^{-1/36} \left( \sum_{m=1}^{\infty} (-1)^{m+1} q^{(2m-1)2/4} \right)
$$

$$
\times \left( \frac{1}{1 + e^{\pi t/q^{1/3}}} + \frac{1}{1 + e^{-\pi t/q^{1/3}}} - 1 \right)
$$

$$
+ \frac{e^{-3\pi t^2/(4n)}}{n} \sum_{m=1}^{\infty} (-1)^{m+1} q^{(2m-1)2/4}
$$

$$
\times \left( \frac{1}{1 + e^{\pi t/q^{3(2m-1)}}} + \frac{1}{1 + e^{-\pi t/q^{3(2m-1)}}} - 1 \right).
$$


Proving these identities is equivalent to proving the following two theorems.

**Theorem 9.1.1.** For each \( n \) with positive real part, set \( q = e^{-3\pi n} \) and \( q_1 = e^{-\pi/(3n)} \). For \( t \in \mathbb{C} \), assume that \( \text{Re}(t/n) \not\in \frac{1}{3} \mathbb{Z} \) and \( \text{Re}(t/n) \not\in \frac{2}{3} \mathbb{Z} \). Then

\[
\frac{2}{\sqrt{3}} \int_0^\infty \frac{e^{-\pi nx^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx
= q^{1/18} \sum_{m=1}^{\infty} \frac{q^{(2m-1)^2/6}}{(-e^{\pi t/q^{1/3}}; q^{2/3})_m (-e^{8\pi t/q^{2/3}}; q^{1/3})_m}
+ \frac{e^{-\pi t^2/(4n)} q_1^{1/2}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{3(2m-1)^2/2}}{(-e^{\pi it/n q^{1/3}}; q_1^{3})_m (-e^-\pi it/n q^{1/3}; q_1^{6})_m}.
\]  

(9.1.6)

**Theorem 9.1.2.** For each \( n \) with positive real part, set \( q = e^{-3\pi n} \) and \( q_1 = e^{-\pi/(3n)} \). Then

\[
q^{1/18} \sum_{m=1}^{\infty} \frac{q^{(2m-1)^2/6}}{(-e^{\pi t/q^{1/3}}; q^{2/3})_m (-e^{8\pi t/q^{2/3}}; q^{1/3})_m}
+ \frac{e^{-\pi t^2/(4n)} q_1^{1/2}}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{q_1^{3(2m-1)^2/2}}{(-e^{\pi it/n q^{1/3}}; q_1^{3})_m (-e^-\pi it/n q^{1/3}; q_1^{6})_m}
= q^{-1/36} \frac{(q^{2/3}; q^{2/3})_\infty}{(q^{1/3}; q^{1/3})_\infty} \left\{ \sum_{m=1}^{\infty} \left( -1 \right)^{m+1} q^{(2m-1)^2/4} \times \left( \frac{1}{1 + e^{\pi t/q^{1/3}} q^{(2m-1)/3}} + \frac{1}{1 + e^{-\pi t/q^{1/3}} q^{(2m-1)/3} - 1} \right) + \frac{e^{-3\pi t^2/(4n)}}{n} \sum_{m=1}^{\infty} \left( -1 \right)^{m+1} q_1^{3(2m-1)^2/4} \times \left( \frac{1}{1 + e^{\pi it/n q^{1/3}} q_1^{3(2m-1)}} + \frac{1}{1 + e^{-\pi it/n q_1^{3(2m-1)} - 1}} \right) \right\}. \]

Following Choi [58], define the function \( f(\alpha, z; q) \) by

\[
f(\alpha, z; q) := \sum_{m=0}^{\infty} q^{m^2 - 3mz^2 m} \frac{z^m}{(-z; q)_m (-\alpha z; q; q)_m}.
\]

If we set \( \alpha = z = q \), we see that \( f(q, q; q) \) reduces to Ramanujan’s third order mock theta function \( f(q) := f_0(q) \), defined in (2.1.1). We can rewrite the right-hand side of the equation in Theorem 9.1.1 in terms of \( f(\alpha, z; q) \), namely,

\[
q^{2/9} f(e^{-2\pi t/q^{1/3}}; q^{1/3}; q^{2/3}) + \frac{e^{-3\pi t^2/(4n)} q_1^2}{\sqrt{n}} f(e^{-2\pi it/n q^{1/3}}; e^{\pi it/n q^{1/3}}; q^{1/3}; q_1^{6}).
\]
Ramanujan’s identities also involve the basic hypergeometric series
\[
\sum_{m=1}^{\infty} \frac{q((2m-1)^2/6)}{(-e^{\pi t} q^{1/3}; q^{2/3})_m (-e^{-\pi t} q^{1/3}; q^{2/3})_m}
\]
and
\[
\sum_{m=1}^{\infty} \frac{q^{3(2m-1)^2/2}}{(-e^{\pi t/n} q^{1}; q^{3})_m (-e^{-\pi t/n} q^{3}; q^{1})_m},
\]
which are special cases of the function, in the notation of [78],
\[
g_3(z, q) := \sum_{m=1}^{\infty} \frac{q^m(m-1)}{(z; q)_m(q/z; q)_m}. \quad (9.1.7)
\]
D. Hickerson [86] did not use the function (9.1.7) in his proofs of the mock theta conjectures, but briefly remarked that he could express the mock theta conjectures in terms of \(g_3(z, q)\), Ramanujan’s fifth order mock theta functions, and classical theta functions. Later, B. Gordon and R. McIntosh [78] called this function a universal mock theta function \(g_3(z, q)\) and rewrote the mock theta conjectures in terms of universal mock theta functions. The function \(g_3(z, q)\) also satisfies a modular transformation [78]. With the function \(g_3(z, q)\), we can rewrite the right-hand side of Ramanujan’s first equation (9.1.6) as
\[
q^{2/9} g_3(-e^{\pi t} q^{1/3}, q^{2/3}) + \frac{e^{3\pi t^2/(4n)} q^{2}}{\sqrt{n}} g_3(-e^{\pi t/n} q^{3}; q^{6}).
\]
In his thesis [149], S. Zwegers studied the normalized Appell–Lerch sum, which is defined by
\[
\mu(u, v; \tau) = \frac{1}{f(-e^{2\pi iv}, -e^{2\pi i(\tau - 2\pi iv)})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{\pi im(m+1)\tau + 2\pi imu}}{1 - e^{2\pi im\tau + 2\pi imu}},
\]
where \(u, v \notin Z\tau + Z\) and \(\tau \in \mathcal{H}\). He established symmetry, elliptic transformations, and modular transformations satisfied by Appell-Lerch sums. One of his modular transformations contains the Mordell integral, namely,
\[
\left( \frac{\tau}{i} \right)^{-1/2} e^{\pi i(u-v)^2/\tau} \mu \left( \frac{u}{\tau}, \frac{v}{\tau}, -\frac{1}{\tau} \right) = -\mu(u, v; \tau) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{\pi i x^2 \tau - 2\pi x (u-v)}}{\cosh \pi x} dx.
\]
With these properties, he explained that \(\mu(u, v; \tau)\) behaves nearly like a Jacobi form of weight \(1/2\).

Further papers on transformations of Mordell integrals have been written by B. Chern and R. C. Rhoades [50].

In Section 9.2, we prove Lemmas 9.2.1 and 9.2.2, featuring generalized Lambert series that are Appell–Lerch sums. The transformation for the Appell–Lerch sum in [149] plays a central role in the proofs of Lemmas 9.2.1
and 9.2.2. In Section 9.3, we relate one of Choi’s two proofs of Theorem 9.1.1 [59]. The proof that we provide uses Lemmas 9.2.1 and 9.2.2, Mordell’s formula, the modular transformation for a theta function \( \theta_1 \), and the evaluations of contour integrals. In his second proof of Theorem 9.1.1, Choi begins by first proving the identity

\[
\frac{1}{\sqrt{3}} \int_{-\infty}^{\infty} \frac{e^{\pi i x^2}}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx = g(z; \tau) + \frac{e^{3\pi i z^2/\tau}}{\sqrt{-i\tau}} g \left( \frac{-z}{\tau}; -\frac{1}{\tau} \right)
\]  

(9.1.8)

where

\[
g(z; \tau) := \frac{e^{2\pi i z/3}}{(e^{2\pi i z}; e^{2\pi i \tau})_{\infty}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{3\pi i m(m+1)}}{1 + e^{2\pi i z + 2\pi i (m+1/2)}}. 
\]  

(9.1.9)

To prove (9.1.8), he uses the elliptic transformation properties of \( g(z; \tau) \), the evaluation of certain contour integrals, and Liouville’s theorem. In Section 9.4, we prove Theorem 9.1.2 by using equation (9.2.3) below and some portions in the proof of Theorem 9.1.1.

### 9.2 Two Lemmas

As indicated above, to prove Theorems 9.1.1 and 9.1.2, we require the following lemmas.

**Lemma 9.2.1.** For any complex number \( q \) with \( |q| < 1 \),

\[
\sum_{m=1}^{\infty} \frac{q^{2m(m-1)/3}}{(-tq^{1/3}; q^{2/3})_m (-t^{-1} q^{1/3}; q^{2/3})_m} = \frac{tq^{2/3}}{f(-t^{3/2} q^{4/3}, -t^{-3} q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3} q^{3m}}{1 + q^{2m+1}} \\
+ \frac{t^{-1} q^{2/3}}{f(-t^{-3} q^{4/3}, -t^{3} q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3} q^{-3m}}{1 + q^{2m+1}} \\
+ \frac{(q^2; q^2)_\infty}{(q^{2/3}; q^{2/3})_\infty f(t^3 q, -t^{-3} q)} + \frac{q^{1/3} (q^2; q^2)_\infty f(q^{1/3}, q^{5/3})}{(q^{2/3}; q^{2/3})_\infty f(q, q) f(t^{-3} q, t^3 q)} \times \left( \frac{t^{-2} f(-t^{-3} q^2, -t^3)}{f(-t^{-3} q^{4/3}, -t^3 q^{2/3})} + \frac{t^2 f(-t^3 q^2, -t^{-3})}{f(-t^3 q^{4/3}, -t^{-3} q^{2/3})} \right).
\]  

(9.2.1)

**Proof.** F.G. Garvan [73] showed that, for \( |q| < |z| < |q|^{-1} \) and \( z \neq 1 \),

\[
z^{-1} \left( 1 + \sum_{m=0}^{\infty} \frac{q^m}{(z; q)_{m+1} (q/z; q)_m} \right) = \frac{1}{(q; q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{3m(m+1)/2}}{1 - q^m z}.
\]
Hickerson [86, p. 649] remarked that

\[
z^{-1} \left( -1 + \sum_{m=0}^{\infty} \frac{q^{m^2}}{(z;q)_{m+1}(q/z;q)_m} \right) = \sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(z;q)_m(q/z;q)_m}. \tag{9.2.2}
\]

Then, combining the two results above, we find that

\[
\sum_{m=1}^{\infty} \frac{q^{m(m-1)}}{(z;q)_m(q/z;q)_m} = \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{3m(m+1)/2}}{1 - q^m}, \tag{9.2.3}
\]

which was also derived in [78, p. 104]. Replacing \( q \) and \( z \) by \( q^{2/3} \) and \(-t^{1/3}\), respectively, in (9.2.3), we deduce that

\[
\sum_{m=1}^{\infty} \frac{q^{2m(m-1)/3}}{(-t^{1/3}; q^{1/3})_m(t^{-1}q^{1/3}; q^{2/3})_m} = \frac{1}{(q^{2/3}; q^{2/3})_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1 + q^{(2m+1)/3}t}. \tag{9.2.4}
\]

In his Lost Notebook, Ramanujan recorded the identity [128, p. 59], [21, p. 72, Entries 4.2.1, 4.2.2]

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^m(m+1)}{1 - q^{2m}z} = \frac{(q^2; q^2)_\infty}{(z; q^2)_\infty(q^2/z; q^2)_\infty}, \tag{9.2.5}
\]

which was perhaps first proved by Choi [55, p. 378]. Using (9.2.5) with \( z \) replaced by \(-t^{3/4}q\), and using the Jacobi triple product identity (4.1.2), we deduce that

\[
\begin{align*}
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^m(m+1)}{1 + q^{(2m+1)/3}t} &= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^m(m+1)(1 - q^{(2m+1)/3}t + q^{(4m+2)/3}t^2)}{1 + q^{2m+1}t^3} \\
 &= \frac{(q^2; q^2)_\infty}{f(t^{3/4}q, t^{-3/4})} + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+(7m+2)/3}t^{-2}}{1 + q^{2m+1}t^{-3}} \\
 &\quad + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+(7m+2)/3}t^2}{1 + q^{2m+1}t^3}. \tag{9.2.6}
\end{align*}
\]

The two sums on the right side of (9.2.6) are Appell–Lerch sums. In his thesis [149], Zwegers showed that the normalized Appell–Lerch sums satisfy the identity

\[
\begin{align*}
\frac{z}{f(-hz, -q/(hz))} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)/2}(hz)^m}{1 - q^mtz} &\quad - \frac{1}{f(-h, -q/h)} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)/2}h^m}{1 - q^mt} \\
&= \frac{(q; q)_3^3 f(-htz, -q/(htz)) f(-z, -q/z)}{f(-t, -q/t) f(-h, -q/h) f(-tz, -q/(tz)) f(-hz, -q/(hz))}, \tag{9.2.7}
\end{align*}
\]
where \( q = e^{2\pi i \tau}, \ h = e^{2\pi i \tau}, \ t = e^{2\pi i u}, \) and \( z = e^{2\pi i z'}, \) such that \( \nu, u, z' \notin \mathbb{Z} \) and \( u, \nu, u + z', \nu + z' \notin \mathbb{Z} \tau + \mathbb{Z}. \) Apply (9.2.7) with \( q, \ t, \ h, \) and \( z \) replaced by \( q^2, \ -q, \ t^{-3}q^{4/3}, \) and \( t^3, \) respectively. We also invoke the Jacobi triple product identity (4.1.2) and note from (4.1.3) that \( f(q^{-1/3}, q^{7/3}) = q^{-1/3}f(q^{5/3}, q^{1/3}). \) Hence, we find that

\[
\begin{align*}
\frac{t^3}{(q^{2/3}; q^{2/3})_{\infty}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}}{1 + q^{2m+1}t^3} \\
- \frac{1}{f(-t^{-3}q^{4/3}, -t^3q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}t^{-3m}}{1 + q^{2m+1}} \\
= - \frac{q^{-1/3} f(q^{5/3}, q^{1/3}) f(-t^3, -q^2t^{-3})}{f(q, q) f(-t^{-3}q^{4/3}, -t^{-3}q^{2/3}) f(t^3q, t^{-3}q)(q^{2/3}; q^{2/3})_{\infty}}.
\end{align*}
\]  

(9.2.8)

Similarly, if we apply (9.2.7) with \( q, \ t, \ h, \) and \( z \) replaced by \( q^2, \ -q, \ t^3q^{4/3}, \) and \( t^{-3}, \) respectively, we find that

\[
\begin{align*}
\frac{t^{-3}}{(q^{2/3}; q^{2/3})_{\infty}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}}{1 + q^{2m+1}t^{-3}} \\
- \frac{1}{f(-t^3q^{4/3}, -t^{-3}q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}t^{3m}}{1 + q^{2m+1}} \\
= - \frac{q^{-1/3} f(q^{5/3}, q^{1/3}) f(-t^{-3}, -q^2t^3)}{f(q, q) f(-t^{-3}q^{4/3}, -t^{-3}q^{2/3}) f(t^3q, t^{-3}q)(q^{2/3}; q^{2/3})_{\infty}}.
\end{align*}
\]  

(9.2.9)

We now return to (9.2.6), and for each of the sums on the right-hand side of (9.2.6), we substitute the expressions derived in (9.2.8) and (9.2.9). We now take the newly derived formula for the left-hand side of (9.2.6) and substitute it in (9.2.4). Hence, we deduce that

\[
\sum_{m=1}^{\infty} \frac{q^{2m(m-1)/3}}{(-tq^{1/3}; q^{2/3})_m(-t^{-1}q^{1/3}; q^{2/3})_m} = \frac{(q^2; q^2)_{\infty}^3}{(q^{2/3}; q^{2/3})_{\infty} f(t^3q, t^{-3}q)}
\]

\[
\begin{align*}
&+ \frac{tq^{2/3}}{f(-t^3q^{4/3}, -t^{-3}q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}t^{-3m}}{1 + q^{2m+1}} \\
&- \frac{tq^{1/3} (q^2; q^2)_m (q^{5/3}, q^{1/3}) f(-t^{-3}, -t^3q^2)}{f(q, q) f(-t^{-3}q^{4/3}, -t^{-3}q^{2/3}) f(t^3q, t^{-3}q)(q^{2/3}; q^{2/3})_{\infty}} \\
&+ \frac{t^{-1}q^{2/3}}{f(-t^{-3}q^{4/3}, -t^{-3}q^{2/3})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+7m/3}t^{3m}}{1 + q^{2m+1}} \\
&- \frac{t^{-1}q^{1/3} (q^2; q^2)_m (q^{5/3}, q^{1/3}) f(-t^3, -t^{-3}q^2)}{f(q, q) f(-t^{-3}q^{4/3}, -t^{-3}q^{2/3}) f(t^3q, t^{-3}q)(q^{2/3}; q^{2/3})_{\infty}}.
\end{align*}
\]  

(9.2.10)

We are almost finished with the proof. It remains to combine the two quotients of theta functions on the right-hand side of (9.2.10). To do this, we make two
further applications of (4.1.3), namely,
\[ f(-t^{-3}, -t^3q^2) = -t^{-3}f(-t^{-3}q^2, -t^3), \]
\[ f(-t^3, -t^{-3}q^2) = -t^3f(-t^3q^2, -t^{-3}). \]

Substituting the two identities above into (9.2.10) and simplifying, we complete the proof of Lemma 9.2.1. □

**Lemma 9.2.2.** Let \( \omega = e^{2\pi i/3} \), and let \( q \) be an arbitrary complex number such that \( |q| < 1 \). Then

\[
q^2 \sum_{m=1}^{\infty} \frac{q^{6m(m-1)}}{(-tq^3; q^6)_m (-t^{-1}q^3; q^6)_m} + \frac{i}{\sqrt{3}} \left\{ \frac{1}{f(-\omega^2t^{-1}q^2, -\omega^{-2}t)} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+m} \omega^{2m} t^{-m}}{1 + q^{2m+1}} \right. \\
+ \frac{1}{f(-\omega^2tq^2, -\omega^{-2}t^{-1})} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m^2+m} \omega^{2m} t^m}{1 + q^{2m+1}} - 1 \right\} = -\frac{(q^2 q^6)^3 \infty}{3(q^6 q^6) \infty \omega f(tq, t^{-1}q)} \\
+ \frac{i}{\sqrt{3}} \left( 1 - \omega^2 \right) f(q, q) f(q, q^6) f(tq, t^{-1}q) \\
\times \left( \frac{1}{f(-\omega^2t^{-1}q^2, -\omega t)} - \frac{1}{f(-\omega^2tq^2, -\omega t^{-1})} \right). \\
(9.2.11)
\]

**Proof.** We first consider the left side of (9.2.11); some elementary formulas for theta functions will be necessary in our examination. Employing the Jacobi triple product identity (4.1.2) and (4.1.3), we easily verify that

\[
f(-\omega^2, -\omega q^2) = (1 - \omega^2)(q^6 q^6) \infty, \tag{9.2.12}\\
f(\omega^2 q, \omega q) = (q^2 q^6) \infty (q^6 q^6) \infty, \tag{9.2.13}\\
f(-\omega^2 t^{-1}, -\omega tq^2) = -\omega^2 t^{-1} f(-\omega^2 t^{-1} q^2, -\omega t). \tag{9.2.14}
\]

Employing (9.2.7) with \( q, h, t, \) and \( z \) replaced by \( q^2, \omega^2 t^{-1}, -q, \) and \( t \), respectively, applying (9.2.12)–(9.2.14), and rearranging terms, we find that

\[
\frac{1}{f(-\omega^2 t^{-1}q^2, -\omega t)} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{2m} t^{-m}}{1 + q^{2m+1}} \\
= \frac{1}{(1 - \omega)(q^6 q^6) \infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{2m}}{1 + q^{2m+1} t} \\
+ \frac{(q^2 q^6)^4 f(-t, -t^{-1}q^2)}{(1 - \omega^2) f(q, q) f(q, q^6) f(-\omega^2 t^{-1}q^2, -\omega t) f(tq, t^{-1}q)}. \tag{9.2.15}
\]
Replacing the index \( m \) of summation by \(-m - 1\) and utilizing \((9.2.12)\), we find that

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{2m}}{1 + q^{2m+1} t^{-1}} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{m+1} (q^{2m+1} t + 1 - 1)}{1 + q^{2m+1} t} \\
= \sum_{m=-\infty}^{\infty} (-1)^{m+1} q^{m(m+1)} \omega^m + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{m+1}}{1 + q^{2m+1} t} \\
= -\omega f(-\omega q^2, -\omega^2) + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^m}{1 + q^{2m+1} t} \\
= (1 - \omega)(q^6; q^6)_\infty + \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^m}{1 + q^{2m+1} t}.
\]

(9.2.16)

Apply \((9.2.15)\) with \( t \) replaced by \( t^{-1} \), and then add the two equations together. Using also \((9.2.16)\), we see that

