Math 441, Solutions, Homework #3

2.4, #22 (a) $y_1(t) = 1-t$, $y_1' = -1$, $y_1(2) = 1-2 = -1$

$$\frac{-t+\sqrt{t^2+4(1-t)}}{2} = \frac{-t+\sqrt{(t-2)^2}}{2} = \frac{-t+t-2}{2} = -1$$

Thus, $y_1(t)$ is a solution of

$$y' = \frac{-t+\sqrt{t^2+4y}}{2}, \quad y(2) = -1. \qquad (*)$$

$y_2(t) = -t^2/4$, $y_2'(t) = -t/2$, $y_2(2) = -2/2 = -1$

$$\frac{-t+\sqrt{t^2+4\cdot -t^2/4}}{2} = \frac{-t+0}{2} = -\frac{t}{2}$$

Thus, $y_2(t)$ is also a solution of $(*)$.

(b) Here $f(t,y) = \dfrac{-t+\sqrt{t^2+4y}}{2}$

$$\frac{\partial f}{\partial y} = \frac{\frac{1}{2}\cdot 4}{2}(t^2+4y)^{-1/2} = (t^2+4y)^{-1/2}$$

at $(2,-1)$, $t^2+4y = 4-4 = 0$.

Thus, $\frac{\partial f}{\partial y}$ does not exist at $(2,-1)$. In particular, $\frac{\partial f}{\partial y}$ is not continuous at $(2,-1)$, and so the hypotheses of Theorem 2.4.2 are not satisfied.

(c) Let $y(t) = ct+c^2$. So, $y' = c$

$$\frac{-t+\sqrt{t^2+4y}}{2} = \frac{-t+\sqrt{t^2+4(ct+c^2)}}{2} = \frac{-t+\sqrt{(t+2c)^2}}{2} = \frac{-t+t+2c}{2} = c$$

Thus, the differential equation $(*)$ is satisfied. If $c = -1$, we obtain the solution $y_1(t)$ of part (a).

$y(t) = ct+c^2$ is linear in t, while $y_2(t) = -t^2/4$ is quadratic in t. Thus, $\nexists c$ such that $ct+c^2 = -t^2/4$.

Lastly, observe that the expression under the square root sign must be nonnegative. Thus, we need $t+2c \geq 0$, or $t \geq -2c$.

Section 2.4, # 31 $\frac{dy}{dt} = (\Gamma \cos t + T)y - y^3$

which is a Bernoulli d.e. with $n = 3$. Let $u = y^{-2}$. Thus,

$$\frac{du}{dt} = -2y^{-3}\frac{dy}{dt}$$

$$\therefore -\frac{y^3}{2}\frac{du}{dt} = (\Gamma \cos t + T)\, u y^3 - y^3$$

Cancel y^3. Multiply both sides by -2. Thus,

$$\frac{du}{dt} + 2(\Gamma \cos t + T)u = 2$$

This is linear, first order. An integrating factor is

$$e^{2\int(\Gamma \cos t + T)dt} = e^{2(\Gamma \sin t + Tt)} =: \mu(t)$$

Multiply both sides by this integrating factor. So,

$$\frac{d}{dt}\left(e^{2\Gamma \sin t + 2Tt} u\right) = 2e^{2\Gamma \sin t + 2Tt}$$

$$\mu(t)u(t) = 2\int \mu(t)dt + C$$

$$y^{-2} = u(t) = \frac{1}{\mu(t)}\left[2\int \mu(t)dt + C\right]$$

$$\therefore y = \sqrt{\frac{\mu(t)}{2\int \mu(t)dt + C}}$$