MATH 441, EXAM 2, SOLUTIONS

1 $y'' - y' - 2y = 3e^{2t}$

$r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r = 2, -1$

$\therefore y_h(t) = c_1 e^{2t} + c_2 e^{-t}$

2 is a root of order 1. Thus, $y_p(t) = Ate^{2t}$

$y_p' = Ae^{2t} + 2Ate^{2t} = Ae^{2t}(1+2t)$

$y_p'' = 2Ae^{2t}(1+2t) + 2Ae^{2t} = 4Ae^{2t} + 4Ate^{2t}$

$L[y] = 4Ae^{2t} + 4Ate^{2t} - (1+2t)Ae^{2t} - 2Ate^{2t}$

$te^{2t}: \; 4A - 2A - 2A = 0$

$e^{2t}: \; 4A - A = 3A = 3 \Rightarrow A = 1$

$\therefore y_p(t) = te^{2t}$

$y = c_1 e^{2t} + c_2 e^{-t} + te^{2t}$

2, $y''' - 2y'' + y' - 2y = 3t\cos t + 5 + t^3 e^{+2t}$

$r^3 - 2r^2 + r - 2 = (r^2+1)(r-2) = 0 \Rightarrow r = \pm i, 2$

$\therefore y_h(t) = C_1 \cos t + C_2 \sin t + C_3 e^{2t}$

$\pm i$ are roots of order 1, Thus,

$y_{p1}(t) = t(At+B)\cos t + t(Ct+D)\sin t$

0 is not a root, Thus, $y_{p2}(t) = E$

2 is a root of order 1. Thus, $y_{p3}(t) = t(Ft^3 + Gt^2 + Ht + I)e^{2t}$

$\therefore y_p(t) = t(At+B)\cos t + t(Ct+D)\sin t + E$

$+ t(Ft^3 + Gt^2 + Ht + I)e^{2t}$

3. $y'' + x^3 y' + x^2 y = 0$

a. x^3, x^2 are analytic everywhere. Thus, 0 is an ordinary point and

b. power series solutions in x thus converge everywhere, i.e, $R = \infty$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$L[y] = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2}$$

Let $n \to n+4$ in the first series. Then

$$L[y] = \sum_{n=-2}^{\infty} a_{n+4}(n+4)(n+3) x^{n+2} + \sum_{n=1}^{\infty} a_n n x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Equate coeffs. to 0. Let $a_0 = 1, a_1 = 0$

x^0: $a_2 \cdot 2 \cdot 1 = 0 \Rightarrow a_2 = 0$

x^1: $a_3 \cdot 3 \cdot 2 = 0 \Rightarrow a_3 = 0$

x^2: $a_4 \cdot 4 \cdot 3 + a_0 = 0 \Rightarrow a_4 = \frac{-1}{4 \cdot 3}$

For $n \geq 1$,

x^{n+2}: $a_{n+4}(n+4)(n+3) + a_n[n+1] = 0$

$$a_{n+4} = \frac{-a_n(n+1)}{(n+4)(n+3)}$$

$a_5 = \frac{-a_1 \cdot 2}{5 \cdot 4} = 0$, $a_9 = \frac{-a_5 \cdot 6}{9 \cdot 8} = 0$ We see $a_{4n+1} = 0, n \geq 0$

$a_6 = \frac{-a_2 \cdot 3}{6 \cdot 5} = 0$, $a_{10} = \frac{-a_6 \cdot 7}{10 \cdot 9} = 0$. We see $a_{4n+2} = 0, n \geq 0$

$a_7 = \frac{-a_3 \cdot 4}{7 \cdot 6} = 0$, $a_{11} = 0, \ldots, a_{4n+3} = 0$

$a_8 = \frac{-a_4 \cdot 5}{8 \cdot 7} = \frac{1 \cdot 5}{8 \cdot 7 \cdot 4 \cdot 3}$, $a_{12} = \frac{-a_8 \, 9}{12 \cdot 11} = \frac{+1 \cdot 5 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 3 \cdot 7 \cdot 11}$