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{2m} t^{-m}}{1 + q^{2m+1}} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} \omega^{2m} t^{-m}}{1 + q^{2m+1}} - 1 \\
= \frac{1}{(1 - \omega)(q^6; q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} (\omega^2 m + \omega^m + 1)}{1 + q^{2m+1} t} \\
+ \frac{(q^6; q^6)^4 f(-t, -t^{-1} q^2)}{(1 - \omega^2) f(q, q) f(q, q^5) f(tq, t^{-1} q^2)} \\
\times \frac{1}{(1 - \omega^2 t^2 q^2, -\omega t) - \frac{t^{-1}}{f(-\omega^2 t^2 q^2, -t^{-1} \omega)}}.
\]

(9.2.17)

We now consider

\[
S := \sum_{m=1}^{\infty} \frac{q^{6m(m-1)+2}}{(-tq^3; q^6)_m (-t^{-1} q^3; q^6)_m} \\
+ \frac{i}{\sqrt{3}(1 - \omega)(q^6; q^6)_\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)} (\omega^2 m + \omega^m + 1)}{1 + q^{2m+1} t}.
\]

(9.2.18)

By elementary calculations, we observe that

\[
\frac{1 + \omega}{1 - \omega} = \frac{i}{\sqrt{3}} \quad \text{and} \quad \frac{\omega^2}{1 - \omega} = -\frac{i}{\sqrt{3}}.
\]

Using these identities, we find that
\[
\frac{1}{1 - \omega} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}(\omega^{2m} + \omega^{m+1})}{1 + q^{2m+1} t} = \frac{i}{\sqrt{3}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{3m^2+3m}}{1 + q^{3m+1} t} \\
+ \frac{2i}{\sqrt{3}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1 + q^{6m+3} t} + \frac{i}{\sqrt{3}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{9m^2+15m+6}}{1 + q^{6m+5} t}.
\] (9.2.19)

Therefore, applying (9.2.3) with \(q\) and \(z\) replaced by \(q^6\) and \(-tq^3\), respectively, (9.2.19), and (9.2.5) with \(z\) replaced by \(-tq\), we deduce from (9.2.18) that

\[
S = \frac{1}{(q^6;q^6)_{\infty}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1 + q^{6m+3} t} - \frac{1}{3(q^6;q^6)_{\infty}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{3m^2+3m}}{1 + q^{6m+1} t} \\
- \frac{2}{3(q^6;q^6)_{\infty}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{9m^2+15m+6}}{1 + q^{6m+5} t} - \frac{1}{3(q^6;q^6)_{\infty}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{9m^2+9m+2}}{1 + q^{6m+3} t} + \frac{i}{\sqrt{3}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{3m^2+3m}}{1 + q^{6m+1} t} \\
- \frac{1}{3(q^6;q^6)_{\infty}} \sum_{m = -\infty}^{\infty} \frac{(-1)^m q^{m^2+m}}{1 + q^{2m+1} t} \\
= - \frac{1}{3(q^6;q^6)_{\infty}} f(tq, t^{-1}q) .
\] (9.2.20)

In conclusion, combining (9.2.17) and (9.2.20), we complete the derivation of Lemma 9.2.2.

\[\Box\]

### 9.3 Proof of Theorem 9.1.1

**Proof of Theorem 9.1.1.** We shall assume that \(t\) and \(n\) are positive real numbers. At the conclusion of the proof, we appeal to analytic continuation. Because the integrand is an even function of \(x\), we can write

\[
\int_{0}^{\infty} \frac{e^{-\pi x^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} dx = \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-\pi x^2/3}(e^{\pi tx} + e^{-\pi tx})}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} dx.
\] (9.3.1)

Since

\[
\frac{1}{e^{2\pi x} + 1 + e^{-2\pi x}} = \frac{e^{2\pi x}}{(e^{2\pi x} + e^{\pi i/3})(e^{2\pi x} + e^{-\pi i/3})} = \frac{1}{i\sqrt{3}} \left( e^{\pi i/3} (e^{2\pi x} + e^{\pi i/3}) - e^{-\pi i/3} (e^{2\pi x} + e^{-\pi i/3}) \right) ,
\]

we can write the integral in (9.3.1) as
\[
\int_0^\infty \frac{e^{-\pi x^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx \\
= \frac{\sqrt{3}}{4i} \left\{ e^{\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-\pi x^2 + 3\pi itx}}{e^{2\pi x} + e^{\pi i/3}} \, dx + e^{\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-\pi x^2 - 3\pi itx}}{e^{2\pi x} + e^{\pi i/3}} \, dx \right. \\
- e^{-\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-\pi x^2 + 3\pi itx}}{e^{2\pi x} + e^{-\pi i/3}} \, dx - e^{-\pi i/3} \left. \int_{-\infty}^{\infty} \frac{e^{-\pi x^2 - 3\pi itx}}{e^{2\pi x} + e^{-\pi i/3}} \, dx \right\}. \tag{9.3.2}
\]

Replacing \( x \) by \( u + it/(2n) \), we find that

\[
\int_{-\infty}^{\infty} \frac{e^{-\pi n x^2 + 3\pi i tx}}{e^{2\pi x} + e^{\pi i/3}} \, dx = e^{-3\pi i^2/(4n)} \int_{-\infty}^{\infty} \frac{e^{-\pi n(u - it/(2n))^2}}{e^{2\pi x} + e^{\pi i/3}} \, dx \\
= e^{-3\pi i^2/(4n)} \int_{-\infty}^{\infty} \frac{e^{-3\pi n u^2}}{e^{2\pi(u - it/(2n))} + e^{\pi i/3}} \, du.
\]

Similar formulas can be derived for each of the three remaining integrals in (9.3.2). Thus,

\[
\int_0^\infty \frac{e^{-\pi n x^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx \\
= \frac{\sqrt{3}}{4i} e^{-3\pi i^2/(4n)} \left\{ e^{\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi n u^2}}{e^{2\pi(u - it/(2n))} + e^{\pi i/3}} \, du \\
+ e^{\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi n u^2}}{e^{2\pi(u + it/(2n))} + e^{\pi i/3}} \, du \\
- e^{-\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi n u^2}}{e^{2\pi(u - it/(2n))} + e^{-\pi i/3}} \, du \\
- e^{-\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi n u^2}}{e^{2\pi(u + it/(2n))} + e^{-\pi i/3}} \, du \right\}. \tag{9.3.3}
\]

For a sufficiently large positive number \( N \), we consider the integral

\[
\int_{C_N} \frac{e^{-3\pi n u^2}}{e^{2\pi(u + it/(2n))} + e^{\pi i/3}} \, du,
\]

taken around the positively oriented rectangle \( C_N \) with vertices at \( \pm N \) and \( \pm N - it/(2n) \). The integrand has simple poles at \( i(-t/(2n) + 2/3 + k) \) for each integer \( k \). However, if we choose \( n \) sufficiently large, then none of these poles lie inside or on the contour \( C_N \). (At the end of our proof, we can remove this stipulation by analytic continuation.) Hence, by Cauchy’s theorem,

\[
\int_{C_N} \frac{e^{-3\pi n u^2}}{e^{2\pi(u + it/(2n))} + e^{\pi i/3}} \, du = 0.
\]
It is easy to see that the integrals along the vertical sides of \( C_N \) tend to 0 as \( N \to \infty \). Therefore, we conclude that

\[
\int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u+it/(2n))} + e^{\pi i/3}} \, du = \int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} + e^{\pi i/3}} \, du. \tag{9.3.4}
\]

Clearly, we can establish similar results for the other three integrals in (9.3.3).

Using (9.3.4) and the aforementioned three analogues, we find that

\[
\int_{0}^{\infty} \frac{e^{-\pi x^2/3} \cos \pi t x}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx = \frac{\sqrt{3}}{4i} e^{-3\pi t^2/(4n)}
\times \left\{ e^{\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u+it/(2n))} + e^{\pi i/3}} \, du + e^{-3\pi nu^2} \right\}
\times \left\{ e^{-\pi i/3} \int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} + e^{-\pi i/3}} \, du \right\}. \tag{9.3.5}
\]

To evaluate the four integrals on the right-hand side of (9.3.5), we turn to Mordell’s evaluation (9.1.1). We first consider the first integral on the right side of (9.3.5). Replacing \( \tau, x, \) and \( \theta \) by \( 3in, 0, \) and \( 2/3 - t/(2n) \), respectively, in Mordell’s formula (9.1.1), we find that

\[
\int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u+it/(2n))} - e^{4\pi i/3}} \, du = e^{-\pi it/n} \int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} - e^{4\pi i/3}} \, du
\]

\[
= e^{3\pi t^2/(4n)} - 2\pi t - 4\pi i/3 q^{-4/9} F\left(\frac{2}{3} - \frac{t}{2n}, -\frac{1}{3n}\right) - 3n F(2in - \frac{3it}{2}, 3in)
\]

\[
3in \theta_{11}(2in - \frac{3it}{2}, 3in).
\]

The remaining three integrals in (9.3.5) are similarly established, and we find that

\[
\int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} - e^{4\pi i/3}} \, du
\]

\[
= e^{3\pi t^2/(4n) + 2\pi t - 4\pi i/3 q^{-4/9} F\left(\frac{2}{3} - \frac{1}{2n}, -\frac{1}{3n}\right) - 3n F(2in + \frac{3it}{2}, 3in)}
\]

\[
3in \theta_{11}(2in + \frac{3it}{2}, 3in),
\]

\[
\int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} - e^{4\pi i/3}} \, du
\]

\[
= e^{3\pi t^2/(4n) - \pi t - 2\pi i/3 q^{-1/9} F\left(\frac{1}{3} - \frac{1}{2n}, -\frac{1}{3n}\right) - 3n F(in - \frac{3it}{2}, 3in)}
\]

\[
3in \theta_{11}(in - \frac{3it}{2}, 3in),
\]

\[
\int_{-\infty}^{\infty} \frac{e^{-3\pi nu^2}}{e^{2\pi(u-it/(2n))} - e^{4\pi i/3}} \, du
\]

\[
= e^{3\pi t^2/(4n) + \pi t - 2\pi i/3 q^{-1/9} F\left(\frac{1}{3} + \frac{1}{2n}, -\frac{1}{3n}\right) - 3n F(in + \frac{3it}{2}, 3in)}
\]

\[
3in \theta_{11}(in + \frac{3it}{2}, 3in).
\]
Using these four evaluations in (9.3.5), we deduce that

\[
\int_0^\infty \frac{e^{-\pi n^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx
\]
\[
= -\frac{\sqrt{3}}{4i} \left\{ -e^{-2\pi t_q^{-4/9}} F\left(\frac{2}{3} - \frac{t}{2n}, -\frac{1}{3n}\right) - 3n \frac{F(2in - \frac{3it}{2}, 3in)}{3in \theta_{11}(2in - \frac{3it}{2}, 3in)} 
+ e^{2\pi t_q^{-4/9}} F\left(\frac{2}{3} + \frac{t}{2n}, -\frac{1}{3n}\right) - 3n \frac{F(2in + \frac{3it}{2}, 3in)}{3in \theta_{11}(2in + \frac{3it}{2}, 3in)} 
- e^{-\pi t_q^{-1/9}} F\left(\frac{1}{3} - \frac{t}{2n}, -\frac{1}{3n}\right) - 3n \frac{F(in - \frac{3it}{2}, 3in)}{3in \theta_{11}(in - \frac{3it}{2}, 3in)} 
- e^{\pi t_q^{-1/9}} F\left(\frac{1}{3} + \frac{t}{2n}, -\frac{1}{3n}\right) - 3n \frac{F(in + \frac{3it}{2}, 3in)}{3in \theta_{11}(in + \frac{3it}{2}, 3in)} \right\}. 
\]

(9.3.6)

We want to simplify (9.3.6). First, it is easily checked that \( F(x, \tau) \) and \( \theta_{11}(x, \tau) \) satisfy the transformation formulas

\[
\begin{align*}
\theta_{11}(x, \tau) &= -\theta_{11}(x + 1, \tau) = -\theta_{11}(-x, \tau) = -e^{\pi t(x+\tau)} \theta_{11}(x + \tau, \tau), \\
F(x, \tau) &= -F(x + 1, \tau) = -F(x, \tau + 1) + \theta_{11}(x, \tau) \\
&= -F(-x, \tau, \tau) = F(-x, \tau) + \theta_{11}(x, \tau).
\end{align*}
\]

(9.3.7)

Second, recall that \( \theta_{11} \) satisfies the transformation formula [145, p. 475]

\[
\theta_{11}\left(\frac{x}{\tau}, -\frac{1}{\tau}\right) = -i \sqrt{-i\tau} \frac{e^{ix^2/\tau}}{e^{x^2/\tau}} \theta_{11}(x, \tau). 
\]

(9.3.8)

In particular, we need two special cases of (9.3.8), namely,

\[
\begin{align*}
\theta_{11}\left(2in - \frac{3it}{2}, 3in\right) &= \frac{i}{\sqrt{3n}} e^{\pi n/3 - 2\pi t + 3\pi t^2/(4n)} \theta_{11}\left(\frac{2}{3} - \frac{t}{2n}, -\frac{1}{3n}\right), \\
\theta_{11}\left(in - \frac{3it}{2}, 3in\right) &= \frac{i}{\sqrt{3n}} e^{\pi n/3 - \pi t + 3\pi t^2/(4n)} \theta_{11}\left(\frac{1}{3} - \frac{t}{2n}, -\frac{1}{3n}\right).
\end{align*}
\]

(9.3.9)

(9.3.10)

Employing (9.3.9), (9.3.10), and (9.3.7) in (9.3.6), we find that

\[
\int_0^\infty \frac{e^{-\pi n^2/3} \cos \pi tx}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} \, dx
\]
\[
= -\frac{\sqrt{3}}{4} \left\{ -e^{-2\pi t_q^{-4/9}} F\left(2in - \frac{3it}{2}, 3in\right) \theta_{11}(2in - \frac{3it}{2}, 3in) + e^{2\pi t_q^{-4/9}} F\left(2in + \frac{3it}{2}, 3in\right) \theta_{11}(2in + \frac{3it}{2}, 3in) 
- e^{-\pi t_q^{-1/9}} F\left(in - \frac{3it}{2}, 3in\right) \theta_{11}(in - \frac{3it}{2}, 3in) - e^{\pi t_q^{-1/9}} F\left(in + \frac{3it}{2}, 3in\right) \theta_{11}(in + \frac{3it}{2}, 3in) \right\}
\]
Thus, (9.3.11) can be transformed into the shape
\[
\frac{2}{\sqrt{3}} \int_0^\infty e^{-\pi x^2/3} \cos \pi t x \, dx
= \frac{e^{-\pi t/3}}{2} \frac{8/9}{f(-e^{-3\pi t} q^{4/3}, -e^{-3\pi t} q^{2/3})} \sum_{m=-\infty}^\infty \frac{(-1)^m q^{m^2 + 7m/3} e^{3m \pi t}}{1 + q^{2m+1}}
+ \frac{e^{-\pi t/3}}{2} \frac{8/9}{f(-e^{-3\pi t} q^{4/3}, -e^{-3\pi t} q^{2/3})} \sum_{m=-\infty}^\infty \frac{(-1)^m q^{m^2 + 7m/3} e^{-3m \pi t}}{1 + q^{2m+1}}
\times \left\{ \frac{1}{f(-\omega, e^{-\pi it/n}, q_1^{2/3}, -\omega^{-2} e^{-\pi it/n})} \sum_{m=-\infty}^\infty \frac{(-1)^m q_1^{m^2 + m + 2m \omega^m e^{m \pi it/n}}}{1 + q_1^{2m+1}} - 1 \right\}.
\]
Now take the new right-hand sides of (9.2.1) and (9.2.11) and add them together. From (4.1.3), $f(-e^{3\pi t} q^2, -e^{-3\pi t}) = -e^{-3\pi t} f(-e^{3\pi t}, -e^{-3\pi t} q^2)$, and so with the use of this identity, we find that this sum equals

$$R := q^{2/9} \left\{ \frac{(q^2;q^2)_\infty^3}{(q^2/3;q^2/3)_\infty} f(e^{3\pi t} q, e^{-3\pi t} q) + (q^{1/3};q^{2/3})_\infty f(q^{1/3}, q^{5/3}) f(-e^{-3\pi t} q^2, -e^{3\pi t}) f(-e^{-3\pi t} q^2, -e^{3\pi t}) \right\}$$

$$+ e^{-3\pi t^2/(4n)} \left\{ \frac{q^{2/3}(q_1^2;q_1^2)_\infty}{1/q^{2/3}} f(-q^{2/3}, -e^{-3\pi t/3}) - f(-e^{-3\pi t} q^2, -e^{3\pi t} q^2/3) \right\}$$

We prove that $R$ is identically equal to zero.

Recall that $q_1 = e^{-\pi i/\tau}$ and $q = e^{\pi i \tau}$. If we translate (9.3.8) via (9.3.12), we find that

$$f(-q^2 e^{2\pi i x/\tau}, -e^{-2\pi i x/\tau}) = -i \sqrt{-1} e^{e^{i\pi x} x^2 \tau - 2\pi i/\tau} (q/q_1)^{1/4} f(-q^2 e^{2\pi i x}, -e^{-2\pi i x})$$

(We remark that in his notebooks [127], [28, p. 36], Ramanujan wrote (9.3.8) or (9.3.16) in the equivalent form

$$\sqrt{\alpha} f(e^{-\alpha^2 + n\alpha}, e^{-\alpha^2 - n\alpha}) = e^{n^2/4} \sqrt{\beta} f(e^{-\beta^2 + in\beta}, e^{-\beta^2 - in\beta}),$$
where $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \pi$, and $\text{Re} \alpha^2 > 0$.) From (9.3.16), with $\tau = 3i$, we can easily deduce that
\[
f(-e^{2\pi i}q^2, -e^{-2\pi i}) = \frac{i}{\sqrt{3n}} q^{-1/4} e^{-\pi i x - \pi (x-1/2)^2 / (3n)} f(-e^{2\pi i x / (3n)} q_1^3, -e^{-2\pi i x / (3n)}).
\] (9.3.17)

Since
\[
\lim_{x \to 0} \frac{1 - e^{-2\pi i x / (3n)}}{1 - e^{-2\pi i x}} = -\frac{i}{3n},
\]
dividing both sides of (9.3.17) by $1 - e^{-2\pi i x}$ and letting $x$ tend to 0, we find that
\[
(q^2; q^2)_\infty = \frac{1}{3n}\sqrt{3n} q^{-1/4} q_1^{1/4} (q_1^2; q_1)_\infty.
\] (9.3.18)

(The identity (9.3.18) is simply the 'cube' of the transformation formula for the Dedekind eta-function, which Ramanujan recorded more than once in his notebooks [127], [28, p. 43, Entry 27(iii)].) Applying (9.3.18), (9.3.18) a second time with $q$ replaced by $q^{1/3}$, and (9.17) with $x = \frac{1}{x} - \frac{3}{x} i n - \frac{3}{2} i t$, we find that
\[
\frac{q^{2/3} (q^2; q^2)_\infty}{(q^{2/3}; q^{2/3})_\infty} f(e^{3\pi i t} q, e^{-3\pi i t} q) = \frac{e^{-3\pi i t^2 / (4n)}}{\sqrt{n}} \frac{3(q_1^6, q_1^6)_\infty}{3(q_1^6; q_1^6)_\infty} f(e^{i\pi i t / n} q_1, e^{-i\pi i t / n} q_1).
\] (9.3.19)

Apply (9.3.17) with $x$ replaced by $\frac{1}{2} - \frac{1}{2} i n, \frac{3}{2} i t, \frac{1}{2} - \frac{3}{2} i n, \frac{3}{2} i t - i n$, and $\frac{3}{2} i t - i n$, respectively. We furthermore use (9.2.13) to deduce (9.3.20) and (4.1.3) to deduce (9.3.21). Accordingly, we find that
\[
f(q^{5/3}, q^{1/3}) = \frac{1}{\sqrt{3n}} q^{-1/9} f(-e^{-\pi i / 3} q_1, -e^{\pi i / 3} q_1)
\]
\[
= \frac{1}{\sqrt{3n}} q^{-1/9} (q_1^{2/3}; q_1^{2/3})_\infty (q_1^6; q_1^6)_\infty f(q_1, q_1),
\] (9.3.20)

\[
f(-e^{-3\pi i t} q^2, -e^{3\pi i t}) = -\frac{i q^{-1/4} q_1^{1/4} e^{3\pi i t / 2 + 3\pi i t^2 / (4n) - i\pi i t / (2n)}}{\sqrt{3n}}
\[
\times f(-e^{i\pi i t / n} q_1, -e^{-i\pi i t / n} q_1^2),
\] (9.3.21)

\[
f(q, q) = \frac{1}{\sqrt{3n}} f(q_1, q_1),
\] (9.3.22)

\[
f(-e^{3\pi i t} q^{4/3}, -e^{-3\pi i t} q^{2/3}) = \frac{i q^{-1/36} q_1^{1/4} e^{-\pi i / 3 - \pi i t / 2 + 3\pi i t^2 / (4n) - i\pi i t / (2n)}}{\sqrt{3n}}
\[
\times f(-e^{i\pi i t / n} \omega q_1^2, -e^{-i\pi i t / n} \omega),
\] (9.3.23)

\[
f(-e^{-3\pi i t} q^{4/3}, -e^{3\pi i t} q^{2/3}) = \frac{i q^{-1/36} q_1^{1/4} e^{-\pi i / 3 + \pi i t / 2 + 3\pi i t^2 / (4n) + i\pi i t / (2n)}}{\sqrt{3n}}
\]
Employing (9.3.19)–(9.3.24) and using the fact that $e^{\pi i/3} = \sqrt{3}i/(1 - \omega^2)$, we conclude that

\[
\frac{q^{5/9}(q^2; q^2)_\infty^3 f(q^{1/3}, q^{5/3}) f(-e^{3\pi t} q^2, -e^{3\pi t})}{(q^{2/3}; q^2)_\infty^3 f(q, q) f(e^{-3\pi t} q, e^{3\pi t} q)} 
\times e^{-\pi t} \frac{e^{-\pi t}}{f(-e^{3\pi t} q^{4/3}, -e^{3\pi t} q^{2/3}) - f(-e^{3\pi t} q^{4/3}, -e^{3\pi t} q^{2/3})}
\times \frac{-e^{3\pi t^2/(4n)} \sqrt{n} (1 - \omega^2) f(q, q) f(q^{1/3}, q^{1/3}) f(e^{\pi t/n} q, e^{-\pi t/n} q)}{f(-\omega^2 e^{-\pi t/n} q, -\omega e^{\pi t/n} q) - f(-\omega^2 e^{-\pi t/n} q, -\omega e^{\pi t/n} q)}. 
\] (9.3.25)

If we insert (9.3.25) and (9.3.19) into (9.3.15), we conclude that $R = 0$. Thus, from (9.3.14), $L = 0$. Therefore, comparing (9.3.13) and (9.3.14), we see that we have proved Theorem 9.1.1.

