EXTRA PROBLEM

$L[y] = (1-x^3)y'' + 2x^2y' - 2xy = 0$

0 is an ordinary point. Let $y = \sum_{n=0}^{\infty} a_n x^n$, then

$$L[y] = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n+1} + 2\sum_{n=1}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= \sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)^{n+1} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n+1} + 2\sum_{n=1}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1}$$

Equate coefficients,

x^0: $a_2 \cdot 2 \cdot 1 = 0 \Rightarrow a_2 = 0$

x^1: $a_3 \cdot 3 \cdot 2 - 2a_0 = 0 \Rightarrow a_3 = \frac{a_0}{3} = \frac{2a_0}{2 \cdot 3}$

x^2: $a_4 \cdot 4 \cdot 3 + 2a_1 - 2a_1 = 0 \Rightarrow a_4 = 0$

For $n \geq 2$

$a_{n+3}(n+3)(n+2) = a_n[n(n-1) - 2n + 2] = a_n(n-2)(n-1)$.

Thus, $a_{n+3} = \frac{a_n(n-2)(n-1)}{(n+3)(n+2)}$, $n \geq 2$

Let $n = 2$. $a_5 = a_2 \frac{0 \cdot 1}{5 \cdot 4} = 0$

Let $n = 3$. $a_6 = \frac{a_3 \cdot 1 \cdot 2}{6 \cdot 5} = \frac{a_0 4 \cdot 2}{6 \cdot 5 \cdot 3} = \frac{a_0 \cdot 2^2}{6 \cdot 5 \cdot 3 \cdot 2}$

Let $n = 4$ $a_7 = \frac{a_4 2 \cdot 3}{7 \cdot 6} = 0$

Let $n = 5$ $a_8 = \frac{a_5 \cdot 3 \cdot 4}{8 \cdot 7} = 0$

Let $n = 6$ $a_9 = a_6 \frac{4 \cdot 5}{9 \cdot 8} = a_0 \frac{1 \cdot 2^2 \cdot 4 \cdot 5}{6 \cdot 5 \cdot 3 \cdot 9 \cdot 8 \cdot 2}$

Note, from the recurrence relation, $a_2 = a_5 = a_8 = \cdots = a_{3n+2} = 0$

Also, from the recurrence relation, $a_4 = a_7 \cdots = a_{3n+1} = 0$, $n \geq 1$

Now let $a_0 = 1$, $a_1 = 0$. We have a factor of 2 in each term a_{3n}, $n \geq 1$. In the numerator we omit each multiple of 3, and the last integer is $3n-4$. In the denominator, we eliminate each integer of the form $3n-2$. Thus for $n=1$, we eliminate 1. For $n=2$, we omit 1 and 4. For $n=3$, we eliminate 1, 4, 7. Thus, #2

$$y_1(x) = 1 + 2\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 4\cdot 5\cdots(3n-5)(3n-4)}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)} x^{3n}.$$

Now let $a_0 = 0$, $a_1 = 1$. Thus, $a_2 = a_3 = a_4 = 0$. By the recurrence relation, $a_{3n} = a_{3n+1} = a_{3n+2} = 0$, $n \geq 1$. Thus,

$$y_2(x) = x,$$

which is a trivial solution that we can see immediately. The distance from 0 to the nearest singular point is 1. So $y_1(x)$, $y_2(x)$ converge at least for $|x|<1$. But $y_2(x)$ obviously converges for all x. By the ratio test, for $y_1(x)$,

$$\lim_{n\to\infty}\left|\frac{a_{3n+3}}{a_{3n}}\right| = \lim_{n\to\infty}\left|\frac{(3n-2)(3n-1)(3n-1)(3n)}{(3n+2)(3n+3)(3n-5)(3n-4)}x^3\right|$$

$$= |x^3| < 1.$$

Thus, the radius of convergence is 1 for $y_1(x)$.

