#45, Section 3.4 $t^2y'' + 5ty' + 13y = 0$, Euler's d.e. (1)

$$r(r-1) + 5r + 13 = r^2 + 4r + 13 = 0$$

$$r = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$$

thus,

$$y = c_1 t^{-2}\cos(3\log t) + c_2 t^{-2}\sin(3\log t)$$

#20, Section 3.5 $y'' + 2y' + 5y = 4e^{-t}\cos(2t)$, $y(0)=1$, $y'(0)=0$.

$$r^2 + 2r + 5 = 0 \Longrightarrow r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\therefore \quad y_h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

As $-1 \pm 2i$ are characteristic roots of order 1,

$$y_p(t) = t[Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)]$$

$$y_p' = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$$
$$+ t[-Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t)]$$
$$= Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$$
$$+ t[(-A+2B)e^{-t}\cos(2t) + (-2A-B)e^{-t}\sin(2t)]$$

$$y_p'' = -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t)$$
$$+ 2Be^{-t}\cos(2t)$$
$$+ (-A+2B)e^{-t}\cos(2t) + (-2A-B)e^{-t}\sin(2t)$$
$$+ t[-(-A+2B)e^{-t}\cos(2t) - 2(-A+2B)e^{-t}\sin(2t)$$
$$- (-2A-B)e^{-t}\sin(2t) + 2(-2A-B)e^{-t}\cos(2t)]$$
$$= (-2A+4B)e^{-t}\cos(2t) + (-4A-2B)e^{-t}\sin(2t)$$
$$+ t[(-3A-4B)e^{-t}\cos(2t) + (4A-3B)e^{-t}\sin(2t)]$$

$$L[y_p] = (-2A+4B)e^{-t}\cos(2t) + (-4A-2B)e^{-t}\sin(2t)$$
$$+ t[(-3A-4B)e^{-t}\cos(2t) + (4A-3B)e^{-t}\sin(2t)]$$
$$+ 2Ae^{-t}\cos(2t) + 2Be^{-t}\sin(2t)$$
$$+ t[(-2A+4B)e^{-t}\cos(2t) + (-4A-2B)e^{-t}\sin(2t)]$$
$$+ 5t[Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)]$$

$$= t\left[0\cdot e^{-t}\cos(2t) + 0\cdot e^{-t}\sin(2t)\right]$$

$$+ 4Be^{-t}\cos(2t) - 4A\,e^{-t}\sin(2t) = 4e^{-t}\cos(2t)$$

$$= 4Be^{-t}\cos(2t) = 4e^{-t}\cos(2t) \Rightarrow B = 1$$

$$\therefore -4A = 0 \Rightarrow A = 0$$

$$\therefore y_p(t) = te^{-t}\sin(2t)$$

$$\therefore y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + te^{-t}\sin(2t),$$

$$y'(t) = -c_1 e^{-t}\cos(2t) - 2c_1 e^{-t}\sin(2t) - c_2 e^{-t}\sin(2t)$$
$$+ 2c_2 e^{-t}\cos(2t) + e^{-t}\sin(2t) - te^{-t}\sin(2t) + 2te^{-t}\cos(2t)$$

$$y(0) = c_1 = 1,$$

$$y'(0) = -c_1 + 2c_2 = 0 \Rightarrow c_2 = 1/2.$$

Thus,

$$y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + te^{-t}\sin(2t).$$