Solutions

1. This is a linear first order equation that we can solve by finding an integrating factor, which is
\[ \mu(t) = \exp \left( - \int \frac{2}{t} \, dt \right) = e^{-2 \log t} = e^{\log t^{-2}} = t^{-2}. \]

Hence,
\[ \frac{d}{dt}(yt^{-2}) = t^{3}. \]

Integrating both sides, we find that
\[ yt^{-2} = \frac{t^{4}}{4} + c \quad \Rightarrow \quad y = \frac{1}{4} t^{6} + ct^{2}, \quad c = \text{constant}. \]

2. We write the equation in the form
\[ M(x, y)dx + N(x, y)dy = 0. \] (1)

Then
\[ M_y = 6x^2y, \quad N_x = 6x^2y. \]

Thus, (1) is exact, and there exists a function \( \Psi(x, y) \) such that
\[ \frac{\partial \Psi}{\partial x} = M(x, y) = (3x^2y^2 + x^2), \quad \frac{\partial \Psi}{\partial y} = N(x, y) = (2x^3y + y^2). \] (2)

Integrating the first equation of (2), we find that
\[ \Psi(x, y) = \int (3x^2y^2 + x^2) \, dx + g(y) = x^3y^2 + \frac{1}{3}x^3 + g(y). \] (3)

Differentiating (3) with respect to \( y \) and using the second equation of (2), we see that
\[ \frac{\partial \Psi}{\partial y} = 2x^3y + g'(y) = 2x^3y + y^2. \]

Thus,
\[ g'(y) = y^2 \quad \Rightarrow \quad g(y) = \frac{1}{2}y^3 + c. \]

Hence,
\[ \Psi(x, y) = x^3y^2 + \frac{1}{3}x^3 + \frac{1}{2}y^3 + c, \]

Hence the solutions are given by
\[ x^3y^2 + \frac{1}{3}x^3 + \frac{1}{2}y^3 + c = 0. \]

3. a. We separate variables to find that
\[ y^3 \, dy = \frac{t}{t^2 - 4} \, dt. \]

Integrating, we get
\[ \frac{1}{4}y^4 = \frac{1}{2} \log(t^2 - 4) + c' \quad \Rightarrow \quad y^4 = 2 \log(t^2 - 4) + c. \]

b. Using the initial condition, we have
\[ 16 = 2 \log(5 - 4) + c \quad \Rightarrow \quad c = 16. \]
Hence,
\[ y^4 = 2 \log(t^2 - 4) + 16 \Rightarrow y = (2 \log(t^2 - 4) + 16)^{1/4}. \]
We must take the positive fourth root because the value of the initial condition, i.e., 2, is positive.

c. Our basic existence and uniqueness theorem guarantees that there exists an interval 
\[-h < t - \sqrt{5} < h,\] 
where our solution is valid. Since there exists a point of discontinuity at 
t = 2 we see that our solution is valid for 
\[-(\sqrt{5} - 2) < t - \sqrt{5} < (\sqrt{5} - 2).\] 
In particular, 
t > 2 is the lower bound. However, an examination of our solution tells us that we can 
actually replace the upper bound \(\sqrt{5} - 2\) by \(\infty\).

4. Note that
\[ f(t, y) = \cos(y^2 e^t) \quad \text{and} \quad \frac{\partial f}{\partial y} = -2ye^2 \sin(y^2 e^t) \]
are continuous for \(-\infty < t, y < \infty\). Thus, by our basic existence and uniqueness theorem, 
for any initial condition, in particular, for \(y(0) = \pi^{13}\), there exists a unique solution. 
Hence, Jasper was wrong.

5. a. The characteristic equation is
\[ r^2 - r - 6 = (r - 3)(r + 2) = 0 \Rightarrow r = 3, -2. \]
Thus, a fundamental set is \(e^{3t}, e^{-2t}\).

b. Set
\[ y(t) = c_1 e^{3t} + c_2 e^{-2t} \Rightarrow y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}. \]
So,
\[ y(0) = c_1 + c_2 = 3, \]
\[ y'(0) = 3c_1 - 2c_2 = -1. \]
Solving this pair of equations, we get \(c_1 = 1, c_2 = 2\). Hence,
\[ y = e^{3t} + 2e^{-2t}. \]

6. a.
\[ L[t^2] = t^2 \cdot 2 - t^2 = 0, \quad L[1/t] = t^2 \frac{2}{t^3} - \frac{2}{t} = 0. \]
Thus, \(y_1\) and \(y_2\) are solutions. To show that they are a fundamental set, we calculate
\[ W(y_1, y_2) = \begin{vmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{vmatrix} = -3 \neq 0. \]
Since \(W(y_1, y_2) \neq 0\), we can conclude that \(t^2, 1/t\) form a fundamental set.

b. \(t = 0\) is a point of discontinuity for the equation \(y'' - \frac{2y}{t^2} = 0\). Thus, our solutions 
are valid on either \((0, \infty)\) or \((-\infty, 0)\).

7. a. In the usual notation for a linear second order differential equation,
\[ p(t) = \frac{3}{t(1 - t)}, \quad q(t) = \frac{-2 - t}{t(1 - t)}. \]
Both \(p(t)\) and \(q(t)\) have discontinuities at \(t = 0, 1\). Thus, \(t_0 \neq 0, 1\).
b. $p(t)$ and $q(t)$ are continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. Thus, by our basic existence and uniqueness theorem, there will be a unique solution of our differential equation on each of these three intervals, depending on the value of $t_0$.

8. Clearly, $\lim_{n \to \infty} f_n(t) = 0$. Let $\epsilon > 0$ be given. We want to show that there exists an integer $N$ such that for $n \geq N$ and all $t > 0$,

$$|f_n(t) - 0| < \epsilon.$$ 

So, let $N > 1/\epsilon^2$. Then

$$|f_n(t) - 0| = \left| \frac{\cos(nt)}{\sqrt{n}} - 0 \right| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \frac{1}{\sqrt{1/\epsilon^2}} = \epsilon.$$ 

Thus, we have shown what is desired for uniform convergence on $(0, \infty)$. 