Solutions, HW Set # 8

Section 5.2 # 13 (a), (d) $2y'' + xy' + 3y = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$.

Thus, $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

$$2y'' + xy' + 3y = 2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + 3\sum_{n=0}^{\infty} a_n x^n.$$

$$= 2\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + 3\sum_{n=0}^{\infty} a_n x^n$$

Thus,

(1) $2a_2 \cdot 2 + 3a_0 = 0$ $\therefore$ if $a_0 = 1$, then $a_2 = -\frac{3}{2 \cdot 2}$

(2) $2a_3 \cdot 3 \cdot 2 + a_1 \cdot 1 + 3a_1 = 0$ $\therefore$ if $a_1 = 0$, $a_3 = 0$.

If $n \geq 2$,

$$2a_{n+2}(n+2)(n+1) + a_n(n+3) = 0.$$

(3) $$a_{n+2} = \frac{-a_n(n+3)}{2(n+2)(n+1)}$$

$$a_4 = \frac{-a_2 \cdot 5}{2 \cdot 4 \cdot 3} = \frac{3 \cdot 5}{2^2 \cdot 4!}$$

$$a_6 = \frac{-a_4 \cdot 7}{2 \cdot 6 \cdot 5} = -\frac{3 \cdot 5 \cdot 7}{2^3\, 6!}$$

In general, for $n \geq 1$

$$a_{2n} = \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n (2n)!}$$

From (3) for odd indices, $a_{2n+1} = 0$, since $a_1 = a_3 = 0$.

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n (2n)!} x^{2n}$$

Now let $a_0 = 0$, $a_1 = 1$. From (1) $a_2 = 0$, and from (3), we see $a_{2n} = 0$.

From (2),

$$a_3 = \frac{-4a_1}{3\cdot 2\cdot 2} = -\frac{1}{3}.$$

From (3),

$$a_5 = \frac{-a_3\cdot 6}{2(5\cdot 4)} = \frac{4\cdot 6}{2^2\, 5!}$$

$$a_7 = \frac{-a_5\cdot 8}{2\cdot 7\cdot 6} = -\frac{4\cdot 6\cdot 8}{2^3\, 7!}$$

In general, for $n \geq 1$,

$$a_{2n+1} = (-1)^n \frac{4\cdot 6\cdots(2n+2)}{2^n\,(2n+1)!}$$

Thus,

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{4\cdot 6\cdots(2n+2)}{2^n(2n+1)!}\, x^{2n+1}$$