How to Find Power Series Solutions

1. First note the point x_0 around which you want to find solutions.
2. Determine if x_0 is an ordinary point or a singular point.
3. If x_0 is a singular point, check to see if it is regular or irregular.
4. If x_0 is a regular singular point, as a check on subsequent work, calculate the indicial polynomial and its roots.
5. Remember before you substitute power series in the differential equation, everything in the differential equation must be expressed in terms of series in $x - x_0$.
6. Substitute

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \qquad \text{or} \qquad \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r},$$

according as x_0 is an ordinary point or a regular singular point.

7. If x_0 is an ordinary point, there always exist two power series solutions about x_0. The initial coefficients a_0 and a_1 will be arbitrary. You can find these two solutions by setting $a_0 = 1, a_1 = 0$, and then reversing the roles.
8. If x_0 is a regular singular point, only a_0 will be arbitrary.
9. After substituting, make all powers of $x - x_0$ the same, by changing appropriate indices of summation.
10. Extract those terms which do not appear in all of your series. Then combine all series together.
11. Equate coefficients of like powers of $x - x_0$.
12. Obtain a recurrence formula, and, if possible, write the recurrence formula in factored form. This will help you more easily determine a pattern for the coefficients. In your recurrence formula, you will always want to solve for the a_n with the largest index.
13. Calculate coefficients until you see a pattern. Remember to keep together those factors arising from the same factor in the recurrence formula.
14. Write down the general formula for the coefficients that you observed above. If you have doubts, check it with your calculations of the particular coefficients. If everything checks out and you still have doubts, you may want to try to prove the proposed general formula by induction on n.
15. Substitute your general formula into the original series to get your series solution in each case. Remember a_n is the coefficient of $(x - x_0)^n$, so if, for example, you found all odd indexed coefficients to be 0, don't forget to take this into account in your final solution.
16. If x_0 is a regular singular point and the indicial roots differ by an integer, there may only exist one power series solution. If the roots are unequal, a power series solution will exist for the larger root. The second solution may involve a logarithm, and you don't know if it will or will not until you start to solve the differential equation. If the indicial roots are equal, the second solution always involves a logarithm.
17. When the indicial roots differ by an integer, the second solution has the form

$$\sum_{n=0}^{\infty} b_n(x - x_0)^{n+r_2} + Cy_1(x)\log x,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $y_1(x)$ is the first solution corresponding to the larger root r_1, and r_2 is the smaller indicial root.

18. Substitute the series above into the differential equation and solve for the coefficients as described above. In all cases, the logarithmic terms will cancel out after you substitute.

19. Finally, check where your solutions converge. You have two options. You can use the ratio test, or you can use the theorem on the nearest singularity to x_0. The latter test gives you only a minimal interval of convergence. However, except when the series terminates, the latter test will generally give you the exact interval of convergence.