1. a. \( x^2y'' - 7xy' + 16y = 0 \) is an Euler's ODE. For \( y'' - \frac{7}{x} y' + \frac{16}{x^2} = 0 \).

\( n(n-1) - 7n + 16 = 0 \Rightarrow n^2 - 8n + 16 = (n-4)^2 = 0 \Rightarrow n = 4 \).

We have a double root of the indicial polynomial. Thus,

\[ y = c_1 x^4 + c_2 x^4 \log x \]

b. \( xp(x) = x \cdot -\frac{7}{x} = -7 \) is analytic everywhere.

c. \( x^2q(x) = x^2 \cdot \frac{16}{x^2} = 16 \) is analytic everywhere.

\( x = 0 \) is a regular singular point.

2. We first solve \( y'' - 3y' - 18y = 0 \). The characteristic polynomial is

\[ n^2 - 3n - 18 = (n-6)(n+3) = 0 \Rightarrow n = 6, -3 \]

\( \therefore y_1(x) = c_1 e^{6x} + c_2 e^{-3x} \)

6 is a root of order 1. Thus,

\[ x(x^2 + Bx + C) e^{6x} \]

is a particular solution. Thus, the general solution is

\[ y(x) = c_1 e^{6x} + c_2 e^{-3x} + x(Ax^2 + Bx + C) e^{6x} + (Dx + E) \cos(2x) + (Fx + G) \sin(2x) \]
3. a. \((2-x)x^2y'' - 2xe^x y' + x^2(x+5)y = 0\)
\[y'' - \frac{2e^x}{x(2-x)} y' + \frac{x+5}{2-x} y = 0\]

\(x = 0, 2\) are singular points.

\(x p(x) = \frac{-2e^x}{2-x} = -1 + \ldots\) is analytic at \(x = 0\).

\(x^2 q(x) = \frac{x^2(x+5)}{2-x} = 0 + \ldots\) is analytic at \(x = 0\).

Thus, \(x = 0\) is a regular singular point.

\((x-2)p(x) = \frac{2e^x}{x} \) is analytic at \(x = 2\).

\((x-2)^2 q(x) = -(x+5)(x-2) \) is analytic at \(x = 2\).

Thus, \(x = 2\) is a regular singular point.

b. As indicated above, \(p_0 = -1\), (constant term in \(\sum_{n=0}^{\infty} p_n x^n\))

\(q_0 = 0\), (constant term in \(\sum_{n=0}^{\infty} q_n x^n\)).

Thus, \(n(n-1) - n = n^2 - 2n = 0 \Rightarrow n = 0, 2\).

c. \(y_1(x) = x^\ell \sum_{n=0}^{\infty} a_n x^n\)

\(y_2(x) = x^{\ell + 2} \sum_{n=0}^{\infty} b_n x^n\)
4 a. \( L [y] = (x^2 - 3)y'' - xy' - 3y = 0, \) obtain ordinary point

Let \( y = \sum_{n=0}^{\infty} a_n x^n, \) then,

\[
L [y] = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n n x^n - 3 \sum_{n=0}^{\infty} a_n x^n
\]

Replace \( n \) by \( n+2 \) in the 2nd sum.

\[
L [y] = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)x^n
\]

\[
= -3a_2 x^0 - 3a_0 x + [ -3a_3 x + a_1 x - 3a_1 ] x
\]

\[
+ \sum_{n=2}^{\infty} \left[ a_n (n(n-1)) - a_n n - 3a_n - 3a_{n+2} (n+2)(n+1) \right] x^n
\]

\( = 0. \) Thus,

\[
a_2 = \frac{-a_0}{2}, \quad a_3 = \frac{-4a_1}{3 \cdot 3 \cdot 2} = \frac{-2a_1}{3^2}
\]

For \( n \geq 0, \)

\[
a_{n+2} = \frac{a_n (n(n-1) + 1)(n+1)}{3(n+2)(n+1)} = a_n (n-3)(n+1)
\]

\[
\therefore \quad a_{n+2} = \frac{a_n (n-3)(n+1)}{3(n+2)(n+1)} = \frac{a_n (n-3)}{3(n+2)}, \quad n \geq 0 \quad (*)
\]

Let \( a_0 = 1, a_1 = 0. \) From \((*)\), we see that \( a_3 = a_5 = \cdots = 0 \)

\[
a_2 = -\frac{1}{2}, \quad a_4 = \frac{-1}{3.4} a_2 = \frac{1}{3.2.4}, \quad a_6 = \frac{1}{3.6} a_4 = \frac{1}{3.2.4.6}
\]

\[
a_4 = \frac{3}{3.8} a_6 = \frac{4.3}{3^3.2.4.8} \quad a_{2n} = \frac{1.3 \cdots (2n-5)}{3^{n-1} 2.4 \cdots (2n)}
\]

\[
y_1(x) = 1 - \frac{1}{2} x^2 + \frac{4}{3.2.4} x^4 + \sum_{n=3}^{\infty} \frac{1.3 \cdots (2n-5)}{3^{n-1} 2^n n!} x^{2n}
\]
Let \( a_0 = 0, a_1 = 1 \). Then \( a_2 = a_4 = \ldots = 0 \).
\[
a_3 = \frac{a_1(-2)}{3} = -\frac{2}{3}, \quad a_5 = \frac{a_3 \cdot 0.4}{5 \cdot 3} = 0.
\]
We then see that \( a_7 = a_9 = \ldots = 0 \).

Thus,
\[
y_2(x) = x - \frac{2}{3^2}x^3.
\]

b. There are singular points at \( x^2 - 3 = 0 \) or \( x = \pm \sqrt{3} \). Thus, \( y_1(x) \) converges for \( |x| < \sqrt{3} \).

Clearly, \( y_2(x) \) converges everywhere.