Sect. 7.8 # 19

$$x' = Ax = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (1)$$

$$|A - rI| = \begin{vmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{vmatrix} = (5-r)\left[-3+2r+r^2\right] + 3\left[8-8r\right] - 2\left[4-4r\right]$$

$$= -15 + 10r + 5r^2 + 3r - 2r^2 - r^3 + 24 - 24r - 8 + 8r$$

$$= -r^3 + 3r^2 - 3r + 1 = -(r-1)^3$$

$\therefore$ 1, 1, 1 are the characteristic roots. Thus,

$$(A - I)\xi = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 3x_2 - 2x_3 = 0$$

The other 2 equations are multiples of this equation. Take $x_1 = 1, x_2 = 0, x_3 = 2$ or $x_1 = 0, x_2 = 2, x_3 = -3$. Thus, we get 2 eigenvectors

$$\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \xi^{(2)} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

and 2 solutions

$$x^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^t, \quad x^{(2)} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} e^t$$

For a third solution, write

$$x = \xi t e^t + \eta e^t$$

where ξ can be taken as $\xi^{(1)}$ or $\xi^{(2)}$ and η is to be determined. Substitute into (1) to get

$$\xi e^t + \xi t e^t + \eta e^t = A(\xi t e^t + \eta e^t)$$

Cancel e^t. Thus,

$$\xi = t(A - I)\xi + (A - I)\eta$$

But ξ is an eigenvector. So

$$(A - I)\xi = 0. \qquad (2)$$

Thus

$$(A - I)\eta = \xi. \qquad (3)$$

Operate by $A - I$ on both sides. Thus, by (2),

$$(A - I)^2\eta = (A - I)\xi = 0$$

An elementary calculation gives

$$(A - I)^2 = 0 \text{ (3×3 matrix of 0's)}$$

Thus, η is arbitrary. Following the authors, let $\eta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus, we have from (3),

$$(A - I)\eta = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}.$$

Thus, our third solution is

$$x^{(3)} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} t e^{t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t}.$$

Note that $$\xi = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix},$$

i.e. $\xi = -2\,\xi^{(1)} - 2\,\xi^{(2)}$.

We take our three solutions to form a fundamental matrix. Factor out e^{t}. Thus

$$\Psi(t) = e^{t} \begin{bmatrix} 1 & 0 & -2t \\ 0 & 2 & -4t \\ 2 & -3 & 2t+1 \end{bmatrix}$$