2.8, #14 (a) If $x = 0$, $\phi_n(x) = 0$ for all $n \ge 0$. If $x > 0$, then

$$\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \frac{2nx}{e^{nx^2}} = 0$$

by L'Hôspital's Rule. Thus,

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = \int_0^1 0\,dx = 0.$$

(b) $$\int_0^1 2nxe^{-nx^2}dx = -e^{-nx^2}\Big|_0^1 = -e^{-n}+1$$

Thus, $$\lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx = \lim_{n\to\infty}(-e^{-n}+1) = 1,$$

$\therefore$ $$\int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx = 0 \ne \lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx = 1.$$

2.8 #15 We assume that D is a closed rectangle. Otherwise the problem is incorrect. Let t be fixed, so that $f(t,y)$ is a function of y. Assume that if $\alpha \le y \le \beta$ on D, ~~then and that~~ then also $\alpha \le y_1, y_2 \le \beta$. By the mean value theorem, $\exists\, y^*$, $y_1 \le y^* \le y_2 \ni$:

$$|f(t,y_1) - f(t,y_2)| = \left|\frac{\partial f}{\partial y}(t,y^*)\,(y_1-y_2)\right|$$

Now $\frac{\partial f}{\partial y}$ is continuous on D and D is closed. Thus, $\left|\frac{\partial f}{\partial y}\right|$ is bounded, say by K, on D. Thus,

$$|f(t,y_1) - f(t,y_2)| \le K|y_1 - y_2|$$

2.9 #46 Let $v = y'$. Then $\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Thus,

$$yy'' - (y')^3 = y \cdot v\frac{dv}{dy} - v^3 = 0 \quad \text{or} \quad y\frac{dv}{dy} - v^2 = 0$$

Separating variables, we get

$$\frac{dv}{v^2} = \frac{dy}{y} \Rightarrow -\frac{1}{v} = \log y + c_1$$

or

$$v = -\frac{1}{\log y + c_1} \qquad (\text{assume } y > 0)$$

or

$$\frac{dy}{dt} = -\frac{1}{\log y + c_1}$$

Separate variables

$$(\log y + c_1)dy = -dt$$

Integrate both sides, with an integration by parts on the left side. Thus,

$$\int(\log y + c_1)dy = (\log y + c_1)y - \int \frac{y\,dy}{y}$$

$$= (\log y + c_1)y - y = -t + c_2$$

or

$$(\log y + c_1)y - y + t = c_2$$