Math 441 Exam No. 1

Solutions

1. This is a linear first order equation that we can solve by finding an integrating factor, which is

$$\mu(t) = \exp\left(-\int \frac{2}{t}dt\right) = e^{-2\log t} = e^{\log t^{-2}} = t^{-2}.$$

Hence,

$$\frac{d}{dt}(yt^{-2}) = t^3.$$

Integrating both sides, we find that

$$yt^{-2} = \frac{t^4}{4} + c \qquad \Rightarrow \qquad y = \frac{1}{4}t^6 + ct^2, \quad c = \text{constant}.$$

2. We write the equation in the form

$$M(x,y)dx + N(x,y)dy = 0. \tag{1}$$

Then

$$M_y = 6x^2y, \quad N_x = 6x^2y.$$

Thus, (1) is exact, and there exists a function $\Psi(x,y)$ such that

$$\frac{\partial\Psi}{\partial x} = M(x,y) = (3x^2y^2 + x^2), \qquad \frac{\partial\Psi}{\partial y} = N(x,y) = (2x^3y + y^2). \tag{2}$$

Integrating the first equation of (2), we find that

$$\Psi(x,y) = \int(3x^2y^2 + x^2)dx + g(y) = x^3y^2 + \tfrac{1}{3}x^3 + g(y). \tag{3}$$

Differentiating (3) with respect to y and using the second equation of (2), we see that

$$\frac{\partial\Psi}{\partial y} = 2x^3y + g'(y) = 2x^3y + y^2.$$

Thus,

$$g'(y) = y^2 \qquad \Rightarrow \qquad g(y) = \tfrac{1}{3}y^3 + c.$$

Hence,

$$\Psi(x,y) = x^3y^2 + \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + c,$$

Hence the solutions are given by

$$x^3y^2 + \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + c = 0.$$

3. a. We separate variables to find that

$$y^3dy = \frac{t}{t^2-4}dt.$$

Integrating, we get

$$\tfrac{1}{4}y^4 = \tfrac{1}{2}\log(t^2-4) + c' \qquad \Rightarrow \qquad y^4 = 2\log(t^2-4) + c.$$

b. Using the initial condition, we have

$$16 = 2\log(5-4) + c = c \qquad \Rightarrow \qquad c = 16.$$

Hence,

$$y^4 = 2\log(t^2-4)+16 \qquad \Rightarrow \qquad y = +\left(2\log(t^2-4)+16\right)^{1/4}.$$

We must take the positive fourth root because the value of the initial condition, i.e., 2, is positive.

c. Our basic *existence and uniqueness theorem* guarantees that there exists an interval $-h < t-\sqrt{5} < h$ where our solution is valid. Since there exists a point of discontinuity at $t=2$ we see that our solution is valid for $-(\sqrt{5}-2) < t-\sqrt{5} < (\sqrt{5}-2)$. In particular, $t > 2$ is the lower bound. However, an examination of our solution tells us that we can actually replace the upper bound $\sqrt{5}-2$ by ∞.

4. Note that

$$f(t,y) = \cos(y^2 e^{t^2}) \quad \text{and} \quad \frac{\partial f}{\partial y} = -2ye^{t^2}\sin(y^2 e^{t^2})$$

are continuous for $-\infty < t, y < \infty$. Thus, by our basic *existence and uniqueness theorem*, for any initial condition, in particular, for $y(0) = \pi^{13}$, there exists a unique solution. Hence, Jasper was wrong.

5. a. The characteristic equation is

$$r^2 - r - 6 = (r-3)(r+2) = 0 \qquad \Rightarrow \qquad r = 3, -2.$$

Thus, a fundamental set is e^{3t}, e^{-2t}.

b. Set

$$y(t) = c_1 e^{3t} + c_2 e^{-2t} \qquad \Rightarrow \qquad y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}.$$

So,

$$y(0) = c_1 + c_2 = 3,$$
$$y'(0) = 3c_1 - 2c_2 = -1.$$

Solving this pair of equations, we get $c_1 = 1, c_2 = 2$. Hence,

$$y = e^{3t} + 2e^{-2t}.$$

6. a.

$$L[t^2] = t^2\cdot 2 - t^2 = 0, \qquad L[1/t] = t^2\frac{2}{t^3} - \frac{2}{t} = 0.$$

Thus, y_1 and y_2 are solutions. To show that they are a fundamental set, we calculate

$$W(y_1,y_2) = \begin{vmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{vmatrix} = -3 \neq 0.$$

Since $W(y_1,y_2) \neq 0$, we can conclude that $t^2, 1/t$ form a fundamental set.

b. $t=0$ is a point of discontinuity for the equation $y'' - \dfrac{2y}{t^2} = 0$. Thus, our solutions are valid on either $(0,\infty)$ or $(-\infty,0)$.

7. a. In the usual notation for a linear second order differential equation,

$$p(t) = \frac{3}{t(1-t)}, \qquad q(t) = -\frac{2-t}{t(1-t)}.$$

Both $p(t)$ and $q(t)$ have discontinuities at $t = 0, 1$. Thus, $t_0 \neq 0, 1$.

b. $p(t)$ and $q(t)$ are continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. Thus, by our basic *existence and uniqueness theorem*, there will be a unique solution of our differential equation on each of the these three intervals, depending on the value of t_0.

8. Clearly, $\lim_{n\to\infty} f_n(t) = 0$. Let $\epsilon > 0$ be given. We want to show that there exists an integer N such that for $n \geq N$ and all $t > 0$,

$$|f_n(t) - 0| < \epsilon.$$

So, let $N > 1/\epsilon^2$. Then

$$|f_n(t) - 0| = \left|\frac{\cos(nt)}{\sqrt{n}} - 0\right| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \frac{1}{\sqrt{1/\epsilon^2}} = \epsilon.$$

Thus, we have shown what is desired for uniform convergence on $(0, \infty)$.