Section 7.5 # 29 $ay'' + by' + cy = 0$. Let $x_1 = y$, $x_2 = y'$

Homework Set # ~~12~~ 13

Then

$$x_1' = y' = x_2$$

$$x_2' = y'' = -\frac{b}{a} y' - \frac{c}{a} y = -\frac{b}{a} x_2 - \frac{c}{a} x_1 .$$

or

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

, i.e. set $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{bmatrix}$$

But,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus,

$$a_{12} = 1, a_{11} = 0, a_{21} = -c/a, a_{22} = -b/a$$

So

$$A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$$

$$|A - rI| = \begin{bmatrix} -r & 1 \\ -c/a & -b/a - r \end{bmatrix} = r^2 + \frac{b}{a} r + \frac{c}{a} = 0$$

or $ar^2 + br + c = 0$

7.4 6a $W(x^{(1)}, x^{(2)}) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$ ②

6b $W(x^{(1)}, x^{(2)}) = 0$ only if $t = 0$.
Thus, $x^{(1)}, x^{(2)}$ are linearly independent on any interval excluding $t = 0$.

6c Since the Wronskian is 0 or never 0 if the coefficient matrix is continuous, it must happen that at least 1 of the 4 entries is discontinuous at $t = 0$.

6d Consider $x^{(1)}$.

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore \quad \begin{aligned} p_{11} t + p_{12} &= 1 \\ p_{21} t + p_{22} &= 0 \end{aligned}$$

Consider $x^{(2)}$.

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} 2t \\ 2 \end{bmatrix} \quad \therefore \quad \begin{aligned} p_{11} t^2 + p_{12} 2t &= 2t \\ p_{21} t^2 + p_{22} 2t &= 2 \end{aligned}$$

Solve: $p_{11} t + p_{12} = 1 \quad \Rightarrow \quad p_{11} t^2 + p_{12} t = t$
$p_{11} t^2 + 2t p_{12} = 2t$

Subtract: $0 + t p_{12} = t \Rightarrow p_{12} = 1$

$p_{11} t + 1 = 1 \Rightarrow p_{11} = 0$

Solve: $p_{21} t^2 + p_{22} \cdot 2t = 2$
$p_{21} t + p_{22} = 0 \Rightarrow p_{21} t^2 + p_{22} t = 0$

Subtract: $0 + p_{22} t = 2 \Rightarrow p_{22} = 2/t$

$p_{21} t + 2/t = 0 \Rightarrow p_{21} = -2/t^2$

$$\therefore \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix}$$

which indeed is discontinuous at $t = 0$.

Sect. 7.7 # 15 (a) Following the author's hint, let s be fixed, and t a variable. We are given that $\Phi(t)$ is a solution of

$$Z'(t) = A Z(t), \quad Z(0) = I. \qquad (1)$$

i.e.

$$\Phi'(t) = A\Phi(t), \quad \Phi(0) = I.$$

$$\Rightarrow \quad \Phi'(t)\Phi(s) = A\Phi(t)\Phi(s). \qquad (2)$$

But, as s is a constant,

$$(\Phi(t)\Phi(s))' = \Phi'(t)\Phi(s).$$

Thus, from (2)

$$(\Phi(t)\Phi(s))' = A(\Phi(t)\Phi(s)),$$

and,

$$(\Phi(t)\Phi(s))\big|_{t=0} = I\,\Phi(s) = \Phi(s).$$

On the other hand, by replacing t by $t+s$, we have

$$\Phi'(t+s) = A\Phi(t+s), \quad \Phi(t+s)\big|_{t=0} = \Phi(s).$$

Thus, $\Phi(t+s)$ and $\Phi(t)\Phi(s)$ satisfy the same system with initial condition at $t=0$ being $\Phi(s)$. Hence, by the uniqueness theorem, we must have $\Phi(t+s) = \Phi(t)\Phi(s)$

Second solution of (a).

$$\Phi(t)\Phi(s) = e^{At}e^{As} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{j=0}^{\infty} \frac{A^j s^j}{j!}.$$

We collect together all terms where the power of A is equal to n.

$$\Phi(t)\Phi(s) = \sum_{n=0}^{\infty} A^n \sum_{k=0}^{n} \frac{t^k s^{n-k}}{k!\,(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} t^k s^{n-k} \binom{n}{k} \quad \text{by the binomial coefficient}$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} (t+s)^n \quad \text{by the binomial theorem}$$

$$= e^{A(t+s)} = \Phi(t+s)$$

b. Let $s = -t$ in (a). Thus

$$\Phi(t-t) = \Phi(0) = I = \Phi(t)\Phi(-t).$$

But by definition,

$$I = \Phi(t)\Phi^{-1}(t).$$

Thus, as the inverse of a matrix is unique,

$$\Phi^{-1}(t) = \Phi(-t) \qquad (*)$$

c. Replace s by $-s$ in part (a)

$$\Phi(t-s) = \Phi(t)\Phi(-s) = \Phi(t)\Phi^{-1}(s) \quad \text{by } (*)$$