Choi [59] gives a second proof of Theorem 9.1.1. He shows that both the left and right sides of (9.1.6) satisfy the same functional equation. He then considers the difference of the left and right sides and proves that it is a bounded entire function. Using Liouville’s theorem, he shows that this difference equals 0.

9.4 Proof of Theorem 9.1.2

Proof of Theorem 9.1.2. Using (9.2.3) with $t$ replaced by $-e^{\pi t}$ and $q$ replaced by $q^{2/3}$, we find that

\[
q^{2/9} \sum_{m=1}^\infty \frac{q^{2m(m-1)/3}}{(-e^{\pi t} q^{1/3}; q^{2/3})_m (-e^{\pi t} q^{1/3}; q^{2/3})_m}
= \frac{q^{2/9}}{(q^{2/3}; q^{2/3})_\infty} \sum_{m=-\infty}^\infty \frac{(-1)^m q^{m(m+1)}}{1 + q^{2m+1+1/3} e^{\pi t}}. 
\] (9.4.1)

Employing (9.2.3) with $t$ replaced by $-e^{\pi it/n}$ and $q$ replaced by $q_1^6$, we deduce that

\[
\frac{e^{-3\pi t^2/(4n)} q_1^2}{\sqrt{n}} \sum_{m=1}^\infty \frac{q_1^{6m(m-1)}}{(-e^{\pi t/n} q_1; q_1)_m (-e^{\pi t/n} q_1; q_1)_m}
= \frac{e^{-3\pi t^2/(4n)} q_1^2}{(q_1^6; q_1^6)_\infty \sqrt{n}} \sum_{m=-\infty}^\infty \frac{(-1)^m q_1^{6m+2+3m}}{1 + q_1^{6m+3} e^{\pi it/n}}. 
\] (9.4.2)
Replacing \( m \) by \( m - 1 \) and \( m \) by \(-m\), respectively, in the two sums on the right-hand side in the first step below, we find that

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1 + q^{(2m+1)/3} e^{\pi it}} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)}}{1 + q^{(2m+1)/3} e^{\pi it}} \quad (9.4.3)
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1 + q^{(2m-1)/3} e^{\pi it}} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m-1)}}{1 + q^{(2m-1)/3} e^{\pi it}}
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1 + q^{(2m-1)/3} e^{\pi it}} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m-1)}(q^{(2m-1)/3} e^{-\pi it} + 1 - 1)}{1 + q^{(2m-1)/3} e^{-\pi it}}
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{m(m-1)}}{1 + q^{(2m-1)/3} e^{\pi it}} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m-1)}}{1 + q^{(2m-1)/3} e^{-\pi it}} - \sum_{m=1}^{\infty} (-1)^{m+1} q^{m(m-1)}.
\]

With \( q \) replaced by \( q_1^3 \) and \( e^{\pi t} \) replaced by \( e^{\pi it/n} \) in (9.4.3), we deduce that

\[
\sum_{m=-\infty}^{\infty} \frac{(-1)^m q_1^{m^2+9m}}{1 + q_1^{6m+3} e^{\pi it/n}} \quad (9.4.4)
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q_1^{m^2-9m}}{1 + q_1^{6m-3} e^{\pi it/n}} + \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{m^2-9m}}{1 + q_1^{6m-3} e^{-\pi it/n}} - \sum_{m=1}^{\infty} (-1)^{m+1} q_1^{m^2-9m}.
\]

In (9.3.18), if we replace \( q \) by \( q^{1/3} \), so that \( q_1 \) is replaced by \( q_1^3 \) and \( n \) is replaced by \( \frac{1}{3} \), we find that

\[
(q^{2/3}; q^{2/3})_{\infty} = \frac{1}{\sqrt{n}} q^{-1/36} q_1^{1/4} (q_1^6; q_1^6)_{\infty}.
\]

Thus,

\[
\frac{q_1^2}{\sqrt{n} (q_1^6; q_1^6)_{\infty}} = \frac{q^{-1/36} q_1^{9/4}}{(q^{2/3}; q^{2/3})_{\infty} \sqrt{n}} \quad (9.4.5)
\]

We now add (9.4.1) and (9.4.2). Then we substitute (9.4.3) and (9.4.4) on the right-hand side of our sum. Lastly, in order to complete the proof of Theorem 9.1.2, we need to use (9.4.5) \( \square \)

In conclusion, we remark that Choi [59] employs Theorem 9.1.1 to derive several transformation formulas for third order mock theta functions. For example,

\[
2 \int_{-\infty}^{\infty} \frac{e^{-\pi n x^2/3} \cos \pi n x e^{\pi n x}}{e^{2\pi x/3} + 1 + e^{-2\pi x/3}} dx = q^{2/9} \left( 1 - \frac{1}{2} f(q) \right) + \frac{q^{1/4} q_1^2}{\sqrt{n}} \omega(q_1^3),
\]

where \( f(q) \) and \( \omega(q) \) are the usual third order mock theta functions.
10

Ramanujan’s Last Letter to Hardy

10.1 Introduction

In one of the most enigmatic, important, and influential mathematical letters in the history of mathematics, written to Hardy on January 12, 1920, Ramanujan mistily described his theory of mock theta functions [34, pp. 220–223]. For nearly a century, mathematicians have been attempting to penetrate into the meanings of the letter’s assertions. Although enormous progress has been made in fitting Ramanujan’s claims into contemporary mathematics, a dark certain has not allowed us to view what is really on stage, i.e., Ramanujan’s thoughts behind his discoveries. In this chapter, we examine the claims made in Ramanujan’s last letter to Hardy and relate the remarkable discoveries that have emerged, especially in recent years.

10.2 Radial Limits

Ramanujan wrote [34, p. 221],

I have proved that if

\[
f(q) = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)(1 + q^2)^2} + \cdots
\]

then

\[
f(q) + (1 - q)(1 - q^3)(1 - q^5) \cdots (1 - 2q + 2q^4 - 2q^8 + \cdots) = O(1)
\]

at all the points \( q = -1, q^3 = -1, q^5 = -1, q^7 = -1, \ldots \), and at the same time

\[
f(q) - (1 - q)(1 - q^3)(1 - q^5) \cdots (1 - 2q + 2q^4 - \cdots) = O(1)
\]
at all the points \( q^2 = -1, q^4 = -1, q^6 = -1, \ldots \). Also obviously \( f(q) = O(1) \) at all the points \( q = 1, q^3 = 1, q^5 = 1, \ldots \). And so \( f(q) \) is a Mock \( \theta \) function. When \( q = -e^{-t} \) and \( t \to 0 \)

\[
f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4. \tag{10.2.1}
\]

The coefficient of \( q^n \) in \( f(q) \) is

\[
(-1)^{n-1} \frac{\exp\left(\frac{\pi}{2}\sqrt{\frac{n^2}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2} \pi \sqrt{\frac{n^2}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}}\right). \tag{10.2.2}
\]

It is inconceivable that a simple \( \vartheta \) function could be found to cut out the singularities of \( f(q) \).

The function \( f(q) \) is Ramanujan’s famous third order mock theta function, which we have denoted by (1.3.2). The two assertions immediately following the definition of \( f(q) \) were first proved by A. Folsom, K. Ono, and R.C. Rhoades [70], [71]. In fact, they proved much more, as we now relate.

Define

\[
R(w; q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n(q/w; q)_n}, \tag{10.2.3}
\]

\[
U(w; q) := \sum_{n=0}^{\infty} (wq; q)_n(q/w; q)_nq^{n+1}, \tag{10.2.4}
\]

\[
C(w; q) := \frac{(q; q)_\infty}{(wq; q)_\infty(q/w; q)_\infty}. \tag{10.2.5}
\]

We note that if \( w = -1 \), then

\[
R(-1; q) = f(q), \tag{10.2.6}
\]

\[
U(-1; q) = \sum_{n=0}^{\infty} (-q; q)_n^2q^{n+1} =: u(q), \tag{10.2.7}
\]

\[
C(-1; q) = \frac{(q; q)_\infty}{(-q; q)_\infty} = \frac{\varphi(-q)}{(-q; q)_\infty} =: b(q), \tag{10.2.8}
\]

by (2.1.14). We also note that \( R(w; q) \) is the generating function for ranks of partitions, and that \( C(w; q) \) is the generating function for cranks of partitions.

**Theorem 10.2.1** ([70], [71]). Let \( a, b, h, \) and \( m \) be integers such that \( 1 \leq a < b, 1 \leq h < m, \) \( \gcd(a, b) = \gcd(h, m) = 1, \) and \( b|m. \) Let \( h' \) denote an inverse of \( h \) modulo \( m, \) i.e., \( hh' \equiv 1 (\text{mod } m). \) Then, if \( \zeta_m \) is an \( m \)th root of unity, as \( q \) radially approaches \( \zeta_m^h \) within the unit disc,

\[
\lim_{q \to \zeta_m^h} \left\{ R(\zeta_m^a; q) - \zeta_m^h \frac{a^2}{2} C(\zeta_m^a; q) \right\} = -(1 - \zeta_m^a)(1 - \zeta_m^{-a})U(\zeta_m^a; \zeta_m^h). \tag{10.2.9}
\]
Using the notation (10.2.6)–(10.2.8), we state separately Ramanujan’s claim, which is the special case $a = 1$, $b = 2$, and $m = 2k$, where $k$ is a positive integer.

**Theorem 10.2.2.** If $\zeta$ is a primitive root of unity of order $2k$, then as $q$ radially approaches $\zeta$ within the unit disc,

$$\lim_{q \to \zeta} \{f(q) - (-1)^k b(q)\} = -4u(\zeta). \quad (10.2.10)$$

The expression immediately to the right of $f(q)$ in (10.2.1) arises from the transformation formulas for the theta functions comprising $b(q)$. W. Zudilin [148] discovered a short proof of (10.2.10) based on theorems in Ramanujan’s lost notebook, and so we will provide his proof here.

**Proof.** We begin by recalling a special case of a result found on page 15 in the lost notebook [128], namely [20, p. 67, Entry 3.4.7],

$$R(w; q) + (1 - w)(1 - 1/w)U(w; q) = \frac{1 - 1/w}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-w)^n q^{n(n+1)/2}}{1 - q^n/w}. \quad (10.2.11)$$

Second, we rewrite (1.3.17), which we proved in [19, p. 264, Entry 12.2.2], in the form

$$C(w; q) = \frac{1 - 1/w}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - q^n/w}. \quad (10.2.12)$$

From (10.2.6), (10.2.7), and (10.2.11), we see that

$$f(q) + 4u(q) = R(-1; q) + 4U(-1; q) = \frac{2}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1 + q^n}. \quad (10.2.13)$$

Furthermore, from (10.2.8) and (10.2.12), we find that

$$b(q) = C(-1, q) = \frac{2}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}. \quad (10.2.14)$$

From (10.2.13) and (10.2.14), we can then deduce that

$$f(q) + 4u(q) - b(q) = \frac{4}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(2n-1)}}{1 + q^{2n-1}}, \quad (10.2.15)$$

$$f(q) + 4u(q) + b(q) = \frac{4}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(2n+1)}}{1 + q^{2n}}. \quad (10.2.16)$$

Moreover [19, p. 264, equations (12.2.7), (12.2.6)],
Thus, we can recast (10.2.15) and (10.2.16) in the forms

\[
\frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(2n+1)}}{1 + q^{2n}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n(q; q^2)_n q^{n^2}}{(-q^2; q^2)_n^2},
\]

\[
\frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(2n-1)}}{1 + q^{2n}} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(q; q^2)_{n-1} q^{n^2}}{(-q^2; q^2)_{n-1}^2}.
\]

We now observe that as \(q \to \zeta\) radially, the infinite products on the right-hand side of (10.2.17) and (10.2.18) each tend to 0, while the infinite series on the right-hand sides each tend to a finite number when \(k\) is even and odd, respectively. This completes the proof of Theorem 10.2.2. \(\square\)

We see that Appell–Lerch basic bilateral series play a key role in Zudilin’s proof of Theorem 10.2.2. E. Mortenson observed that results that he had established on Appell–Lerch series [116] could be utilized to prove further results of this sort [117]. Since many mock theta functions are special cases of the ‘universal’ mock theta function \(g_2(x; q)\), defined in (7.4.5), R. Rhoades [132] asked how one can explicitly determine modular forms \(f_{a,b,A,B,h,k}\) in a uniform way such that

\[
g_2(\zeta_k^a q^A; q^B) - f_{a,b,A,B,h,k}(q) = O(1),
\]

as \(q \to \zeta_k\) radially, where \(\zeta_k\) is a \(k\)th root of unity. He further asked how one can determine the constants

\[
Q_{a,b,A,B,h,k} := \lim_{q \to \zeta_k} \left\{ g_2(\zeta_k^n q^A; q^B) - f_{a,b,A,B,h,k}(q) \right\}.
\]

K. Bringmann and L. Rolen completely answered Rhoades’s queries in [46].

### 10.3 Formulas for the Taylor Series Coefficients of \(f(q)\)

We now discuss the assertion in (10.2.2). L. Dragonette [65] first proved that if

\[
f(q) =: \sum_{n=0}^{\infty} a(n) q^n,
\]

then
10.3 Formulas for the Taylor Series Coefficients of $f(q)$

$$a(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \frac{(-1)^{(k+1)/2} A_{2k}(n - k(1 + (-1)^k)/4) \sinh \left( \frac{\pi}{k} \sqrt{\frac{n}{6} - \frac{1}{144}} \right)}{\sqrt{k(n - \frac{1}{24})}} + O(n^{\frac{1}{2} + \epsilon}),$$

(10.3.2)

for each $\epsilon > 0$, where $A_k(n)$ denotes the same sum that appears in the Hardy–Ramanujan–Rademacher formula for the partition function $p(n)$, i.e.,

$$A_k(n) = \sum_{h \equiv 0 \pmod{k}, \gcd(h,k)=1} \omega_{h,k} e^{-2\pi i h n/k},$$

where $\omega_{h,k}$ is a certain $24k$th root of unity. See [8, pp. 70–71] for a more precise definition of $\omega_{h,k}$ and the complete formula for $p(n)$. Dragonette suggested that the error term was much smaller than what is indicated in (10.3.2) and that it may be that when the main sum is rounded off to the nearest integer, the sum would give the value of $a(n)$ exactly. Andrews [2] improved Dragonette’s result by replacing the error term by $O(n^{\epsilon})$, for each $\epsilon > 0$. He furthermore conjectured that the series in (10.3.2) actually converges, i.e., that the error term is $o(1)$, as $n \to \infty$. Readers should consult Andrews’ paper [16], which also contains numerical evidence.

The conjecture of Dragonette and Andrews was proved in 2007 by K. Bringmann and K. Ono [39]. They constructed a weak Maass–Poincaré series with holomorphic part $q^{-1}f(q)$. As a consequence, in finding a formula analogous to the exact Hardy–Ramanujan–Rademacher exact formula for $p(n)$, they obtain an exact formula for $a(n)$ and consequently exact formulas for $N_o(n)$ and $N_e(n)$, the number of partitions of $n$ with odd rank and the number of partitions of $n$ of even rank, respectively.
11

Euler Products in Ramanujan’s Lost Notebook

11.1 Introduction

In his famous paper, *On certain arithmetical functions* [125], Ramanujan offers for the first time the Euler product for what is now known as Ramanujan’s Dirichlet series. More precisely, if \( \tau(n) \) denotes Ramanujan’s tau-function, then [125, p. 153]

\[
\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{1-2s}}, \quad \sigma = \text{Re } s > \frac{13}{2},
\]

(11.1.1)

where the product is over all primes \( p \). It should be emphasized that (11.1.1) is prefaced by the words, “For it appears that”. Thus, at the time he wrote [125], Ramanujan did not have a proof of (11.1.1), and we are uncertain if he later devised a proof or not. In the same article, Ramanujan states further Euler products for the Dirichlet series associated with special cases of

\[
q\{(1 - q^{24/\alpha})(1 - q^{48/\alpha})(1 - q^{72/\alpha}) \cdots \}^\alpha =: \sum_{n=1}^{\infty} \psi_\alpha(n)q^n, \quad \alpha \mid 24,
\]

for which he also does not provide proofs. Proofs were given by Mordell in 1917 [111, p. 121].

Published with the lost notebook is a more complete list of Ramanujan’s discoveries about such Euler products [128, pp. 233–235]. In particular, in his paper [125], Ramanujan examines only Euler products corresponding to powers of the Dedekind eta function, while in the manuscript in [128], Ramanujan examines Dirichlet series arising from powers of the eta function multiplied by certain Eisenstein series. This list of 46 modular forms with their corresponding Euler products is examined in the last section of the present paper. It is to be emphasized that in this list, the associated modular form contains a power of only one eta function.
At scattered places in his lost notebook, Ramanujan offers further examples of Euler products that we relate below. Most of these were claimed to have been proven by S.S. Rangachari [129] using the theory of modular forms, but his proofs are incomplete, and he failed to notice that several of Ramanujan’s claims need corrections. See also Rangachari’s paper [130] for a discussion of some of these results. S. Raghavan [122] disproved one of Ramanujan’s claims, but did not offer a corrected version. Being in the original lost notebook, these scattered results were most likely discovered during the last year of Ramanujan’s life, after he had returned to India. Thus, from time to time, he was clearly seeking further theorems along the lines of what he wrote in [125], and it is unfortunate that he did not live longer to further develop his ideas.

The results are dispersed somewhat randomly, and we shall examine them in the order in which they appear in [128]. After we examine the scattered claims, we focus our attention on the list given on pages 233–235 of [128]. It is doubtful that few, if any, of our arguments coincide with those found by Ramanujan. In particular, we use ideas, for example, from the theory of modular forms, with which Ramanujan would have been unfamiliar. We have attempted to present proofs that are as elementary as possible, but even these proofs are unlikely to be close to any found by Ramanujan. It would be of enormous interest if one could discern how Ramanujan discovered these beautiful theorems on Euler products.

In our accounts that follow, we employ Ramanujan’s notations for theta functions and Eisenstein series. As usual, set

\[(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.\]

Ramanujan’s function \(f(-q)\) is defined by

\[f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q; q)_\infty = q^{-1/24}\eta(z), \quad q = e^{2\pi iz}, z \in \mathbb{H},\]

(11.1.2)

where the second equality above is the pentagonal number theorem, where \(\eta(z)\) denotes the Dedekind eta function, and where \(\mathbb{H} = \{z : \text{Im } z > 0\}\).

Ramanujan’s Eisenstein series are defined for \(|q| < 1\) by

\[P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n},\]

(11.1.3)

\[Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n},\]

(11.1.4)

\[R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^n}.\]

(11.1.5)
11.2 Scattered Entries on Euler Products

Entry 11.2.1 (p. 54). If \( f(-q) \) is defined by (11.1.2) and

\[
\sum_{n=1}^{\infty} a_n q^n := q f^3(-q) f^3(-q^7),
\]

then

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{1 + 7^{1-s}} \prod_p \frac{1}{1 - p^{2(1-s)}} \prod_q \frac{1}{1 + 2c_q q^{-s} + q^{2(1-s)}},
\]

where the first product is over all primes \( p \equiv 3, 5, 6 \pmod{7} \), the second product is over all primes \( q \equiv 1, 2, 4 \pmod{7} \), and

\[
c_q = \begin{cases} 
\frac{3}{2}, & \text{if } q = 2, \\
7v^2 - u^2, & \text{if } q = u^2 + 7v^2.
\end{cases}
\]

Entry 11.2.1 was essentially established by Rangachari [129]; see formula (5) under (b) in his paper. However, like Ramanujan, he failed to see that \( c_2 \) had to be defined separately from the remaining cases when \( q \equiv 1, 2, 4 \pmod{7} \).