Section 5.3, #10 $(1-x^2)y''-xy'+\alpha^2 y=0$.

Let $y=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=1}^{\infty} a_n n x^{n-1}$, $y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

Thus,

$$(1-x^2)y''-xy'+\alpha^2 y$$
$$=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}-\sum_{n=2}^{\infty} a_n n(n-1)x^n-\sum_{n=1}^{\infty} a_n n x^n+\alpha^2\sum_{n=0}^{\infty} a_n x^n$$
$$=\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n+\sum_{n=0}^{\infty} a_n[n(n-1)-n+\alpha^2]x^n$$

Note that we have added some terms to the 2nd + 3rd sums, but these added terms are equal to 0. Thus, for $n\ge 0$,

$$(n+2)(n+1)a_{n+2}=a_n(n^2-\alpha^2)$$

(1) or $$a_{n+2}=\frac{a_n(n^2-\alpha^2)}{(n+2)(n+1)}$$

We see immediately from (1) that if $\alpha=k$, a positive integer, then $a_{k+2}=0$. Hence, $a_{k+4}=0$, etc. All succeeding terms are equal to 0, and we get a polynomial solution.

Suppose $a_0=1$, $a_1=0$. From (1), we see that $a_{2n+1}=0$, $n\ge 0$.

$$a_2=\frac{a_0(-\alpha^2)}{2\cdot 1}=\frac{-\alpha^2}{2!}$$

$$a_4=\frac{a_2(2^2-\alpha^2)}{4\cdot 3}=\frac{-\alpha^2(2^2-\alpha^2)}{4!}$$

$$a_6=\frac{a_4(4^2-\alpha^2)}{6\cdot 5}=\frac{-\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)}{6!}$$

In general,

$$a_{2n}=\frac{-\alpha^2(2^2-\alpha^2)\cdots((2n-2)^2-\alpha^2)}{(2n)!}$$

Thus, $$y_1(x)=1-\sum_{n=1}^{\infty}\frac{+\alpha^2(2^2-\alpha^2)\cdots((2n-2)^2-\alpha^2)}{(2n)!}x^{2n}$$

Suppose $a_0=0$, $a_1=1$. From (1), we see that $a_{2n}=0$, $n\ge 0$

$$a_3=\frac{a_1(1^2-\alpha^2)}{3\cdot 2}=\frac{1^2-\alpha^2}{3!}$$

$$a_5=\frac{a_3(3^2-\alpha^2)}{5\cdot 4}=\frac{(1^2-\alpha^2)(3^2-\alpha^2)}{5!}.$$

In general, for $n \geq 1$,

$$a_{2n+1} = \frac{(1^2-\alpha^2)\cdots((2n-1)^2-\alpha^2)}{(2n+1)!}$$

So,

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(1^2-\alpha^2)\cdots((2n-1)^2-\alpha^2)}{(2n+1)!} x^{2n+1}.$$

Let $\alpha = 0$. Then for $y_1(x)$, $a_2 = 0$, and from (1), $a_{2n} = 0, n \geq 1$.
Thus,

$$y_1(x) = 1$$

Let $\alpha = 2$. Then for $y_1(x)$, $a_4 = 0$, and $a_2 = -\frac{2^2}{2} = -2$. From (1), we see that $a_{2n} = 0$, for $n \geq 2$. Thus,

$$y_1(x) = 1 - 2x^2$$

Let $\alpha = 1$. Then from (1), $a_3 = 0$, $a_{2n+1} = 0$, $n \geq 1$. Thus,

$$y_2(x) = x$$

Let $\alpha = 3$. Then $a_3 = \frac{a_1(1^2-3^2)}{3!} = \frac{-8}{6} = -\frac{4}{3}$.
From (1), $a_5 = 0$, and so $a_{2n+1} = 0$, $n \geq 2$. Thus,

$$y_2(x) = x - \frac{4}{3}x^3.$$