In general,

$$a_{4n} = \frac{(-1)^n \, 1 \cdot 5 \cdots (4n-3)}{4 \cdot 8 \cdots 4n \; 3 \cdot 7 \cdots (4n-1)}, \quad n \geq 1$$

Thus,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n} \, 1 \cdot 5 \cdots (4n-3)}{4^n n! \, 3 \cdot 7 \cdots (4n-1)}$$

Let $a_0 = 0, a_1 = 1$. By same calculation as above, $a_2 = a_3 = 0$

Now

x^2: $a_4 \cdot 4 \cdot 3 + a_0 = 12 a_4 + 0 = 0 \Rightarrow a_4 = 0$

$$a_5 = \frac{-a_1 \cdot 2}{5 \cdot 4} = -\frac{2}{5 \cdot 4}$$

$$a_6 = \frac{-a_2 \, 3}{6 \cdot 5} = 0 \quad \therefore \; a_{4n+2} = 0$$

$$a_7 = \frac{-a_3 \cdot 4}{7 \cdot 6} = 0 \quad \therefore \; a_{4n+3} = 0$$

$$a_8 = \frac{-a_4 \cdot 5}{8 \cdot 7} = 0 \quad \therefore \; a_{4n} = 0$$

$$a_9 = \frac{-a_5 \cdot 6}{9 \cdot 8} = \frac{2 \cdot 6}{4 \cdot 8 \cdot 5 \cdot 9}$$

$$a_{13} = \frac{-a_9 10}{13 \cdot 12} = \frac{-2 \cdot 6 \cdot 10}{4 \cdot 8 \cdot 12 \cdot 5 \cdot 9 \cdot 13}$$

In general

$$a_{4n+1} = \frac{(-1)^n 2 \cdot 6 \cdots (4n-2)}{4 \cdot 8 \cdots 4n \; 5 \cdot 9 \cdots (4n+1)}, \; n \geq 1.$$

$$\therefore \quad y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1} \, 2 \cdot 6 \cdots (4n-2)}{4^n n! \; 5 \cdot 9 \cdots (4n+1)}$$

4. $x^2(x-1)y'' - xy' + 2y = 0$

a. $y' - \frac{1}{x(x-1)} y' + \frac{2}{x^2(x-1)} y = 0$

$x\,p(x) = -\frac{1}{x-1}$, $x^2 q(x) = \frac{2}{x-1}$ Both analytic at $x=0$.

Thus 0 is a regular singular point.

$(x-1)\,p(x) = -\frac{1}{x}$, $(x-1)^2 q(x) = \frac{2(x-1)}{x^2}$ Both analytic at $x=1$

Thus, 1 is a regular singular pt.

b. 1 is the nearest singular pt. to 0. Thus, series solutions around $x=0$ converge at least for $|x|<1$

c. $r(r-1) + r - 2 = r^2 - 2 = 0 \Rightarrow r = \pm\sqrt{2}$ indicial roots

d. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$L[y] = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r+1} - \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r}$$
$$- \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + 2\sum_{n=0}^{\infty} a_n x^{n+r}$$
$$= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)x^{n+r}$$
$$- \sum_{n=0}^{\infty} a_n\left[(n+r)(n+r-1) + (n+r) - 2\right]x^{n+r} = 0$$

Let ~~m = 1. Recall $a_0 = 1$~~. Thus,

$$\cancel{(r)(r-1) - a_1[(r+1)r + (r+1) - 2] = 0}$$

$$a_{n-1}(n+r-1)(n+r-2) = a_n\left[(n+r)^2 - 2\right]$$

or $$a_n = \frac{a_{n-1}(n+r-1)(n+r-2)}{(n+r)^2 - 2}, \quad n \geq 1$$

For $n=0$, $a_0\left[r(r-1) + r - 2\right] = a_0\left[r^2 - 2\right] = 0$

$\therefore\ r = \pm\sqrt{2}$