Ramanujan records another form of Entry 11.2.1 in his manuscript on the partition and tau-functions; in particular, see [128, p. 146] or [21, p. 105]

Entry 11.2.2 (p. 146). Define the coefficients \( a_n, n \geq 1 \), by (11.2.1). Then

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{1 + 7^{1-s}} \prod_{p \equiv 3, 5, 6 \pmod{7}} \frac{1}{1 - p^{2(1-s)}} \prod_{p \equiv 1, 2, 4 \pmod{7}} \frac{1}{1 + C_p p^{-s} + p^{2(1-s)}},
\]

where

\[
C_p = 2p - a^2
\]

with

\[
4p = a^2 + 7b^2.
\]

If \( p \) is odd, then the equality \( p = u^2 + 7v^2 \) implies that \( 4p = (2u)^2 + 7(2v)^2 \). Conversely, if \( 4p = a^2 + 7b^2 \) and \( p \) is odd, then \( a \) and \( b \) are even and \( p = (a/2)^2 + 7(b/2)^2 \). Hence, (11.2.2) is equivalent to (11.2.4) when \( p \) is odd, and, in particular,

\[
C_p = 2p - a^2 = 2(p - 2u^2) = 2(u^2 + 7v^2 - 2u^2) = 2(7v^2 - u^2) = 2c_p.
\]

When \( p \) is even, it is easy to check that \( C_2 \) is equal to \( 2c_2 \). We thus see that Entries 11.2.1 and 11.2.2 are equivalent.

Entry 11.2.2 was discussed by Berndt and K. Ono in their paper [33, equation (8.4)], and they remarked that it could be proved with two applications of Jacobi’s identity.
\[
(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}, \tag{11.2.7}
\]

but they did not supply a complete proof; see also [21, p. 145]. The first complete proof was given by H.H. Chan, S. Cooper, and W.-C. Liaw [47], and it is their proof that we now give below.

**Proof.** As indicated in [129] and [33], using the Dedekind eta function \( \eta(z) \), which is defined in (11.1.2), we see that

\[
F(z) := \sum_{n=1}^{\infty} a_n q^n = \eta^3(z) \eta^3(7z) = q f^3(-q) f^3(-q^7), \quad q = e^{2\pi i z}, \quad z \in \mathbb{H},
\]

is in \( S := S_3 \left( \Gamma_0(7) \right) \), the space of weight 3 cusp forms on \( \Gamma_0(7) \) with the Legendre symbol \( (\cdot|7) \) as the character, which can be deduced by applying Newman’s criterion for \( \eta \)-products [119]. The space \( S \) is one dimensional [61, Théorème 1], and hence \( F(z) \) is an eigenform. Consequently, the corresponding Dirichlet series has an Euler product expansion [96, p. 163]

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s} + \left( \frac{p}{7} \right) p^{2(1-s)}}. \tag{11.2.8}
\]

It remains to determine \( a_p \) for all primes \( p \).

We write Jacobi’s identity (11.2.7) in the form

\[
\eta^3(z) = \sum_{\alpha = \infty}^{\infty} \alpha q^{\alpha^2/8}. \tag{11.2.9}
\]

Therefore,

\[
\eta^3(z) \eta^3(7z) = \sum_{\alpha, \beta = \infty}^{\infty} \alpha \beta q^{(\alpha^2 + 7\beta^2)/8}. \tag{11.2.10}
\]

Hence, for all positive integers \( n \), we have, from (11.2.1) and (11.2.10),

\[
a_n = \sum_{\alpha, \beta = \infty}^{\infty} \alpha \beta. \tag{11.2.11}
\]

First, we take \( n = 2 \) in (11.2.11). The only pair \((\alpha, \beta) \in \mathbb{Z}^2\) satisfying \( 16 = \alpha^2 + 7\beta^2 \) and \((\alpha, \beta) \equiv (1, 1) \pmod{4}\) is \((\alpha, \beta) = (-3, 1)\). Hence, \( a_2 = -3 \). This gives the value of \( c_2 \) in (11.2.3).
Second, we take $n = 7$ in (11.2.11). The only pair $(\alpha, \beta) \in \mathbb{Z}^2$ satisfying $56 = \alpha^2 + 7\beta^2$ and $(\alpha, \beta) \equiv (1, 1) \pmod{4}$ is $(\alpha, \beta) = (-7, 1)$. Hence, $a_7 = -7$. This gives the first factor in (11.2.2).

Now let $n$ denote a prime $p \equiv 3, 5, 6 \pmod{7}$, so that \((-7/p)\) denotes the Legendre symbol. In this case, $8p \neq \alpha^2 + 7\beta^2$ for any odd integers $\alpha$ and $\beta$, for otherwise, $8p = \alpha^2 + 7\beta^2$ and thus

\[
\left( -\frac{7}{p} \right) = \left( -\frac{\alpha^2 - 8p}{p} \right) = \left( \frac{\alpha^2}{p} \right) = 1,
\]

which is a contradiction. Thus, $a_p = 0$ for $p \equiv 3, 5, 6 \pmod{7}$, and we obtain the second product on the right-hand side of (11.2.2).

Finally, we let $n$ denote a prime $p \equiv 1, 2, 4 \pmod{7}$, so that $\left( -\frac{7}{p} \right)$ is odd. In this case there exist integers $A$ and $B$ such that $p = A^2 + 7B^2$ [89, p. 309, Exercise 8]. Since $p$ is odd, all such pairs $(A, B)$ satisfy $A + B \equiv 1 \pmod{4}$. The mapping $(A, B) \mapsto (A - 7B, A + B)$ shows that half of these pairs satisfy $A + B \equiv 1 \pmod{4}$, with the other half satisfying $A + B \equiv 3 \pmod{4}$. Let

\[
S := \{ (A, B) \in \mathbb{Z}^2 : p = A^2 + 7B^2, A + B \equiv 1 \pmod{4} \}
\]

and

\[
T := \{ (\alpha, \beta) \in \mathbb{Z}^2 : 8p = \alpha^2 + 7\beta^2, \alpha \equiv \beta \equiv 1 \pmod{8} \}.
\]

We note that if $(\alpha, \beta) \in T$, then $\alpha \equiv \beta \pmod{8}$. The mapping $\lambda : S \mapsto T$ given by

\[
\lambda ((A, B)) = (A - 7B, A + B)
\]

is a bijection. Hence, applying (11.2.12) in (11.2.11), we obtain

\[
a_p = \sum_{(\alpha, \beta) \in T} \alpha\beta = \sum_{(A, B) \in S} (A - 7B)(A + B)
\]

\[
= \sum_{(A, B) \in \mathbb{Z}^2} (A - 7B)(A + B) = \frac{1}{2} \sum_{p = A^2 + 7B^2} (A - 7B)(A + B).
\]

(11.2.13)

If $(A, B) \in \mathbb{Z}^2$ is any solution of $p = A^2 + 7B^2$, all the solutions are $(A, B), (A, -B), (-A, B), (-A, -B)$, so that by (11.2.13),

\[
a_p = \frac{1}{2} ((A - 7B)(A + B) + (A + 7B)(A - B) + (-A - 7B)(-A + B) + (-A + 7B)(-A - B)) = 2(A^2 - 7B^2).
\]

This completes the proof of (11.2.3) and the derivation of the third factor in (11.2.2) for primes $p \neq 2$. \qed
Entry 11.2.3 (p. 207). If
\[
\sum_{n=1}^{\infty} q(n)q^n := qf^4(-q)f^4(-q^5),
\] (11.2.14)
then
\[
\sum_{n=1}^{\infty} \frac{q(n)}{n^s} = \frac{1}{1 + 5^{1-s}} \prod_p \frac{1}{1 - q(p)p^{-s} + p^{3-2s}},
\] (11.2.15)
where the product is over all primes \( p \) except \( p = 5 \). Furthermore, \( q(2) = -4, q(3) = 2, q(5) = -5, q(7) = 6 \), and generally \( q^2(p) < 4p^3 \).

Entry 11.2.3 was not discussed by Rangachari [129]. However, it does fall under the theory outlined in his paper. We have corrected a misprint in the lost notebook [128]; Ramanujan had written \( p^3 - 2s \) instead of \( p^{3-2s} \) in the last term in the denominator on the right side of (11.2.15). The values of \( q(n), n = 2, 3, 5, 7 \), calculated by Ramanujan are correct. It is doubtful that Ramanujan had a proof of the inequality \( q^2(p) < 4p^3 \), which follows from the deep work of P. Deligne [63]. It would be extremely interesting to know how Ramanujan deduced it. At the end of our proof of Entry 11.2.3, we provide an elementary proof of a much weaker result.

Proof. The coefficients \( q(n), n \geq 1 \), are defined by (11.2.14). By a paper by Y. Martin [104, p. 4852, Table 1], \( q(n) \) is a multiplicative function of \( n \). From a paper of M. Newman [118, p. 487], \( q(p^\alpha) \), where \( p \) is a prime not equal to 5, satisfies the recurrence relation
\[
q(p^\alpha) - q(p)q(p^{\alpha-1}) + p^3q(p^{\alpha-2}) = 0, \quad \alpha \in \mathbb{N}, \quad \alpha \geq 2,
\] (11.2.16)
and
\[
q(5^\alpha) = (-5)^\alpha, \quad \alpha \in \mathbb{N}_0.
\] (11.2.17)
Hence, for \( p \neq 5 \), by (11.2.16),
\[
(1 - q(p)p^{-s} + p^{3-2s}) \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}}
\]
\[
= \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} - q(p) \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{(\alpha+1)s}} + p^3 \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{(\alpha+2)s}}
\]
\[
= \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} - q(p) \sum_{\alpha=1}^{\infty} \frac{q(p^{\alpha-1})}{p^{\alpha s}} + p^3 \sum_{\alpha=2}^{\infty} \frac{q(p^{\alpha-2})}{p^{\alpha s}}
\]
\[
= q(1) + \left( \frac{q(p)}{p^s} - \frac{q(p)q(1)}{p^s} \right) + \sum_{\alpha=2}^{\infty} \frac{q(p^\alpha) - q(p)q(p^{\alpha-1}) + p^3q(p^{\alpha-2})}{p^{\alpha s}}
\]
\[
= 1.
\] (11.2.18)
It follows from (11.2.18) that
\[
\sum_{\alpha=0}^{\infty} q(p^{\alpha}) = \frac{1}{1 - q(p)p^{-s} + p^{3-2s}}, \quad p \neq 5.
\] (11.2.19)

Also, from (11.2.17),
\[
\sum_{\alpha=0}^{\infty} q(5^{\alpha}) = \sum_{\alpha=0}^{\infty} (-5^{1-s})^\alpha = \frac{1}{1 + 5^{1-s}}.
\] (11.2.20)

Thus, as \(q(n)\) is multiplicative, we deduce from (11.2.19) and (11.2.20) that
\[
\sum_{n=1}^{\infty} q(n)n^s = \sum_{\alpha=0}^{\infty} q(5^{\alpha}) \prod_{p \neq 5} \sum_{\alpha=0}^{\infty} q(p^{\alpha}) = \frac{1}{1 + 5^{1-s}} \prod_{p \neq 5} \frac{1}{1 - q(p)p^{-s} + p^{3-2s}},
\] and so the proof of Entry 11.2.3 is complete, except for the inequality for \(q(p)\).

To obtain an elementary bound for \(q(n)\), we use Ramanujan's two famous identities [21, p. 98]
\[
\frac{(q; q^5)_{\infty}}{(q^5; q^5)_{\infty}} = 1 - 5 \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \frac{5}{d} \right) d \right) q^n
\]
and
\[
\frac{q(q^5; q^5)_{\infty}}{(q; q)_{\infty}} = \sum_{m=1}^{\infty} \left( \sum_{e|m} \left( \frac{5}{m/e} \right) e \right) q^m,
\]
where \(\left( \frac{m}{n} \right)\) denotes the Kronecker symbol, to deduce that
\[
q(q; q)_{\infty}^4 \frac{(q^5; q^5)_{\infty}}{q(q^5; q^5)_{\infty}} = \sum_{m=1}^{\infty} \left( \sum_{e|m} \left( \frac{5}{m/e} \right) e \right) q^m
\]
\[= -5 \sum_{N=1}^{\infty} \left( \sum_{m,n=1}^{\infty} \sum_{d|n} \sum_{e|m} \left( \frac{5}{dm/e} \right) de \right) q^N.
\]

Hence, writing \(d = a, e = b, m = by, \) and \(n = ax,\) we deduce that, for \(N \geq 1,\)
\[
q(N) = \sum_{d|N} \left( \frac{5}{N/d} \right) d - 5 \sum_{(a,b,x,y) \in \mathbb{N}^4} \sum_{ax+by=N} \left( \frac{5}{ay} \right) ab.
\]
Thus, using below an evaluation from [125], [126, p. 146, Table IV, no. 1], we find that
\[ |q(N)| < \sum_{d \mid N} d + 5 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \ni ax + by = N}} ab \]

\[ = \sigma(N) + 5 \sum_{r=1}^{N-1} \sigma(r)\sigma(N-r) \]

\[ = \sigma(N) + 5 \left( \frac{5}{12} \sigma_3(N) + \left( \frac{1}{12} - \frac{1}{2} N \right) \sigma(N) \right) \]

\[ = \frac{25}{12} \sigma_3(N) + \left( \frac{17}{12} - \frac{5}{2} N \right) \sigma(N) \]

\[ < \frac{25}{12} \sigma_3(N). \]

In particular, if \( N = p \) is a prime,

\[ |q(p)| < \frac{25}{12} \sigma_3(p) = \frac{25}{12} (1^3 + p^3) < \frac{25}{6} p^3, \]

which is much weaker than Ramanujan’s assertion.

**Entry 11.2.4 (p. 247).** If

\[ \sum_{n=1}^{\infty} \phi(n)q^n := qf^{12}(-q^2), \quad (11.2.21) \]

then

\[ \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \prod_p \frac{1}{1 - \phi(p)p^{-s} + p^{s-2}}, \]

where the product is over all odd primes \( p \).

Entry 11.2.4 is actually a special case of a general claim made without proof in Ramanujan’s paper [125, p. 162]. Entry 11.2.4 was proved by Mordell [111, p.121]; it is the case \( a = 6 \) in Mordell’s paper. A proof was also given by Rangachari [129].

**Entry 11.2.5 (p. 247).** If

\[ \sum_{n=1}^{\infty} \phi(n)q^n := qf(-q^2)f(-q^{22}), \quad (11.2.23) \]

then

\[ \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{1}{1 - 11^{-s}} \prod_p \frac{1}{1 + p^{-s} + p^{-2s}} \prod_q \frac{1}{1 - q^{-2s}} \prod_r \frac{1}{(1 - r^{-s})^2}, \quad (11.2.24) \]

where the first product is over all primes \( p \) such that \( p \equiv 1, 3, 4, 5, 9 \) (mod 11), the second product is over all odd primes \( q \) such that \( q \equiv 2, 6, 7, 8, 10 \) (mod 11), and the third product is over all primes \( r \) such that \( r \) can be written in the form \( r = 11A^2 + B^2 \).
Entry 11.2.5 is formula (2) under part (b) in Rangachari’s paper. Unfortunately, Rangachari failed to notice that Entry 11.2.5 is incorrect, and so therefore is his claim to a proof. Entry 11.2.5 was first proved in its corrected form by Z.-H. Sun and Williams [141, p. 386, Theorem 7.2].

**Entry 11.2.6 (p. 247).** If

\[
\sum_{n=1}^{\infty} \phi(n)q^n := qf(-q^3)f(-q^{21}), \tag{11.2.25}
\]

then

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{1}{1 + 7^{-s}} \prod_p \frac{1}{1 + p^{-2s}} \prod_q \frac{1}{1 - q^{-2s}} \prod_r \frac{1}{(1 + r^{-s})^2} \prod_t \frac{1}{(1 - t^{-s})^2}, \tag{11.2.26}
\]

where the first product is over all odd primes \( p \) such that \( p \equiv 2, 8, 11 \pmod{21} \), the second product is over all primes \( q \) such that \( q \equiv 5, 17, 20 \pmod{21} \), the third product is over all primes \( r \) such that \( r \) can be written in the form \( r = 9A^2 + 7B^2 \), and the fourth product is over all primes \( t \) such that \( t \) can be written in the form \( t = A^2 + 63B^2 \).

Entry 11.2.6 is formula (1) under part (b) in Rangachari’s paper [129]. Entry 11.2.6 is incorrect, and so therefore is Rangachari’s proof. The first proof of a corrected version of Entry 11.2.6 was given by Sun and Williams [141, p. 388, Theorem 8.2(i)]. The proofs of corrected versions of Entries 11.2.5 and 11.2.6 by Sun and Williams are discussed in more detail in Section 11.3, which is devoted to their methods.

**Entry 11.2.7 (p. 249).** Define

\[
\sum_{n=1}^{\infty} \phi(n)q^n := qf^4(-q^6)R(q^6), \tag{11.2.27}
\]

where \( R(q) \) is defined in (11.1.5). Then

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \prod_p \frac{1}{1 - \phi(p)p^{-s} + p^2 - 2s^{-1}}, \tag{11.2.28}
\]

where the product is over all primes \( p \) exceeding 3 and

\[
\phi(p) = \begin{cases} 
0, & \text{if } p \equiv -1 \pmod{6}, \\
(A + iB\sqrt{3})^7 + (A - iB\sqrt{3})^7, & \text{if } p \equiv 1 \pmod{6}.
\end{cases} \tag{11.2.29}
\]

where \( A \) and \( B \) are determined by \( p = A^2 + 3B^2 \).
Entry 11.2.7 can also be found under List I on pages 233–235 of [128] or in List I in Section 11.4 below.

In Ramanujan’s formulation, he wrote “where A and B are the same as before.”

Proof. Let

\[ \omega = \frac{1 + \sqrt{-3}}{2}. \]

For the convenience of future calculations, we record the values

\[ \omega^2 = \frac{-1 + \sqrt{-3}}{2} = -1 + \omega, \quad \omega^4 = \frac{-1 - \sqrt{-3}}{2} = -\omega, \quad \omega^5 = \frac{1 - \sqrt{-3}}{2} = 1 - \omega, \]

\[ \sqrt{-3} = -1 + 2\omega, \quad \omega\sqrt{-3} = -2 + \omega, \quad \omega^2\sqrt{-3} = -1 - \omega. \]

We briefly review basic facts about the ring of integers \( \mathbb{Z} + \mathbb{Z}\omega = \{ x + y\omega | x, y \in \mathbb{Z} \} \) in the imaginary quadratic field \( \mathbb{Q}(\sqrt{-3}) \), which has discriminant \(-3\). It is well known that \( \mathbb{Z} + \mathbb{Z}\omega \) is a unique factorization domain. The group of units in \( \mathbb{Z} + \mathbb{Z}\omega \) is the cyclic group of order 6 generated by \( \omega \). The Eisenstein integer \( 2 + 2\omega \) is the product of two irreducible integers, namely, \( 2 \) and \( 1 + \omega \).

We now define a character \( \chi \) on \( \mathbb{Z} + \mathbb{Z}\omega \) modulo \( 2 + 2\omega \). Let \( x + y\omega \in \mathbb{Z} + \mathbb{Z}\omega \).

First, observe that

\[ 2| \gcd(x + y\omega, 2 + 2\omega) \iff x \equiv y \equiv 0 \pmod{2}, \]

\[ (1 + \omega)| \gcd(x + y\omega, 2 + 2\omega) \iff x \equiv y \pmod{3}. \]

Hence,

\[ \gcd(x + y\omega, 2 + 2\omega) = 1 \iff (x, y) \equiv (0, 1), (1, 0), \text{ or } (1, 1) \pmod{2} \]

and \( x \neq y \pmod{3} \).

For those \( x + y\omega \) coprime with \( 2 + 2\omega \),

\[ x + y\omega \equiv 1 \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (1, 0) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \]

\[ x + y\omega \equiv \omega \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (0, 1) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}, \]

\[ x + y\omega \equiv \omega^2 \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (1, 1) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \]

\[ x + y\omega \equiv \omega^3 \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (1, 0) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}, \]

\[ x + y\omega \equiv \omega^4 \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (0, 1) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \]

\[ x + y\omega \equiv \omega^5 \pmod{2 + 2\omega}, \text{ if } (x, y) \equiv (1, 1) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}. \]

Hence, we can define a character \( \chi \) of order 6 on \( \mathbb{Z} + \mathbb{Z}\omega \) modulo \( 2 + 2\omega \) by

\[ \chi(x + y\omega) = \begin{cases} \omega^{-\ell}, & \text{if } x + y\omega \equiv \omega^\ell \pmod{2 + 2\omega} \text{ for some } \ell \in \{0, 1, 2, 3, 4, 5\}, \\ 0, & \text{if } \gcd(x + y\omega, 2 + 2\omega) \neq 1. \end{cases} \]
In particular, note that 
\[ \chi(\omega) = \omega^{-1} = \omega. \]
If \(\epsilon\) is a unit in \(\mathbb{Z} + \mathbb{Z}\omega\), then \(\epsilon = \omega^\ell\) for some \(\ell \in \{0, 1, 2, 3, 4, 5\}\) and 
\[ \chi(\epsilon)\epsilon^7 = \chi(\omega^\ell)\omega^{7\ell} = \chi(\omega^{\ell-1})\omega^\ell = 1. \]
Thus, by [97, equations (5.8), (5.9)], the Hecke theta series \(\theta_8(-3, \chi, z)\) is given by 
\[ \theta_8(-3, \chi, z) = \frac{1}{6} \sum_{x + y\omega \in \mathbb{Z} + \mathbb{Z}\omega} \chi(x + y\omega)(x + y\omega)^7 e^{2\pi i (x + y\omega)(x + y\omega) z}, \tag{11.2.30} \]
where \(z \in \mathbb{H}\). Noting that \((x + y\omega)(x + y\omega) = x^2 + xy + y^2\), we define, for each \(\ell \in \{0, 1, 2, 3, 4, 5\}\),
\[ A_\ell := \omega^{-\ell} \sum_{(x, y) \in \mathbb{Z}^2} \chi(x + y\omega)(x + y\omega)^7 q^{x^2 + xy + y^2}, \quad q = e^{2\pi i z}. \]
Then,
\[ \frac{1}{6} \sum_{\ell=0}^{5} A_\ell = \frac{1}{6} \sum_{\ell=0}^{5} \omega^{-\ell} \sum_{(x, y) \in \mathbb{Z}^2} \chi(x + y\omega)(x + y\omega)^7 q^{x^2 + xy + y^2} \]
\[ = \frac{1}{6} \sum_{\ell=0}^{5} \sum_{(x, y) \in \mathbb{Z}^2} \chi(x + y\omega)(x + y\omega)^7 q^{x^2 + xy + y^2} \]
\[ = \frac{1}{6} \sum_{\gcd(x + y\omega, 2 + 2\omega) = 1} \chi(x + y\omega)(x + y\omega)^7 e^{2\pi i (x + y\omega)(x + y\omega) z} \]
\[ = \theta_8(-3, \chi, z), \tag{11.2.31} \]
by (11.2.30).

Next, we evaluate \(A_0, A_1, \ldots, A_5\) by making the indicated changes of variable in the corresponding series:
- \(A_0\) \( (x, y) = (r - s, 2s) \),
- \(A_1\) \( (x, y) = (2s, r - s) \),
- \(A_2\) \( (x, y) = (-r + s, r + s) \),
- \(A_3\) \( (x, y) = (-r - s, 2s) \),
- \(A_4\) \( (x, y) = (2s, -r - s) \),
- \(A_5\) \( (x, y) = (r + s, -r + s) \).
We calculate one of the sums; the other calculations are similar. To that end,
\[ A_5 = \omega^{-5} \sum_{(x,y) \in \mathbb{Z}^2, x \equiv y \equiv 2 \pmod{3}} (x + y \omega)q^{x^2 + xy + y^2} \]
\[ = \omega \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{3}, r \equiv 1 \pmod{2}} (r - s \omega^2 \sqrt{-3}) q^{r^2 + 3s^2} \]
\[ = \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{3}, r \equiv 1 \pmod{2}} (r + s \sqrt{-3}) q^{r^2 + 3s^2}. \]

In summary, we find that
\[ A_0 = A_4 = A_5 = \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{2}, r \equiv 1 \pmod{3}} (r + s \sqrt{-3}) q^{r^2 + 3s^2}, \]
\[ A_1 = A_2 = A_3 = \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{2}, r \equiv 1 \pmod{3}} (r - s \sqrt{-3}) q^{r^2 + 3s^2}. \]

Hence, by (11.2.32) and (11.2.33),
\[ \sum_{\ell=0}^{5} A_{\ell} = 3 \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{2}, r \equiv 1 \pmod{3}} ((r + s \sqrt{-3}) q^{r^2 + 3s^2} + (r - s \sqrt{-3}) q^{r^2 + 3s^2}), \]
and so, by (11.2.31),
\[ \theta_8(-3, \chi, z) = \frac{1}{2} \sum_{(r,s) \in \mathbb{Z}^2, r-s \equiv 1 \pmod{2}, r \equiv 1 \pmod{3}} ((r + s \sqrt{-3}) q^{r^2 + 3s^2} + (r - s \sqrt{-3}) q^{r^2 + 3s^2}). \]

Now [97, p. 122],
\[ \theta_8(-3, \chi, z) = q f^4(-q^6) R(q^6). \]

Thus, from (11.2.27), (11.2.35), and (11.2.34),
\[ \phi(n) = \frac{1}{2} \sum_{(r,s) \in \mathbb{Z}^2 \atop r-s \equiv 1 \pmod{2}, \atop r \equiv 1 \pmod{3}, \atop r^2 + 3s^2 = n} \left( (r+s\sqrt{-3})^7 + (r-s\sqrt{-3})^7 \right). \quad (11.2.36) \]

As the only solution to \( r^2 + 3s^2 = 1 \), \( r-s \equiv 1 \pmod{2}, r \equiv 1 \pmod{3} \) is \((r,s) = (1,0)\), we find from (11.2.36) that
\[ \phi(1) = 1. \quad (11.2.37) \]

The conditions \( r-s \equiv 1 \pmod{2}, r^2 + 3s^2 = n \) imply that \( n \equiv 1 \pmod{2} \). Hence,
\[ \phi(n) = 0, \quad \text{if} \ 2 \mid n. \quad (11.2.38) \]

The conditions \( r \equiv 1 \pmod{3}, r^2 + 3s^2 = n \) imply that \( n \equiv 1 \pmod{3} \). Hence,
\[ \phi(n) = 0, \quad \text{if} \ 3 \mid n. \quad (11.2.39) \]

If \( p \) is a prime with \( p \equiv -1 \pmod{6} \) and \( n \) is odd, then there are no integers \( r \) and \( s \) such that \( p^n = r^2 + 3s^2 \). Thus,
\[ \phi(p^n) = 0, \quad \text{if} \ p \equiv -1 \pmod{6}, \ n \text{ odd}. \quad (11.2.40) \]

If \( p \) is a prime with \( p \equiv 1 \pmod{6} \), then there are integers \( A \) and \( B \) such that \( p = A^2 + 3B^2 \). Replacing \( A \) by \(-A\), if necessary, we may suppose that \( A \equiv 1 \pmod{3} \). Replacing \( B \) by \(-B\), if necessary, we may suppose that \( B > 0 \). All solutions of \( p = x^2 + 3y^2 \) are given by \((x,y) = (A,B), (A,-B), (-A,B), (-A,-B)\). Thus,
\[
\begin{align*}
\phi(p) &= \frac{1}{2} \left( ((A-B\sqrt{-3})^7 + (A+B\sqrt{-3})^7) \\
&\quad + ((A-B\sqrt{-3})^7 + (A+B\sqrt{-3})^7) \right) \\
&= (A+B\sqrt{-3})^7 + (A-B\sqrt{-3})^7,
\end{align*}
\quad (11.2.41)
\]

where \( A \) and \( B \) are given uniquely by \( p = A^2 + 3B^2, A \equiv 1 \pmod{3}, B > 0 \).

The Hecke theta series
\[ \theta_8(-3, \chi, z) = \sum_{n=1}^{\infty} \phi(n)e^{2\pi i n z}, \quad (11.2.42) \]

where \( \phi(n) \) is given by (11.2.36), is a modular form of weight 8 on the group \( \Gamma_0(36) \) [97, p. 77]. We also know [97, p. 70] that \( \phi(n) \) is multiplicative and satisfies the recursion relation
\[
\phi(p^m) - \phi(p)\phi(p^{m-1}) + p^7 \phi(p^{m-2}) = 0, \quad (11.2.43)
\]
for each integer \( m \geq 2 \) and each prime \( p \) with \( p \equiv 1 \pmod{6} \) [97, equation (5.7)], and the exact formula
220 11 Euler Products in Ramanujan’s Lost Notebook

\[ \phi(p^{2m}) = (-p^7)^m, \quad (11.2.44) \]

for each positive integer \( m \) and each prime \( p \) with \( p \equiv -1 \pmod{6} \) [97, p. 71].

Hence, for \( p \equiv -1 \pmod{6} \), by (11.2.37), (11.2.40), and (11.2.44),

\[
\sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} = \sum_{m=0}^{\infty} \frac{(-p^7)^m}{p^{2ms}} = \sum_{m=0}^{\infty} (-p^{7-2s})^m = \frac{1}{1 + p^7 - 2s} = \frac{1}{1 - \phi(p)p^{-s} + p^7 - 2s},
\]

(11.2.45)

by (11.2.40) once again.

For \( p \equiv 1 \pmod{6} \), by (11.2.43),

\[
(1 - \phi(p)p^{-s} + p^7 - 2s) \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}}
\]

\[
= \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} - \phi(p) \sum_{m=0}^{\infty} \frac{\phi(p^{m+1})}{p^{m+1}s} + p^7 \sum_{m=0}^{\infty} \frac{\phi(p^{m+2})}{p^{m+2}s}
\]

\[
= \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} - \phi(p) \sum_{m=1}^{\infty} \frac{\phi(p^{m-1})}{p^{ms}} + p^7 \sum_{m=2}^{\infty} \frac{\phi(p^{m-2})}{p^{ms}}
\]

\[
= 1 + \frac{\phi(p) - \phi(p)\phi(1)}{p^s} + \sum_{m=2}^{\infty} \frac{\phi(p^m) - \phi(p)\phi(p^{m-1}) + p^7\phi(p^{m-2})}{p^{ms}}
\]

\[ = 1. \quad (11.2.46) \]

Thus, by (11.2.45) and (11.2.46),

\[
\sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} = \frac{1}{1 - \phi(p)p^{-s} + p^7 - 2s}, \quad p > 3.
\]

(11.2.47)

Clearly, by (11.2.38) and (11.2.39),

\[
\sum_{m=0}^{\infty} \frac{\phi(2^m)}{2^{ms}} = 1 = \sum_{m=0}^{\infty} \frac{\phi(3^m)}{3^{ms}},
\]

(11.2.48)

respectively. Thus, as \( \phi(n) \) is multiplicative, by (11.2.47) and (11.2.48),

\[
\sum_{n=0}^{\infty} \frac{\phi(n)}{n^s} = \sum_{m=0}^{\infty} \frac{\phi(2^m)}{2^{ms}} \cdot \sum_{m=0}^{\infty} \frac{\phi(3^m)}{3^{ms}} \cdot \prod_{p \text{ prime}} \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}}
\]

\[ = \prod_{p > 3 \atop p \text{ prime}} \frac{1}{1 - \phi(p)p^{-s} + p^7 - 2s}. \]

This completes the proof of Entry 11.2.7.
Before discussing the next entry, we offer a remark on the convergence of the series and product in (11.2.28). The number of solutions \((x, y) \in \mathbb{Z}^2\) of \(n = x^2 + 3y^2\) is \(\ll \epsilon n^{7/2}\), for each \(\epsilon > 0\) [90, Corollary 9.1(a)]. Hence, by (11.2.36),

\[ |\phi(n)| \ll \epsilon n^{7/2 + \epsilon}, \]

for each \(\epsilon > 0\). Therefore, the Dirichlet series on the left-hand side of (11.2.28) converges absolutely for \(\text{Re } s > \frac{9}{2}\), and, moreover, we see that the product on the right-hand side of (11.2.28) also converges absolutely for \(\text{Re } s > \frac{9}{2}\).

**Entry 11.2.8 (p. 328).** If

\[ \sum_{n=1}^{\infty} a_n q^n := q^3 f^{18}(-q^4), \]  

(11.2.49)

then

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{1 - 78 \cdot 3^{-s} + 38^{-2s}} \frac{1}{1 + 510 \cdot 5^{-s} + 58^{-2s}} \frac{1}{1 + 1404 \cdot 7^{-s} + 78^{-2s}} \cdots
\]

\[
- \frac{1}{1 + 78 \cdot 3^{-s} + 38^{-2s}} \frac{1}{1 - 510 \cdot 5^{-s} + 58^{-2s}} \frac{1}{1 - 1404 \cdot 7^{-s} + 78^{-2s}} \cdots.
\]  

(11.2.50)

Ramanujan writes a plus sign in front of 510 in each of the products above. Since one of the signs is likely to be incorrect, we have accordingly changed the second sign. Raghavan [122] numerically disproved Ramanujan’s formula as he had written it. However, even with our slight change, Entry 11.2.8 is still incorrect. In searching for an appropriate linear combination of products of eta functions and Eisenstein series in order to correct Entry 11.2.8, we are led to

\[ 78q^3 f^{18}(-q^4) + qf^6(-q^4)R(q^4) = q + 78q^3 - 510q^5 - 1404q^6 + \cdots, \]

where \(R(q)\) is defined in (11.1.5). From this calculation, it is possible that Ramanujan thought that \(78q^3 f^{18}(-q^4) + qf^6(-q^4)R(q^4)\) has an Euler-product, which, however, is not the case. Nonetheless, Ramanujan later discovered the correct Euler product involving \(q^3 f^{18}(-q^4)\), which is given in Entry 11.4.3.

We quote Ramanujan in the next entry. We emphasize that he does not provide any product representations.

**Entry 11.2.9 (p. 328).** Presumably there are analogous results for

\[ \sum_{n=1}^{\infty} a_n n^{-s} \]  

where \(\sum_{n=1}^{\infty} a_n q^n\) is any of the functions

\[ q^5 f^{10}(-q^{12}), \]

\[ q^7 f^{14}(-q^{12}), \]

\[ q^5 f^{20}(-q^6), \]

\[ q^{11} f^{22}(-q^{12}). \]
Rangachari [129] established these Euler products, but they can also be found in Ramanujan’s list of 46 products given in Section 11.4. Euler products corresponding to the first, second, and fourth functions above can be found in List IV, while the Euler product associated with the third modular form above can be found in List I.

**Entry 11.2.10 (p. 329).** Define

\[ F(q) := qf^{16}(-q^3) =: \sum_{n=1}^{\infty} A_n q^n \]  
(11.2.51)

and

\[ F_1(q) := qf^{8}(-q^3)Q(q^3) =: \sum_{n=1}^{\infty} a_n q^n, \]  
(11.2.52)

where \( Q(q) \) is defined in (11.1.4). Then

\[
\sum_{n=1}^{\infty} \frac{a_n + 6A_n \sqrt{10}}{n^s} = \frac{1}{1 - 6\sqrt{10} \cdot 2^{-s} + 27^{-2s}} \frac{1}{1 + 96\sqrt{10} \cdot 5^{-s} + 57^{-2s}}
\]
\[
\times \frac{1}{1 - 260 \cdot 7^{-s} + 77^{-2s}} \frac{1}{1 + 1920\sqrt{10} \cdot 11^{-s} + 117^{-2s}} \cdots
\]  
(11.2.53)

and

\[
\sum_{n=1}^{\infty} \frac{a_n - 6A_n \sqrt{10}}{n^s} = \frac{1}{1 + 6\sqrt{10} \cdot 2^{-s} + 27^{-2s}} \frac{1}{1 - 96\sqrt{10} \cdot 5^{-s} + 57^{-2s}}
\]
\[
\times \frac{1}{1 - 260 \cdot 7^{-s} + 77^{-2s}} \frac{1}{1 - 1920\sqrt{10} \cdot 11^{-s} + 117^{-2s}} \cdots
\]  
(11.2.54)

In his definition of \( F(q) \) in (11.2.51), Ramanujan inadvertently wrote the factor \( q^2 \) instead of \( q \) on the right-hand side of (11.2.51). This result is also given in Part II of Section 11.4, where \( a_n \) and \( A_n \) above are replaced by \( \Omega_2''(n) \) and \( \Omega_2'(n) \), respectively. It is noteworthy that Ramanujan had found a linear combination of modular forms having an Euler product, which is work for which E. Hecke later became famous [84], [85, pp. 644–707].

### 11.3 The Approach of Zhi-Hong Sun and Kenneth Williams Through the Theory of Binary Quadratic Forms

As previously noted in Entries 11.2.5 and 11.2.6, on page 247 of his “lost” notebook, Ramanujan [128] recorded without proof Euler products for the
Dirichlet series $\sum_{n=1}^{\infty} a(n)q^{-n}$ and $\sum_{n=1}^{\infty} b(n)q^{-n}$, where the arithmetic functions $a(n)$ and $b(n)$ are defined by

$$\sum_{n=1}^{\infty} a(n)q^{n} = q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{22n}), \quad q \in \mathbb{C}, \quad |q| < 1, \quad (11.3.1)$$

and

$$\sum_{n=1}^{\infty} b(n)q^{n} = q \prod_{n=1}^{\infty} (1 - q^{3n})(1 - q^{21n}), \quad q \in \mathbb{C}, \quad |q| < 1. \quad (11.3.2)$$

In [129] Rangachari outlined proofs of Ramanujan’s formulas for the Euler products using class field theory and modular forms. Unfortunately, Ramanujan’s formulas are incorrect, and so Rangachari’s proofs are invalid. The proofs given by Sun and Williams [141] of corrected forms of Ramanujan’s formulas are based on the classical theory of binary quadratic forms and so are elementary. The corrected forms of Ramanujan’s conjectures [141, Theorems 7.2 and 8.2] are

$$\sum_{n=1}^{\infty} a(n) = \frac{1}{1 - 11^{-s}} \prod_{p \equiv 2, 6, 7, 8, 10 \ (\text{mod} \ 11)} \frac{1}{1 - p^{-2s}} \prod_{p=3x^2+2xy+4y^2} \frac{1}{1 + p^{-s} + p^{-2s}} \prod_{p=x^2+11y^2 \neq 11} \frac{1}{(1 - p^{-s})^2} \quad (11.3.3)$$

and

$$\sum_{n=1}^{\infty} b(n) = \frac{1}{1 + 7^{-s}} \prod_{p \equiv 3, 5, 6 \ (\text{mod} \ 7), p \neq 3} \frac{1}{1 - p^{-2s}} \prod_{p=2, 8, 11 \ (\text{mod} \ 21)} \frac{1}{1 + p^{-2s}} \prod_{p=x^2+y^2+16y^2} \frac{1}{(1 - p^{-s})^2} \prod_{p=4x^2+y^2+4y^2 \neq 7} \frac{1}{(1 + p^{-s})^2}, \quad (11.3.4)$$

which are valid for $s \in \mathbb{C}$ with Re $s > 1$. Before describing the approach taken by Sun and Williams in [141], we describe briefly the results we need from the theory of binary quadratic forms.

A binary quadratic form is a polynomial $ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$. We always assume that $ax^2 + bxy + cy^2$ is positive-definite, equivalently $a > 0$ and $b^2 - 4ac < 0$, and primitive, equivalently gcd$(a, b, c) = 1$. For brevity, we write $(a, b, c)$ for the form $ax^2 + bxy + cy^2$. The discriminant $d$ of the form $(a, b, c)$ is the negative integer $d = b^2 - 4ac$. We note that $d \equiv 0, 1 \ (\text{mod} \ 4)$. For $n \in \mathbb{N}$ we define

$$R((a, b, c); n) := \text{card}\{(x, y) \in \mathbb{Z}^2 \mid ax^2 + bxy + cy^2 = n\}$$
Each form in \( [a, b, c] \) is positive-definite, primitive, and of discriminant \( d \). Clearly the class \( [a, b, c] \) contains the form \( (a, b, c) \). Furthermore, if \( (A, B, C) \in [a, b, c] \), then \( [A, B, C] = [a, b, c] \). Moreover the number of representations of \( n \in \mathbb{N} \) by any form in the class \( [a, b, c] \) is the same, so we can define

\[
R((a, b, c); n) = R((a, b, c); n).
\]

Gauss proved that each positive-definite, primitive, binary quadratic form of discriminant \( d \) belongs to one and only one of a finite set \( H(d) \) of form classes. We denote the number of such form classes by \( h(d) \). With respect to Gaussian composition, which we write multiplicatively, the set \( H(d) \) is a finite abelian group. The identity of the group is \( I = [1, 0, 0] \) if \( d \equiv 0 \pmod{4} \) and \( I = [1, 1, (1 - d)/4] \) if \( d \equiv 1 \pmod{4} \). The inverse \( A^{-1} \) of the form class \( A = [a, b, c] \) is the form class \( [-a, -b, -c] \). As \( H(d) \) is a finite abelian group, there exist \( r \in \mathbb{N} \), \( A_1, \ldots, A_r \in H(d) \), and \( h_1, \ldots, h_r \in \mathbb{N} \) with \( h_1 \cdots h_r = h(d) \), such that

\[
H(d) = \{ A_1^{k_1} \cdots A_r^{k_r} \mid k_1 = 0, 1, \ldots, h_1 - 1, \ldots, k_r = 0, 1, \ldots, h_r - 1 \}.
\]

For \( n \in \mathbb{N} \) and \( M = A_1^{m_1} \cdots A_r^{m_r} \in H(d) \), we define [140, Definition 7.1]

\[
F(M, n) := \frac{1}{w(d)} \sum_{k_1=0,\ldots,k_r=0}^{h_1-1,\ldots,h_r-1} \cos 2\pi \left( \frac{k_1 m_1}{h_1} + \cdots + \frac{k_r m_r}{h_r} \right) R(A_1^{k_1} \cdots A_r^{k_r}; n),
\]

where

\[
w(d) = \begin{cases} 
6, & \text{if } d = -3, \\
4, & \text{if } d = -4, \\
2, & \text{if } d < -4.
\end{cases}
\]

Since each of \( H(-3) \) and \( H(-4) \) is the trivial group, we have \( w(d) = 2 \) if \( H(d) \) is nontrivial. From [140, Theorem 7.2] we know that \( F(M, n) \) is a multiplicative function of \( n \in \mathbb{N} \). Sun and Williams [141, p. 372] proved that in \( \{ s \in \mathbb{C} \mid \text{Re } s > 1 \} \) the Dirichlet series \( \sum_{n=1}^\infty F(M, n)n^{-s} \) converges absolutely, is an analytic function of \( s \), and has an Euler product.

For our purposes we are interested in the function \( F \) when \( H(d) \) is a cyclic group of order 3 or 4 (so that \( w(d) = 2 \)). If \( H(d) \) is a cyclic group of order 3, say, \( H(d) = \{ I, A, A^2 \} \) with \( A^3 = I, A \neq I \), then

\[
F(A, n) = \frac{1}{2} (R(I, n) - R(A, n)) \quad (11.3.5)
\]
is a multiplicative function of \( n \), which is given explicitly in [140, Theorem 10.1]. Similarly, if \( H(d) \) is a cyclic group of order 4, say, \( H(d) = \{ I, A, A^2, A^3 \} \) with \( A^4 = I \), \( A^2 \neq I \), then

\[
F(A, n) = \frac{1}{2}(R(I, n) - R(A^2, n)) \tag{11.3.6}
\]

is a multiplicative function of \( n \), whose value is given in [140, Theorem 11.1]. If \( H(d) \) is a cyclic group of order \( \geq 5 \) generated by \( A \), then \( F(A, n) \) does not have a simple representation such as (11.3.5) and (11.3.6); see [140, Theorem 7.4].

We are now in a position to describe the approach taken by Sun and Williams [141]. To keep notation consistent with that of Ramanujan, we replace their function \( \phi(q) \) by \( f(-q) \), which is defined in (11.1.2). Next they defined, for \( k \in \{ 1, 2, 3, \ldots, 12 \} \), the arithmetic function \( \phi_k : \mathbb{N} \to \mathbb{Z} \) by

\[
qf(-q^k)f(-q^{24-k}) = \sum_{n=1}^{\infty} \phi_k(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1, \tag{11.3.7}
\]

so that we are interested in \( \phi_2(n) = a(n) \) and \( \phi_3(n) = b(n) \). Using (11.1.2) in the left-hand side of (11.3.7), and manipulating the resulting product of series, Sun and Williams [141, Theorem 2.2] found, on equating coefficients, explicit formulas for \( \phi_k(n) \), \( k \in \{ 1, 2, 3, \ldots, 12 \} \), namely,

- \( \phi_1(n) R([1, 1, 6]; n) - R([2, 1, 3]; n) \) \( d = -23, H(d) \simeq \mathbb{Z}/3\mathbb{Z} \),
- \( \phi_2(n) R([1, 0, 11]; n) - R([3, 2, 4]; n) \) \( d = -44, H(d) \simeq \mathbb{Z}/3\mathbb{Z} \),
- \( \phi_3(n) R([1, 1, 16]; n) - R([4, 1, 4]; n) \) \( d = -63, H(d) \simeq \mathbb{Z}/4\mathbb{Z} \),
- \( \phi_4(n) R([1, 0, 20]; n) - R([4, 0, 5]; n) \) \( d = -80, H(d) \simeq \mathbb{Z}/4\mathbb{Z} \),
- \( \phi_5(n) R([1, 1, 24]; n) - R([4, 1, 6]; n) \) \( d = -95, H(d) \simeq \mathbb{Z}/8\mathbb{Z} \),
- \( \phi_6(n) R([1, 0, 27]; n) - R([4, 2, 7]; n) \) \( d = -108, H(d) \simeq \mathbb{Z}/3\mathbb{Z} \),
- \( \phi_7(n) R([1, 1, 30]; n) - R([4, 3, 8]; n) \) \( d = -119, H(d) \simeq \mathbb{Z}/10\mathbb{Z} \),
- \( \phi_8(n) R([1, 0, 32]; n) - R([4, 4, 9]; n) \) \( d = -128, H(d) \simeq \mathbb{Z}/4\mathbb{Z} \),
- \( \phi_9(n) R([1, 1, 34]; n) - R([4, 3, 9]; n) \) \( d = -135, H(d) \simeq \mathbb{Z}/6\mathbb{Z} \),
- \( \phi_{10}(n) R([1, 0, 35]; n) - R([4, 2, 9]; n) \) \( d = -140, H(d) \simeq \mathbb{Z}/6\mathbb{Z} \),
- \( \phi_{11}(n) R([1, 1, 36]; n) - R([4, 1, 9]; n) \) \( d = -143, H(d) \simeq \mathbb{Z}/10\mathbb{Z} \),
- \( \phi_{12}(n) R([1, 0, 36]; n) - R([4, 0, 9]; n) \) \( d = -144, H(d) \simeq \mathbb{Z}/4\mathbb{Z} \).

In each line of the list above, the value of \( d \) is the discriminant of each of the two forms appearing in the formula for \( \phi_k(n) \). For \( k = 1, 2, \ldots, 12 \), we have \( d = k(k - 24) \). The first form class in each line is the identity class of discriminant \( d \). All of the form class groups \( H(k(k - 24)) \), \( k = 1, 2, \ldots, 12 \), are cyclic. Moreover, exactly seven of them have \( H(k(k - 24)) \simeq \mathbb{Z}/3\mathbb{Z} \) or \( \mathbb{Z}/4\mathbb{Z} \).

For \( k = 1, 2, 6 \), we have \( d = -23, -44, -108 \), respectively, and
For these three values of \( k \), we see from the list that

\[
\phi_k(n) = \frac{1}{2} (R(I, n) - R(A, n)) = F(A, n). \tag{11.3.8}
\]

Thus \( \phi_k(n) \) (\( k = 1, 2, 6 \)) is a multiplicative function of \( n \), whose value is given by [140, Theorem 10.1]; see [141, Theorem 4.4]. Using these evaluations, Sun and Williams [141, Theorem 7.2] deduced that

\[
\sum_{n=1}^{\infty} \frac{\phi_1(n)}{n^s} = \frac{1}{1 - 23^{-s}} \prod_{(p^2) = -1} \frac{1}{1 - p^{-2s}} \times \prod_{p = 2s^2 + xy + 3y^2} \frac{1}{1 + p^{-s} + p^{-2s}} \prod_{p = x^2 + xy + 6y^2 \neq 23} \frac{1}{(1 - p^{-s})^2}, \tag{11.3.9}
\]

\[
\sum_{n=1}^{\infty} \frac{\phi_2(n)}{n^s} = \frac{1}{1 - 11^{-s}} \prod_{p \equiv 2, 6, 7, 8, 10 \pmod{11}} \frac{1}{1 - p^{-2s}} \times \prod_{p = 3x^2 + 2xy + 4y^2} \frac{1}{1 + p^{-s} + p^{-2s}} \prod_{p = x^2 + 11y^2 \neq 11} \frac{1}{(1 - p^{-s})^2}, \tag{11.3.10}
\]

and

\[
\sum_{n=1}^{\infty} \frac{\phi_3(n)}{n^s} = \prod_{p \equiv 5 \pmod{6}} \frac{1}{1 - p^{-2s}} \prod_{p = x^2 + 27y^2} \frac{1}{1 - p^{-s} + p^{-2s}} \times \prod_{p = 4x^2 + 2xy + 7y^2} \frac{1}{1 + p^{-s} + p^{-2s}}. \tag{11.3.11}
\]

For \( k = 3, 4, 8, 12 \), we have \( d = -63, -80, -128, -144 \), respectively, and

\[
H(-234I, A, A^2, A^3), \quad \text{where} \quad I = [1, 1, 6], A = [2, 1, 8], A^2 = [4, 1, 4], A^3 = [2, -1, 8], A^4 = I, \tag{11.3.12}
\]

\[
H(-444I, A, A^2, A^3), \quad \text{where} \quad I = [1, 0, 11], A = [3, 2, 4], A^2 = [3, -2, 4], A^3 = I, \tag{11.3.12}
\]

\[
H(-1004I, A, A^2, A^3), \quad \text{where} \quad I = [1, 0, 27], A = [4, 2, 7], A^2 = [4, -2, 7], A^3 = I. \tag{11.3.12}
\]
For these four values of $k$, we see from the list that
\[
\phi_k(n) = \frac{1}{2}(R(I, n) - R(A^2, n)) = F(A, n).
\]
Thus $\phi_k(n)$, $k = 3, 4, 8, 12$, is a multiplicative function of $n$, whose value is given by [140, Theorem 11.1]; see [141, Theorem 4.5]. Using these evaluations, Sun and Williams [141, Theorem 8.2] deduced that
\[
\sum_{n=1}^{\infty} \frac{\phi_3(n)}{n^s} = \frac{1}{1 + 7^{-s}} \prod_{p \equiv 3, 6 \pmod{7}, \ p \neq 3} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 2, 8, 11 \pmod{21}} \frac{1}{1 + p^{-2s}} \prod_{p = x^2 + xy + 16y^2, \ p = 4x^2 + xy + 4y^2 \neq 7} \frac{1}{1 - (1 - p^{-s})^2},
\]
(11.3.13)
\[
\sum_{n=1}^{\infty} \frac{\phi_4(n)}{n^s} = \frac{1}{1 + 5^{-s}} \prod_{p \equiv 11, 13, 17, 19 \pmod{20}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 3, 7 \pmod{20}} \frac{1}{1 + p^{-2s}} \prod_{p = x^2 + 20y^2, \ p = 4x^2 + 5y^2 \neq 5} \frac{1}{1 - (1 - p^{-s})^2},
\]
(11.3.14)
\[
\sum_{n=1}^{\infty} \frac{\phi_8(n)}{n^s} = \prod_{p \equiv 5, 7 \pmod{8}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 3 \pmod{8}} \frac{1}{1 + p^{-2s}} \prod_{p = x^2 + 32y^2, \ p = 4x^2 + 4y^2 + 9y^2 \neq 7} \frac{1}{1 - (1 - p^{-s})^2},
\]
(11.3.15)
and
\[
\sum_{n=1}^{\infty} \frac{\phi_{12}(n)}{n^s} = \prod_{p \equiv 3 \pmod{4}, \ p \neq 3} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 5 \pmod{12}} \frac{1}{1 + p^{-2s}} \prod_{p = x^2 + 36y^2, \ p = 4x^2 + 9y^2} \frac{1}{1 - (1 - p^{-s})^2}.
\]
(11.3.16)

Formulas (11.3.10) and (11.3.13) are the corrected formulas (11.3.3) and (11.3.4) of Ramanujan. Formulas (11.3.9), (11.3.11), (11.3.14), (11.3.15), and (11.3.16) were not stated by Ramanujan.

11.4 A Partial Manuscript on Euler Products

Pages 233–235 in [128] are devoted to a manuscript by Ramanujan on Euler products in four sections, but in the handwriting of G.N. Watson. The original
Euler Products in Ramanujan’s Lost Notebook

manuscript can be found in the library of Trinity College, Cambridge. We copy the manuscript section by section and then offer proofs and commentary after each section. As we shall see, Ramanujan discovered many Euler products associated with linear combinations of modular forms. We do not have any ideas on how Ramanujan found these Euler products without invoking the theory of modular forms. It would be extremely interesting to find a new, more elementary method to attack these formulas. In our transcription, we have taken the liberty of introducing standard notation for $q$-products. Recall also that Ramanujan’s Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ are defined by (11.1.3)–(11.1.5), respectively.

The following proposition is useful in deducing that $\Omega_1(n) = n\Omega_0(n)$ in the first two series of formulas below.

**Proposition 11.4.1.** Let $\alpha$ be a divisor of 24. Then

$$q \frac{d}{dq} f_\alpha(z) = f_\alpha(z)P(\alpha z),$$

where $f_\alpha(z) = \eta^{24/\alpha}(z)$.

**Proof.** Since

$$\log f_\alpha(z) = \log q + \frac{24}{\alpha} \sum_{n=1}^{\infty} \log(1 - q^{\alpha n}),$$

we arrive at

$$q \frac{d}{dq} f_\alpha(z) = f_\alpha(z) \left( 1 - 24 \sum_{n=1}^{\infty} \frac{q^{\alpha n}}{1 - q^{\alpha n}} \right) = f_\alpha(z)P(\alpha z),$$

as desired. \qed

Another simple proof can be constructed using the theory of modular forms. Observe that $q \frac{d}{dq} f_\alpha(z) - f_\alpha(z)P(\alpha z)$ is a modular form of weight $\frac{12}{\alpha} + 2$ with level $\frac{576}{\alpha^2}$ with a proper quadratic character. Using Sturm’s bound, we only need to check that the first few terms vanish. Note that this method only works for modular forms of integral weight, i.e., when $24/\alpha$ is an even number.

In the following four lists, except for the aforementioned notational simplifications, we quote Ramanujan. In our proofs, we frequently appeal to dimensions of certain spaces of modular forms, all of which can be found in [61] or which can be calculated using MAGMA.

**Entry 11.4.1 (List I).** Suppose that $A$ and $B$ are any two integers such that $A^2 + 3B^2 = p$ and $A \equiv 1 \pmod{3}$, $p$ being a prime of the form $6k + 1$. Let

$$\sum_{n=1}^{\infty} \Omega_0(n)q^{n/6} = q^{1/6}(q;q)^4,$$

(11.4.1)
11.4 A Partial Manuscript on Euler Products

\[
\begin{align*}
\sum_{n=1}^{\infty} \Omega_1(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 P(q), \\
\sum_{n=1}^{\infty} \Omega_2(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 Q(q), \\
\sum_{n=1}^{\infty} \Omega_3(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 R(q), \\
\sum_{n=1}^{\infty} \Omega_4'(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 Q'(q), \\
\sum_{n=1}^{\infty} \Omega_4''(n)q^{n/6} &= q^{5/6}(q;q)_{\infty}^{20},
\end{align*}
\]

\[
\Omega_4(n) = \Omega_4'(n) + 288\omega\sqrt{70} \Omega_4''(n), \quad \omega^2 = 1,
\]

\[
\begin{align*}
\sum_{n=1}^{\infty} \Omega_5(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 Q(q)R(q), \\
\sum_{n=1}^{\infty} \Omega_7'(n)q^{n/6} &= q^{1/6}(q;q)_{\infty}^4 Q^2R(q), \\
\sum_{n=1}^{\infty} \Omega_7''(n)q^{n/6} &= q^{5/6}(q;q)_{\infty}^{20} R(q),
\end{align*}
\]

\[
\Omega_7(n) = \Omega_7'(n) + 10080\omega\sqrt{286} \Omega_7''(n), \quad \omega^2 = 1.
\]

In all these cases,

\[
\sum_{n=1}^{\infty} \frac{\Omega_{\lambda}(n)}{n^{s}} = \prod_{p} \frac{1}{1 - \Omega_{\lambda}(p)p^{-s} + p^{2\lambda+1-2s}},
\]

where \( p \) assumes all prime values greater than 3. If \( \lambda = 0, 2, 3, 5 \), then

\[
\Omega_{\lambda}(p) = \begin{cases} 
0, & p \equiv -1 \pmod{6}, \\
(A + iB\sqrt{3})^{2\lambda+1} + (A - iB\sqrt{3})^{2\lambda+1}, & p \equiv 1 \pmod{6}.
\end{cases}
\]

\( \Omega_1(n) = n\Omega_0(n) \) for all values of \( n \). But \( \Omega_4(n) \) and \( \Omega_7(n) \) do not seem to have such simple laws.

In our arguments below, we always work with forms supported only on integral exponents. This enables us to avoid the use of multiplier systems. Moreover, we note that we proceeded in this fashion throughout Sections 11.2 and 11.3.

Proof of (11.4.1). For the following facts, we refer to [64, Chapter 3]. The right side of (11.4.1) equals \( \eta^4(6z) \), and this is a modular form of weight 2.
and level 36 with trivial character. Though the dimension of this space is 12, its new-space has dimension 1 and its basis element is the unique new form \( \eta^4(6z) \). Therefore, its Euler product is given as [91, p. 118]

\[
\sum_{n=1}^{\infty} \frac{\Omega_0(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_0(p)p^{-s} + p^{1-2s}}.
\]

Now, we give an elementary proof of the explicit formula for \( \Omega_0(p) \). First, note that, by the pentagonal number theorem and Jacobi’s identity, respectively,

\[
\eta(z) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{(n-1)/6} q^{n^2/24}}{n \equiv 1 \pmod{6}},
\]

\[
\eta^3(z) = \sum_{n=-\infty}^{\infty} nq^{n^2/8}.
\]

Therefore,

\[
\Omega_0(p) = \sum_{n \equiv 1 \pmod{6}, \ k \equiv 1 \pmod{4}} (-1)^{(n-1)/6} k. \tag{11.4.10}
\]

For integers \( n \) and \( k \) satisfying the conditions in (11.4.10), we define

\[
A := \frac{n + 3k}{4} \quad \text{and} \quad B := \frac{n - k}{4}, \quad \text{if} \ n \equiv 1 \pmod{12},
\]

\[
A := \frac{n - 3k}{4} \quad \text{and} \quad B := \frac{n + k}{4}, \quad \text{if} \ n \equiv 7 \pmod{12}.
\]

Thus, \( A \) and \( B \) are integers satisfying \( A^2 + 3B^2 = p \), with \( A \equiv 1 \pmod{3} \). Therefore, from (11.4.10),

\[
\Omega_0(p) = \sum_{A^2 + 3B^2 = p} (A - B).
\]

Note that if \( (A, B) \) satisfies the foregoing conditions, then so does \( (A, -B) \). Therefore, we deduce that

\[
\Omega_0(p) = \begin{cases} 
2A, & \text{if} \ p \equiv 1 \pmod{6}, \ p = A^2 + 3B^2, \text{and} \ B > 0, \\
0, & \text{otherwise},
\end{cases}
\]

which completes the proof.

The form \( \eta^4(6z) \) is associated with the elliptic curve \( y^2 = x^3 + 1 \), i.e.,

\[
\Omega_0(p) = 1 + p - a(p),
\]
where $a(p)$ is the number of points on this elliptic curve after reducing modulo $p$.

Observe that (11.4.2) follows from Proposition 11.4.1 and (11.4.1).

The remaining Euler products with explicit formulas for the $p$-th coefficients can be derived from the fact that these are modular forms with complex multiplication, or, in other words, newforms associated with a certain Hecke Grössencharacter. For the following description, we refer to K. Ono’s monograph [120].

For the field $K = \mathbb{Q}(\sqrt{-3})$, we can define a Hecke Grössencharacter $\phi$ by

$$\phi((\alpha)) = \alpha^{k-1},$$

where $k \geq 2$ is an integer, and $\alpha$ is a generator of the ideal $(\alpha)$, such that $\alpha \equiv 1 \pmod{\Lambda}$, where $\Lambda = (3)$. Then,

$$\Phi(z) := \frac{1}{2} \sum_{a} \phi(a) q^{N(a)} = \frac{1}{2} \sum_{n=1}^{\infty} a(n) q^{n},$$

is a newform of weight $k$ of level 36 with a trivial character. Moreover, the ideal $(p)$ is inert if $p \equiv 5 \pmod{6}$, and if $p \equiv 1 \pmod{6}$, then $(p)$ splits in the form

$$(p) = (x + i\sqrt{3}y)(x - i\sqrt{3}y),$$

where $x$ and $y$ are integers such that $x \equiv 1 \pmod{3}$. From these, we deduce that

$$a(p) = \phi((x + i\sqrt{3}y)) + \phi((x - i\sqrt{3}y)) = (x + i\sqrt{3}y)^{k-1} + (x - i\sqrt{3}y)^{k-1},$$

which implies Ramanujan’s claim (11.4.3). Actually (11.4.3) is identical to Entry 11.2.7, for which a complete proof was given earlier.

Now we examine the entries that are represented as linear combinations of two forms.

For $\Omega_4(n)$, note that $f_{\Omega_4'}(z) = \eta^4(6z)Q^2(6z)$ and $f_{\Omega_4''}(z) = \eta^{20}(6z)$ are in $S^{\text{new}}_{10}(\Gamma_0(36))$, the new space of cusp forms. The dimension of $S^{\text{new}}_{10}(\Gamma_0(36))$ is 4, but there are only two forms for which the exponents are supported on one residue class modulo 6. Note that if $f(z) = a(n) q^{\delta n} \in S^{\text{new}}_{10}(\Gamma_0(36))$, then $a$ must be coprime to 6.

By a simple calculation for the Hecke operator $T_5$, we see that

$$T_5f_{\Omega_4'}(z) = 5806000f_{\Omega_4''}(z),$$

$$T_5f_{\Omega_4''}(z) = f_{\Omega_4'}(z).$$

The eigenvalues of the matrix

$$\begin{pmatrix} 0 & 5806000 \\ 1 & 0 \end{pmatrix}$$
are \(\pm 288\sqrt{70}\). Therefore,

\[ f_{\Omega_4}(z) = f_{\Omega'_4}(z) \pm 288\sqrt{70}f_{\Omega''_4}(z) \]

is an eigenform for \(T_5\).

Since \(T_p\) and \(T_5\) are commutative, where \(p\) is a prime larger than 5, we can conclude that \(f_{\Omega_4}(z)\) is a Hecke eigenform as was claimed. Thus, the verifications of (11.4.4)–(11.4.6) have been demonstrated.

For \(\Omega_7(n)\), the argument is exactly the same as that above. In particular,

\[ T_5(\eta^4(6z)Q^2(6z)R(6z)) = 29059430400\eta^{20}(6z)R(6z), \]
\[ T_5(\eta^{20}(6z)R(6z)) = \eta^4(6z)Q^2(6z)R(6z). \]

Therefore, the verifications of (11.4.7)–(11.4.9) follow.

Entry 11.4.2 (List II). Suppose that \(A\) and \(B\) are defined as in I and let

\[
\sum_{n=1}^{\infty} \Omega_0(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty, \quad (11.4.11)
\]
\[
\sum_{n=1}^{\infty} \Omega_1(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty P(q), \quad (11.4.12)
\]
\[
\sum_{n=1}^{\infty} \Omega'_2(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty Q(q),
\]
\[
\sum_{n=1}^{\infty} \Omega''_2(n)q^{n/3} = q^{1/3}(q; q)^{16}_\infty,
\]
\[
\Omega_2(n) = \Omega'_2(n) + 6\omega\sqrt{10}\Omega''_2(n), \quad \omega^2 = 1,
\]
\[
\sum_{n=1}^{\infty} \Omega_3(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty R(q),
\]
\[
\sum_{n=1}^{\infty} \Omega'_4(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty Q^2(q),
\]
\[
\sum_{n=1}^{\infty} \Omega''_4(n)q^{n/3} = q^{2/3}(q; q)^{16}_\infty Q(q),
\]
\[
\Omega_4(n) = \Omega'_4(n) + 6\omega\sqrt{70}\Omega''_4(n), \quad \omega^2 = 1,
\]
\[
\sum_{n=1}^{\infty} \Omega'_5(n)q^{n/3} = q^{1/3}(q; q)^{8}_\infty Q(q)R(q),
\]
\[
\sum_{n=1}^{\infty} \Omega''_5(n)q^{n/3} = q^{2/3}(q; q)^{16}_\infty R(q),
\]
\[
\Omega_5(n) = \Omega'_5(n) + 12\omega\sqrt{55}\Omega''_5(n), \quad \omega^2 = 1,
\]
\[
\sum_{n=1}^{\infty} \Omega_7'(n)q^{n/3} = q^{1/3}(q; q)^8Q(q)R(q),
\]
\[
\sum_{n=1}^{\infty} \Omega_7''(n)q^{n/3} = q^{2/3}(q; q)_{\infty}^{16}Q(q)R(q),
\]
\[
\Omega_7(n) = \Omega_7'(n) + 12\sqrt{3}10\Omega_7''(n), \quad \omega^2 = 1.
\]

In all these cases,
\[
\sum_{n=1}^{\infty} \Omega_\lambda(n) q^{ns} = \prod_p \frac{1}{1 - \Omega_\lambda(p)p^{-s} + p^{2\lambda+3-2s}},
\]
where \( p\) assumes all prime values except 3. If \( \lambda = 0 \) or 3, then
\[
\Omega_\lambda(p) = \begin{cases} 
0, & \text{if } p \equiv -1 \text{ (mod 3),} \\
(A + iB\sqrt{3})^{2\lambda+3} + (A - iB\sqrt{3})^{2\lambda+3}, & \text{if } p \equiv 1 \text{ (mod 3).}
\end{cases}
\]
\[
\Omega_1(n) = n\Omega_0(n).
\]

Note that (11.4.12) follows from (11.4.11) and Proposition 11.4.1. First observe that \( \eta^3(3z) \) is a modular form of weight 4 and level 9 with trivial character. Though the dimension of this space is 4, its new-space has dimension 2, and \( \eta^3(3z) \) is a basis element and eigenform.

It should be possible to derive explicit formulas by using Hecke Grössencharacters, which we have used in the previous entry. Note that, except for \( p = 2 \), every prime is congruent to 1 modulo 2. Thus, there is no essential difference between the explicit formulas (11.4.3) in List I and (11.4.11) in List II. For the formulas for \( \Omega_4, \Omega_5, \) and \( \Omega_7 \), the derivations are like those above, and so are omitted.

Here we give another proof for the first entry, (11.4.11). From [94, p. 373],
\[
q(q^3; q^3)^8 = \frac{1}{6} \sum_{(x,y) \in \mathbb{Z}^2} x^3 q^{x^2+3xy+3y^2}.
\]

Therefore, for \( n \in \mathbb{N} \),
\[
\Omega_0(n) = \frac{1}{6} \sum_{(x,y) \in \mathbb{Z}^2} x^3.\]

Let \( n \) be a prime. If \( p = x^2 + 3xy + 3y^2 \), then \( p \equiv x^2 \equiv 0, 1 \text{ (mod 3).} \) Hence, if \( p \equiv 2 \text{ (mod 3), then } p \neq x^2 + 3xy + 3y^2 \) and so \( \Omega_0(p) = 0 \). Now suppose that \( p \equiv 1 \text{ (mod 3).} \) We can define integers \( A \) and \( B \) uniquely by
\[
p = A^2 + 3B^2, \quad A \equiv 1 \text{ (mod 3), } B > 0.
\]
We consider the sum
\[ \sum_{(x,y) \in \mathbb{Z}^2 \atop x \equiv 2 \pmod{3}} x^3. \]

If \( y = 0 \), then \( p = x^2 \), which is not feasible. If \( x + y = 0 \), then \( y = -x \), so \( p = y^2 \), which is also not possible. If \( x + 2y = 0 \), then \( x = -2y \), and so \( p = y^2 \), which leads to a contradiction. Therefore,
\[
y \neq 0, \quad x + y \neq 0, \quad x + 2y \neq 0.
\]

Moreover, as \( p \equiv 1 \pmod{3} \), \((x,y) \not\equiv (0,0) \pmod{2}\). Thus, we arrive at
\[
\begin{align*}
x &= -A - 3B, & y &= 2B, & \text{if } (x,y) \equiv (1,0) \pmod{2} \text{ and } y > 0, \\
x &= -A + 3B, & y &= -2B, & \text{if } (x,y) \equiv (1,0) \pmod{2} \text{ and } y < 0, \\
x &= -A - 3B, & y &= A - B, & \text{if } (x,y) \equiv (1,1) \pmod{2} \text{ and } x + y > 0, \\
x &= -A + 3B, & y &= A + B, & \text{if } (x,y) \equiv (1,1) \pmod{2} \text{ and } x + y < 0, \\
x &= 2A, & y &= -A + B, & \text{if } (x,y) \equiv (0,1) \pmod{2} \text{ and } x + 2y > 0, \\
x &= 2A, & y &= A - B, & \text{if } (x,y) \equiv (0,1) \pmod{2} \text{ and } x + 2y < 0.
\end{align*}
\]

In summary, for \( p \equiv 1 \pmod{3} \),
\[
\sum_{(x,y) \in \mathbb{Z}^2 \atop x \equiv 2 \pmod{3}} x^3 = 12A^3 - 108AB^2,
\]

so \( \Omega_0(p) = \frac{1}{6}(12A^3 - 108AB^2) = 2A^3 - 18AB^2 \). Finally, we can easily check that \( \Omega_0(3) = 0 \), which completes the proof.

**Entry 11.4.3 (List III).** Suppose that \( A \) and \( B \) are integers such that \( A^2 + 4B^2 = p \) where \( p \) is of the form \( 4k + 1 \).

\[
\begin{align*}
\sum_{n=1}^{\infty} \Omega_0(n)q^{n/4} &= q^{1/4}(q;q)_\infty^6, & (11.4.13) \\
\sum_{n=1}^{\infty} \Omega_1(n)q^{n/4} &= q^{1/4}(q;q)_\infty^6P(q), & (11.4.14) \\
\sum_{n=1}^{\infty} \Omega_2(n)q^{n/4} &= q^{1/4}(q;q)_\infty^6Q(q), \\
\sum_{n=1}^{\infty} \Omega_3'(n)q^{n/4} &= q^{1/4}(q;q)_\infty^6R(q), \\
\sum_{n=1}^{\infty} \Omega_3''(n)q^{n/4} &= q^{3/4}(q;q)_\infty^{18}.
\end{align*}
\]
In all these cases,
\[
\sum_{n=1}^{\infty} \Omega_{\lambda}(n) n^{-s} = \Pi_1 \Pi_2,
\]
where
\[
\Pi_1 = \prod_p \frac{1}{1 - \Omega_\lambda(p)p^{-s} - p^{2\lambda+2-2s}},
\]
p assuming prime values of the form \(4k - 1\) and
\[
\Pi_2 = \prod_p \frac{1}{1 - \Omega_\lambda(p)p^{-s} + p^{2\lambda+2-2s}},
\]
p assuming prime values of the form \(4k + 1\). If \(\lambda = 0, 2,\) or \(4\), then
\[
\Omega_\lambda(p) = \begin{cases} 0, & p \equiv -1 \pmod{4}, \\ (A + 2iB)^{2\lambda+2} + (A - 2iB)^{2\lambda+2}, & p \equiv 1 \pmod{4}. \end{cases}
\]

\(\Omega_1(n) = n\Omega_0(n)\).

As before, (11.4.14) is a consequence of (11.4.13) and Proposition 11.4.1.

Proof of (11.4.13). First, \(\eta^6(4z)\) is again the unique newform in the space \(S_3^{new}\left(16, \left(\frac{-1}{-1}\right)\right)\). We derive explicit formulas for each \(p\)-th coefficient. Let \(K = \mathbb{Q}(i)\) and \(\Lambda = (2)\). Define a Hecke Grössencharacter by
\[
\phi((\alpha)) = \alpha^{k-1},
\]
where $k$ is an integer at least equal to 2. Then
\[
\Phi(z) := \frac{1}{4} \sum_a \phi(a) q^N(a) = \frac{1}{4} \sum_{n=1}^{\infty} a(n) q^n
\]
is a newform in the space $S^\text{new}_k(16, (\frac{-4}{\cdot})).$ Now $p \equiv -1 \pmod{4}$ is inert in $K,$ and for the primes $p \equiv 1 \pmod{4},$ we have the splitting
\[
a(p) = (x+iy)^{k-1} + (x-iy)^{k-1},
\]
where $x$ is an odd integer and $y$ is an even integer. (Ramanujan sets $y = 2B.$)

Now we give an elementary proof of the explicit formula for $\eta^6(4z) = \sum_{n=1}^{\infty} \Omega_0(n) q^n.$ Again, we use Jacobi’s identity in the form
\[
\eta^3(z) = \sum_{n \equiv 1 \pmod{4}}^{\infty} n q^{n^2/8}.
\]
Using our previous argument, we arrive at
\[
\Omega_0(p) = \sum_{(C,D) \equiv (1,1) \pmod{4}} CD. \quad (11.4.15)
\]
Now we define
\[
A := \frac{C + D}{2} \quad \text{and} \quad B := \frac{C - D}{2},
\]
which imply that $A$ is an odd number and $B$ is an even number. Thus, from (11.4.15),
\[
\Omega_0(p) = \sum_{(A,B) \equiv (1,0) \pmod{2}}^{A^2 + B^2 = p} A^2 - B^2.
\]
Note that if $(A, B)$ satisfies the conditions in the summand, then $(A, -B)$ is the only other pair satisfying the conditions. In summary, we have deduced that
\[
\Omega_0(p) = \begin{cases} 
2(A^2 - 4B^2), & \text{if } p \equiv 1 \pmod{4} \text{ and } p = A^2 + 4B^2, \\
0, & \text{if } p \equiv -1 \pmod{4},
\end{cases}
\]
which completes the proof. \qed

For $\Omega_3(n),$ $\Omega_5(n),$ and $\Omega_7(n),$ the derivations are similar to those above.

Now, we give Hecke relations for $\Omega_3(n),$ with $\omega^2 = -1$ for these entries. For the Hecke operator $T_3,$
11.4 A Partial Manuscript on Euler Products

\[ T_3(\eta^6(4z)R(q^4)) = -20160\eta^{18}(4z), \]
\[ T_3(\eta^{18}(4z)) = \eta^6(4z)R(q^4). \]

As \( T_3 \) and \( T_p \) commute for all primes \( p > 3 \) and the eigenvalues of the matrix

\[
\begin{pmatrix}
0 & -20160 \\
1 & 0
\end{pmatrix}
\]

are \( \pm 24\sqrt{35} \), we can conclude that

\[ f_{\Omega_3}(q) := \eta^6(4z)R(q^4) + \omega 24\sqrt{35}\eta^{18}(4z) \]

is a Hecke eigenform. The other claims can be derived via exactly the same argument, and so we omit the proofs.

Entry 11.4.4 (List IV). Let

\[ \sum_{n=1}^{\infty} \Omega_0(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2, \quad (11.4.16) \]
\[ \sum_{n=1}^{\infty} \Omega_1(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2P(q), \]
\[ \sum_{n=1}^{\infty} \Omega'_2(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2Q(q), \]
\[ \sum_{n=1}^{\infty} \Omega''_2(n)q^{n/12} = q^{5/12}(q; q)_{\infty}^{10}, \quad (11.4.17) \]
\[ \Omega_2(n) = \Omega'_2(n) + 48\omega \Omega''_2(n), \quad \omega^2 = 1, \]
\[ \sum_{n=1}^{\infty} \Omega'_3(n)q^{n/12} = q^{7/12}(q; q)_{\infty}^{14}, \]
\[ \Omega_3(n) = \Omega'_3(n) + 360\omega\sqrt{3}\Omega''_3(n), \quad \omega^2 = -1, \]
\[ \sum_{n=1}^{\infty} \Omega'_4(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2Q^2(q), \]
\[ \sum_{n=1}^{\infty} \Omega''_4(n)q^{n/12} = q^{5/12}(q; q)_{\infty}^{10}Q(q), \]
\[ \Omega_4(n) = \Omega'_4(n) + 672\omega \Omega''_4(n), \quad \omega^2 = 1, \]
\[ \sum_{n=1}^{\infty} \Omega'_5(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2Q(q)R(q). \]
\[
\sum_{n=1}^{\infty} \Omega_5^{(n)} q^{n/12} = q^{5/12} R(q),
\]
\[
\sum_{n=1}^{\infty} \Omega_5^{(2n)} q^{n/12} = q^{7/12} Q(q),
\]
\[
\sum_{n=1}^{\infty} \Omega_5^{(2n+1)} q^{n/12} = q^{11/12} Q(q),
\]
\[
\Omega_5(n) = \Omega_5'(n) + 96 \omega_1 \sqrt{1045} \cdot \Omega_5''(n) + 216 \omega_2 \sqrt{7315} \Omega_5'''(n) + 103680 \omega_1 \omega_2 \sqrt{72} \Omega_5''''(n),
\]
\[
\Omega_5'(n) = \Omega_5'(n) + 48 \omega_1 \sqrt{910} \cdot 2911 \Omega_5''(n) + 216 \omega_2 \sqrt{5005} \cdot 2911 \Omega_5'''(n) - 471744 \omega_1 \omega_2 \sqrt{22} \Omega_5''''(n),
\]
\[
\omega_1^2 = 1, \quad \omega_2^2 = -1.
\]

Ramanujan did not provide Euler product formulas for the entries in his final list, Entry 11.4.4. However, we can provide the missing product formula with
\[
\sum_{n=1}^{\infty} \frac{\Omega_7(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_7(p)p^{-s} + \chi(p)p^{2\lambda-2s}},
\]
where \(\chi\) is the quadratic character modulo 12 defined by
\[
\chi(p) = \begin{cases} 
1, & \text{if } p \equiv 1, 5 \pmod{12}, \\
-1, & \text{if } p \equiv 7, 11 \pmod{12}.
\end{cases}
\]

These Euler products follow from the general theory of Hecke eigenforms, for example, in [91, p. 118, equation (6.98)].

First, we give an elementary argument for the evaluation of \(a(p)\), where \(\eta^2(12z) = \sum_{n=1}^{\infty} a(n)q^n\).

**Proof of (11.4.16).** Using the pentagonal number theorem in the form
\[
\eta(z) = \sum_{n=-\infty}^{\infty} (-1)^{(n-1)/6} q^{n^2/24},
\]
we see that
\[
a(p) = \sum_{C^2 + D^2 = 2p \atop (C,D) \equiv (1,1) \pmod{6}} (-1)^{(C+D-2)/6}.
\]
Setting
\[
A := \frac{C + D}{2} \quad \text{and} \quad B := \frac{C - D}{2},
\]
we observe that \( A \equiv 1 \pmod{3} \), \( B \equiv 0 \pmod{3} \), and \( p = A^2 + B^2 \). To satisfy these conditions, \( p \) should be congruent to 1 modulo 12, since \( p \equiv 1 \pmod{4} \) and \( p \equiv 1 \pmod{3} \). On the other hand, when \( p \equiv 1 \pmod{12} \) and \( p = A^2 + B^2 \), then \( A \equiv \pm 1 \pmod{3} \) and \( B \equiv 0 \pmod{3} \). By employing an argument similar to that used before, we conclude that
\[
a(p) = \begin{cases} 
2(-1)^{(A-1)/3}, & \text{if } p \equiv 1 \pmod{12} \text{ and } p = A^2 + 9B^2, \ A \equiv 1 \pmod{3}, \\
0, & \text{otherwise}.
\end{cases}
\]

\[\Box\]

J.-P. Serre [137] proved that every \( L \)-series associated to a weight one newform is an Artin \( L \)-function attached to an irreducible two dimensional complex linear representation of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \). In the case of \( qf^2(-q^{12}) \), it is related to the dihedral group, \( D_4 \). Consult [137, pp. 242–244] for further information.

**Proof of** (11.4.16). We give an elementary proof of (11.4.16). Throughout the proof, we use the notation from Section 3. Recall that from (11.3.12),
\[
H(-144) = \{I, A, A^2, A^3\} = <A> \cong \mathbb{Z}/4\mathbb{Z},
\]
where
\[
I = [1, 0, 36], \ A = [5, 4, 8], \ A^2 = [4, 0, 9], \ A^3 = [5, -4, 8], \ \text{and} \ A^4 = I.
\]
By [139, p. 16, Theorem 3.1] with \( a = b = 1 \),
\[
(q; q)_{\infty}^2 = 1 + \sum_{n=1}^{\infty} \frac{1}{2} (R(I, 12n + 1) - R(A^2, 12n + 1)) q^n.
\]
As \( R(I, 1) = 2 \) and \( R(A^2, 1) = 0 \), we deduce that
\[
(q; q)_{\infty}^2 = \sum_{n=0}^{\infty} \frac{1}{2} (R(I, 12n + 1) - R(A^2, 12n + 1)) q^n. \tag{11.4.20}
\]
With the notation in (11.3.8),
\[
(q; q)_{\infty}^2 = \sum_{n=0}^{\infty} \phi_{12}(12n + 1) q^n. \tag{11.4.21}
\]
From (11.4.20) and (11.4.21), we deduce that
\[
\sum_{n=1}^{\infty} \Omega_0(n) q^{(n-1)/12} = \sum_{n=0}^{\infty} \phi_{12}(12n+1) q^n,
\]
so that
\[
\Omega_0(n) = \begin{cases} 
\phi_{12}(n), & \text{if } n \equiv 1 \pmod{12}, \\
0, & \text{if } n \not\equiv 1 \pmod{12}.
\end{cases} \tag{11.4.22}
\]
By [141, p. 371, Theorem 4.5 (iv)],
\[
\phi_{12}(n) = 0, \quad \text{if } n \not\equiv 1 \pmod{12}. \tag{11.4.23}
\]
Thus, from (11.4.22) and (11.4.23), we deduce that
\[
\Omega_0(n) = \phi_{12}(n), \quad n \in \mathbb{N}. \tag{11.4.24}
\]
By (11.4.24) and (11.3.16) ([141, p. 389, Theorem 8.2 (iv)]), we obtain
\[
\sum_{n=1}^{\infty} \frac{\Omega_0(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\phi_{12}(n)}{n^s} = \prod_{p \equiv 3 \pmod{4} \atop p \neq 3} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 5 \pmod{12}} \frac{1}{1 + p^{-2s}} 
\times \prod_{p = x^2 + 36y^2} \frac{1}{(1 - p^{-s})^2} \prod_{p = 4x^2 + 9y^2} \frac{1}{(1 + p^{-s})^2}. \tag{11.4.25}
\]
Let
\[
F_s(p) := \frac{1}{1 - \Omega_0(p) p^{-s} + \chi(p) p^{-2s}}. \tag{11.4.26}
\]
By (11.4.24) and [141, p. 371, Theorem 4.5 (iv)],
\[
\Omega_0(p) = \phi_{12}(p) = \begin{cases} 
2, & \text{if } p \equiv 1 \pmod{12}, p = x^2 + 36y^2, \\
-2, & \text{if } p \equiv 1 \pmod{12}, p = 4x^2 + 9y^2, \\
0, & \text{if } p \not\equiv 1 \pmod{12}.
\end{cases} \tag{11.4.27}
\]
From the definition of \(\chi(p)\) in (11.4.19), (11.4.26), and (11.4.27), we deduce that
\[
F_s(2) = 1, \tag{11.4.28}
F_s(3) = 1, \tag{11.4.29}
F_s(p) = \begin{cases} 
\frac{1}{1 - 2p^{-s} + p^{-2s}} = \frac{1}{(1 - p^{-s})^2}, & \text{if } p \equiv 1 \pmod{12}, p = x^2 + 36y^2, \\
\frac{1}{1 + 2p^{-s} + p^{-2s}} = \frac{1}{(1 + p^{-s})^2}, & \text{if } p \equiv 1 \pmod{12}, p = 4x^2 + 9y^2, \\
\frac{1}{1 + p^{-s}}, & \text{if } p \equiv 5 \pmod{12}, \\
\frac{1}{1 - p^{-2s}}, & \text{if } p \equiv 3 \pmod{4}, p \not\equiv 3.
\end{cases} \tag{11.4.30}
\]
Thus, appealing to (11.4.25), (11.4.28)–(11.4.30), and (11.4.26), we obtain
\[
\sum_{n=1}^{\infty} \frac{\Omega_0(n)}{n^s} = \prod_{p} F_s(p) = \prod_{p} \frac{1}{1 - \Omega_0(p)p^{-s} + \chi(p)p^{-2s}}
\]
as asserted.

We have shown that Ramanujan’s missing product formula for \(\lambda = 0\) in Entry 11.4.4 is the formula (11.3.16) of Sun and Williams. By [140, Theorem 7.4 (ii)], \(\phi_{12}(n)\) is a multiplicative function of \(n\), so by (11.4.24), \(\Omega_0(n)\) is a multiplicative function of \(n\).

For \(\Omega_2, \Omega_3, \text{ and } \Omega_4\), we can verify Ramanujan’s claims by using the same argument, so we omit the proofs here. For \(\Omega_5(n)\), we give a more detailed verification. Note that \(f_1(q) := \eta^2(12z)Q(q^{12})R(q^{12}), f_5(q) := \eta^{10}(12z)R(q^{12}), f_7(q) := \eta^{14}(12z)Q(q^{12})\), and \(f_{11}(q) := \eta^{22}(12z)\) are in the space \(S_{11}^{\text{new}}(\Gamma_0(144), \chi)\), where \(\chi\) is defined in (11.4.19). Moreover, each form \(f_a\) is supported on one residue class \(a\) modulo 12, which is coprime to 12. We can easily observe that \(T_5 f_a\) is supported on the residue class \(pa\) modulo 12 and that
\[
T_5 \begin{pmatrix} f_1 \\ f_5 \\ f_7 \\ f_{11} \end{pmatrix} = \begin{pmatrix} 0 & 963027 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 46080 \\ 0 & 0 & 209 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_5 \\ f_7 \\ f_{11} \end{pmatrix}.
\]
Thus, we find that
\[
f_1 + \omega_1 96\sqrt{1045}f_5 \text{ and } f_7 + \omega_1 ^{96} \sqrt{1045}f_{11}
\]
are eigenforms with eigenvalue \(96^2 \cdot 1045\) under the action of \(T_5\). By a similar calculation, we observe that
\[
f_1 + \omega_2 216\sqrt{7315}f_7 \text{ and } f_5 + \omega_2 ^{216} \sqrt{7315}f_{11}
\]
are eigenforms under the action of \(T_7\), and
\[
f_1 + \omega_3 103680\sqrt{7}f_{11} \text{ and } f_5 + \omega_3 ^{103680} \sqrt{7}f_7
\]
are eigenforms under the action of \(T_{11}\), where \(\omega_3^2 = -1\). Therefore,
\[
f_1 + \omega_1 96\sqrt{1045}f_5 + \omega_2 216\sqrt{7\cdot 1045}(f_7 + \omega_1 ^{96} \sqrt{1045}f_{11})
\]
\[
= f_1 + \omega_2 216\sqrt{7315}f_7 + \omega_1 96\sqrt{1045}(f_5 + \omega_2 ^{216} \sqrt{7\cdot 1045}f_{11})
\]
\[
= f_1 + \omega_3 193680\sqrt{7}f_{11} + \omega_1 96\sqrt{1045}(f_5 + \omega_2 ^{103680} \sqrt{7}f_7)
\]
\[
= f_1 + \omega_1 96\sqrt{1045}f_5 + \omega_2 216\sqrt{7\cdot 1045} + \omega_1 \omega_2 103680\sqrt{7}f_{11}
\]
is the Hecke eigenform as desired. For $\Omega_7$, we can use exactly the same argument, so we omit it. We remark that Rangachari [129] pointed out that the coefficient in the definition of $\Omega_7(n)$ should be 4717440 instead of 471744, as was written by Ramanujan.

We further remark that an approach of H.H. Chan, S. Cooper, and P.C. Toh [48] can be used to derive representations for certain coefficients that Ramanujan did not provide. For example, Theorem 7.1 in [48] implies that

$$\sum_{n=1}^{\infty} \Omega_2'(n)q^{n/12} = \frac{\eta^2(z)Q(q)}{\eta^2(z)} = \frac{1}{4} \sum_{\substack{\alpha \equiv 1 \pmod{6} \\ \beta \equiv 1 \pmod{6}}} (-1)^{(\alpha+\beta+4)/6}(\alpha+i\beta)^4q^{(\alpha^2+\beta^2)/24},$$

where $\Omega_2'(n)$ is defined in List IV, and Theorem 7.4 in [48] implies that

$$\sum_{n=1}^{\infty} \Omega_2(n)q^{n/4} = \frac{\eta^6(z)Q(q)}{\eta^6(z)} = -\frac{1}{8i} \sum_{\substack{\alpha \equiv 1 \pmod{4} \\ \beta \equiv 1 \pmod{4}}} (\alpha+i\beta)^6q^{(\alpha^2+\beta^2)/12},$$

which give another explicit formula for $\Omega_2(n)$ in List III.

The results in [48, Section 7] can also be used to establish some of the formulas in Lists I–III that we derived by employing Hecke Grössencharacters.
12

Continued Fractions

12.1 Introduction

Recall that the Rogers-Ramanujan continued fraction is defined by
\[ R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}, \quad |q| < 1. \] (12.1.1)

The continued fraction \( R(q) \) satisfies two famous and useful identities recorded by Ramanujan in his notebooks [28, p. 265, Entry 11(iii)], namely,
\[ \frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \] (12.1.2)

and
\[ \frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}, \] (12.1.3)

where
\[ f(-q) := (q;q)_\infty := q^{-1/24}\eta(q), \quad q = e^{2\pi i \tau}, \quad \text{Im } \tau > 0; \] (12.1.4)

the function \( \eta(q) \) is the Dedekind eta function.

12.2 Finite and Infinite Rogers-Ramanujan Continued Fractions

The first entry that we examine does not appear to be correct. Appearing in Ramanujan’s purported identity is the expression
\[ (1 - q^{5})^5(1 - q^{10})^{10} \cdots, \] (12.2.1)

so that the \( n \)th term in this product is \( (1 - q^{5n})^{5n} \). We think that (12.2.1) is incorrect, and that Ramanujan mis-recorded the second term of the product.
representation for \( f^5(-q^5) \). On the right-hand side of Ramanujan’s formula, we find the function \( F(q) \), which is not defined by Ramanujan. However, \( F(q) \) is evidently the Rogers-Ramanujan continued fraction \( R(q) \) defined in (12.1.1).

Recall that in his second letter to Hardy, Ramanujan used the same notation, but without the factor \( q^{1/5} \), to denote the Rogers-Ramanujan continued fraction [34, p. 57]. This entry is also difficult to read in [128]. There appears to be a spurious constant, possibly 5, or 25, or \( \sqrt{5} \), before the quotient on the right-hand side of (12.2.3).

**Entry 12.2.1 (p. 56).** Define

\[
Q(q) := q f^5(-q^5) f(-q). \tag{12.2.2}
\]

Then, if \( F(q) = R(q) \),

\[
\sqrt{\frac{Q(q^{1/5})}{Q(q)}} = \sqrt[5]{\left\{ \frac{1}{\sqrt{F(q)}} + \frac{\sqrt{5} - 1}{2} \sqrt{F(q)} \right\}}.
\tag{12.2.3}
\]

**Proof.** For brevity, we set \( t = F(q) = R(q) \) and \( \alpha = (1 - \sqrt{5})/2 \). We employ the identities

\[
\frac{1}{\sqrt{t}} - \alpha \sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^n + q^{2n}}, \tag{12.2.4}
\]

\[
\left( \frac{1}{\sqrt{t}} \right)^5 - (\alpha \sqrt{t})^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^n + q^{2n}}, \tag{12.2.5}
\]

which are found on page 204 in Ramanujan’s lost notebook [128], [19, pp. 21–22]. Now, with the use of (12.2.4), we find that the numerator of the right-hand side of (12.2.3) is equal to

\[
\left\{ \frac{1}{\sqrt{F(q)}} + \frac{\sqrt{5} - 1}{2} \sqrt{F(q)} \right\}^5 = \left\{ \frac{1}{\sqrt{t}} - \alpha \sqrt{t} \right\}^5
\]

\[
= \left( \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^n + q^{2n}} \right)^5. \tag{12.2.6}
\]

In (12.2.5), replace \( q \) by \( q^{1/5} \) and let \( t' = R(q^{1/5}) = F(q^{1/5}) \). Then the denominator on the right-hand side of (12.2.3) equals

\[
\left\{ \frac{1}{\sqrt{F(q^{1/5})}} + \frac{\sqrt{5} - 1}{2} \sqrt{F(q^{1/5})} \right\}^5 = \left( \frac{1}{q^{1/20}} \sqrt{\frac{f(-q^{1/5})}{f(-q^{1/5})^5}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^n + q^{2n}} \right)^5.
\]
\[
\left\{ \frac{1}{\sqrt{F(q^{1/5})}} \right\}^5 + \left\{ \frac{\sqrt{5} - 1}{2} \sqrt{F(q^{1/5})} \right\}^5 = \left( \frac{1}{\sqrt{t'}} \right)^5 - \left( \alpha \sqrt{t'} \right)^5
\]

\[
= \frac{1}{q^{1/10}} \sqrt{\frac{f(-q^{1/5})}{f(-q)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^n/5 + q^{2n}/5}.
\]

(12.2.7)

If we now divide (12.2.6) by (12.2.7), we find that the right-hand side of (12.2.3) is equal to

\[
\sqrt{\frac{f^5(-q)}{f^5(-q^5)}} \frac{q^{1/5} f(-q)}{f(-q^{1/5})} = \sqrt{\frac{Q(q^{1/5})}{Q(q)}},
\]

which establishes (12.2.3).

On page 57 of [128], Ramanujan examines four finite Rogers-Ramanujan continued fractions that we failed to examine in [19]. We state the first as Ramanujan recorded it, although it is perhaps more natural to interchange the hypothesis and conclusion.

**Entry 12.2.2 (p. 57).** If \( x^6 = 1 + x^4 \), then

\[
1 - \frac{ix}{1} - \frac{x^2}{1} + \frac{ix^3}{1} - \frac{x^4}{1} - \frac{ix^5}{1} - \frac{x^6}{1} = 0.
\]

(12.2.8)

Observe that (12.2.8) is simply the sixth partial quotient \( A_6/B_6 \) of \( F(q) := 1/(q^{-1/5} R(q)) \), where \( R(q) \) is the Rogers-Ramanujan continued fraction defined in (12.1.1), and where \( q = -ix \). Ramanujan’s claim is that if \( x^6 = 1 + x^4 \), then \( A_6 = 0 \).

**Proof.** First, for a continued fraction

\[
b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots,
\]

recall the standard recurrence relations for the \( n \)th partial quotient \( A_n/B_n \) [101, p. 6]

\[
A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2},
\]

for \( n \geq 1 \), with initial values \( A_{-1} = 1 \), \( B_{-1} = 0 \), \( A_0 = b_0 \), and \( B_0 = 1 \). We calculate the numerators \( A_n \), \( 2 \leq n \leq 6 \). To that end,

\[
A_2 = (1 - x^2) - ix,
\]

\[
A_3 = (1 - x^2 + x^4) + i(x^3 - x),
\]

\[
A_4 = 1 - x^2 + 2x^4 - x^6 - i(x - x^3 + x^5)
\]

\[
A_5 = (1 - x^2 + 2x^4 - 2x^6 + x^8) - ix(x^4 + 1)(1 - x^2 + x^4),
\]

\[
A_6 = (x - 1)(x + 1)(x^4 + 1)(x^6 - x^4 - 1) - ix(1 - x^2 + x^4)(x^6 - x^4 - 1).
\]
We see that $A_6$ is the first case when the real and imaginary parts have a common factor, which is $(x^6 - x^4 - 1)$. Ramanujan’s Entry 12.2.2 then follows.

Do further numerators contain a common factor? We can extend Ramanujan’s result with the next theorem. We do not have a general theorem, however.

**Theorem 12.2.1.** If $x^4 + 1 = 0$, then $A_7/B_7 = 0$, and if $x^{10} + 1 = 0$ but $x \neq \pm i$, then $A_8/B_8 = 0$.

**Proof.** We note that

$$A_7 = (x^4 + 1)(x^{12} - x^{10} + 2x^8 - 2x^6 + x^4 - x^2 + 1) - ix(x - 1)(x + 1)(x^4 + 1)(x^8 - x^6 + x^4 + 1)$$

and

$$A_8 = (x^4 + 1)(x^8 - x^6 + 1)(x^8 - x^6 + x^4 - x^2 + 1) + ix(x^8 - x^6 + x^4 - x^2 + 1)(x^{10} - x^8 + x^6 - x^4 - 1).$$

Since $(x^8 - x^6 + x^4 - x^2 + 1) \mid (x^{10} + 1)$, the result follows. 

The three remaining entries to be examined here are finite Rogers-Ramanujan continued fractions evaluated at roots of unity. The authors inexplicably failed to address these entries in [19]. All three results are consequences of a table found on page 133 of [19], which is incorrectly labelled, for it was erroneously assumed that Ramanujan was employing the same notation on page 46 of [128] as he was on page 57. To remedy this blunder, we redefine $P_n(x)$ and $Q_n(x)$, for each positive integer $n$, by

$$P_n(a) = Q_n(a) = 1 + ax + ax^2 + \cdots + ax^n. \quad (12.2.9)$$

Then the following table taken from [19, p. 133] is correct. In each evaluation, $x$ is a primitive $n$th root of unity.

<table>
<thead>
<tr>
<th>$n \equiv 1, 4 \pmod{5}$</th>
<th>$P_{n-2}(1)$</th>
<th>$P_{n-2}(x)$</th>
<th>$P_{n-1}(1)$</th>
<th>$P_{n-3}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 2, 3 \pmod{5}$</td>
<td>$x^{(1-\rho n)/5}$</td>
<td>$x^{-(1-\rho n)/5}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$n \equiv 0 \pmod{5}$</td>
<td>0</td>
<td>0</td>
<td>$(x^{-2n/5} + x^{2n/5}) - (x^{-n/5} + x^{n/5})$</td>
<td></td>
</tr>
</tbody>
</table>

**Entry 12.2.3 (p. 57).** If $x$ is a primitive $n$th root of unity, with $n \equiv 2, 3 \pmod{5}$, then

$$1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-1}}{1}} = 0. \quad (12.2.10)$$
\textbf{Proof.} If \( n \equiv 2, 3 \pmod{5} \), then according to the table above, \( P_{n-1}(1) = 0 \), which is precisely the assertion of Entry 12.2.3. \qed

\textbf{Entry 12.2.4 (p. 57).} If \( x \) is a primitive \( n \)th root of unity, with \( n \equiv 1, 4 \pmod{5} \), then
\[
\frac{1}{1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-2}}{1}}} = 0. \tag{12.2.11}
\]

\textbf{Proof.} From the table above, if \( n \equiv 1, 4 \pmod{5} \), then
\[
\frac{P_{n-3}(1)}{Q_{n-3}(1)} = 1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-3}}{1}}} = 0. \tag{12.2.12}
\]

The assertion (12.2.11) is equivalent to \( Q_{n-2}(1) = 0 \). Now,
\[
\frac{Q_{n-2}(1)}{P_{n-2}(1)} = 1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-2}}{1 + \frac{x^{n-2}}{1}}} = 0. \tag{12.2.13}
\]

By (12.2.12), \( P_{n-3}(1) = 0 \), and since the partial numerators of (12.2.12) are identical to those of (12.2.13), it follows that \( Q_{n-2}(1) = 0 \), which is what we sought to prove. \qed

\textbf{Entry 12.2.5 (p. 57).} If \( x \) is a primitive \( n \)th root of unity, then, if \( n \equiv 0 \pmod{5} \),
\[
1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-2}}{1}}} = 0 \tag{12.2.14}
\]
and
\[
1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-1}}{1}}} = 0. \tag{12.2.15}
\]

\textbf{Proof.} According to the table above, when \( n \equiv 0 \pmod{5} \),
\[
\frac{P_{n-2}(1)}{Q_{n-2}(1)} = 1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-2}}{1}}} = 0. \tag{12.2.16}
\]
which is the assertion (12.2.14). On the other hand,
\[
\frac{Q_{n-1}(1)}{P_{n-1}(1)} = 1 + \frac{x}{1 + \frac{x^2}{1 + \cdots + \frac{x^{n-2}}{1 + \frac{x^{n-2}}{1}}} = 0. \tag{12.2.17}
\]

Since, by (12.2.16), \( P_{n-2}(1) = 0 \), and since the partial numerators of (12.2.17) are identical to those of (12.2.16), we conclude that \( Q_{n-1}(1) = 0 \), which is equivalent to the claim (12.2.15). \qed
In the next entry, Ramanujan offers two values for the Rogers-Ramanujan continued fraction \( R(q) \).

**Entry 12.2.6 (p. 204).** Let

\[
t := R(e^{-\pi \sqrt{2}}) \quad \text{and} \quad t' := R(e^{-2\pi \sqrt{2}}).
\]

Then

\[
\frac{1}{t} - t = 1 + \xi \sqrt{5} \tag{12.2.18}
\]

and

\[
\frac{1}{t'} - t' = 1 + \frac{\sqrt{5}}{\xi}, \tag{12.2.19}
\]

where

\[
\xi^2 \frac{1 + \xi}{1 - \xi} = \frac{\sqrt{5} - 1}{2}. \tag{12.2.20}
\]

The identity (12.2.19) was proved by K.G. Ramanathan [123].

**Proof.** Recall the definitions of \( f(-q) \) and \( \eta(\tau) \) in (12.1.4). In both the proofs of (12.2.18) and (12.2.19), we employ the familiar transformation formula for the Dedekind eta function [28, p. 43, Entry 27(iii)]

\[
\eta(-1/\tau) = \sqrt{\tau} / \eta(\tau). \tag{12.2.21}
\]

First, from (12.1.2), with the use of (12.2.21), we find that

\[
\frac{1}{t} - t - 1 = \frac{f(-e^{-\pi \sqrt{2}/5})}{e^{-\pi \sqrt{2}/5} f(e^{-5\pi \sqrt{2}})}
\]

\[
= \frac{\eta(10i/\sqrt{2})}{\eta(5i \sqrt{2}/2)}
\]

\[
= \sqrt{10} \frac{\eta(10i/\sqrt{2})}{\sqrt{2} \eta(5i \sqrt{2}/2)}
\]

\[
= \sqrt{10} \frac{\eta(5i \sqrt{2}/2)}{2^{1/4} \eta(5i \sqrt{2}/2)}
\]

\[
= \frac{\sqrt{5}}{g_{50}} =: \frac{\sqrt{5}}{g},
\]

where \( g_{50} \) is Ramanujan’s class invariant [30, p. 183], [124], and where \( g = g_{50} \) satisfies the equation [30, p. 201], [144, p. ]

\[
\frac{g^3 - g^2}{g + 1} = \frac{\sqrt{5} + 1}{2}. \tag{12.2.22}
\]

If we now take \( \xi = 1/g \), we see that (12.2.22) and (12.2.20) are identical. Thus, the proof of (12.2.18) is complete.
Second, using again (12.1.2) and (12.2.21), we find that

\[
\frac{1}{t' - t} - 1 = \frac{f(-e^{-2\pi\sqrt{\frac{2}{5}}})}{e^{-2\pi\sqrt{\frac{2}{5}}/2}f(e^{-10\pi\sqrt{\frac{2}{5}}})} \\
= \frac{\eta(i\sqrt{\frac{2}{5}})}{\eta(5i\sqrt{\frac{2}{5}})} \\
= \sqrt{\frac{5}{\sqrt{\frac{2}{5}}} \eta(5i\sqrt{\frac{2}{5}}/2)} \\
= \frac{\sqrt{5/2} \eta(5i\sqrt{\frac{2}{5}}/2)}{2^{1/4} \eta(5i\sqrt{\frac{2}{5}})} \\
= \sqrt{5g_{50}} = \sqrt{5g}.
\]

Since \( g = 1/\xi \), we have completed the proof of (12.2.19). \qed

Recall from Ramanujan’s second letter to Hardy [126, p. xxviii], [34, p. 57] that if \( \alpha\beta = 1 \), with \( \alpha, \beta > 0 \), then

\[
\left\{ \frac{\sqrt{5} + 1}{2} + R(e^{-2\pi\alpha}) \right\} \left\{ \frac{\sqrt{5} + 1}{2} + R(e^{-2\pi\beta}) \right\} = \frac{5 + \sqrt{5}}{2}. \tag{12.2.23}
\]

If we let \( \alpha = \sqrt{2} \), so that \( \beta = 1/\sqrt{2} \), then (12.2.23) becomes

\[
\left\{ \frac{\sqrt{5} + 1}{2} + R(e^{-2\pi\sqrt{2}}) \right\} \left\{ \frac{\sqrt{5} + 1}{2} + R(e^{-\sqrt{2}\pi}) \right\} = \frac{5 + \sqrt{5}}{2}. \tag{12.2.24}
\]

Thus, given the value of \( t \) (or \( t' \)), we can use (12.2.24) to determine \( t' \) (or \( t \)). However, this observation does not appear to yield an easier proof than the one that we have given.

Further values for the Rogers-Ramanujan continued fraction can be found in [30, Chapter 32] and [92], for example.
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