5.7 #14 Bessel's d.e. of order 0 is Homework Set #12 ①

$$u^2y''+uy'+u^2y=0.$$

Let $u=ax$. By the chain rule,

$$\frac{dy}{du}=\frac{dy}{dx}\frac{dx}{du}=\frac{dy}{dx}\frac{1}{a}\ ;\ \frac{d^2y}{du^2}=\frac{d^2y}{dx^2}\frac{1}{a^2}$$

Thus,

$$u^2y''+uy'+u^2y=a^2x^2\frac{d^2y}{dx^2}\frac{1}{a^2}+ax\frac{dy}{dx}\frac{1}{a}+a^2x^2y$$

$$=x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+a^2x^2y=0$$

or

$$y''(x)+\frac{1}{x}y'(x)+a^2y(x)=0,$$

or

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+a^2y(x)=0, \qquad (1)$$

i.e. $J_0(ax)$ is a solution of (1). Similarly, $J_0(bx)$ is a solution of

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+b^2y(x)=0. \qquad (2)$$

Multiply both (1) and (2) by x. Thus,

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)=-xa^2y(x), \qquad (3)$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)=-xb^2y(x). \qquad (4)$$

Multiply (3) by $J_0(bx)$; multiply (4) by $J_0(ax)$. Then subtract the second equation from the first to get

$$J_0(bx)\frac{d}{dx}\left(x\frac{dJ_0(ax)}{dx}\right)-J_0(ax)\frac{d}{dx}\left(x\frac{dJ_0(bx)}{dx}\right) \qquad (5)$$

$$=(b^2-a^2)xJ_0(ax)J_0(bx).$$

If we apply the product rule ~~to each term~~ on the left side, we obtain an alternative version of (5), i.e. below,

$$\frac{d}{dx}\left[xJ_0(bx)J_0'(ax)-xJ_0(ax)J_0'(bx)\right]=(b^2-a^2)xJ_0(ax)J_0(bx).$$

(6)

Now integrate both sides of (6) over (0,1) to get

$$(b^2-a^2)\int_0^1 x J_0(ax)J_0(bx)dx$$

$$= \int_0^1 \frac{d}{dx}\left[xJ_0(bx)J_0'(ax) - xJ_0(ax)J_0'(bx)\right]$$

$$= xJ_0(bx)J_0'(ax) - xJ_0(ax)J_0'(bx)\Big|_0^1$$

$$= J_0(b)J_0'(a) - J_0(a)J_0'(b)$$

$$= 0,$$

if a and b are zeros of $J_0(x)$.

<u>7.2 #23</u>

$$x' = \begin{pmatrix}1\\0\end{pmatrix}e^t + 2\begin{pmatrix}1\\1\end{pmatrix}(e^t + te^t) = \begin{pmatrix}3\\2\end{pmatrix}e^t + 2\begin{pmatrix}1\\1\end{pmatrix}te^t$$

$$\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}x + \begin{pmatrix}1\\-1\end{pmatrix}e^t = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\left\{\begin{pmatrix}1\\0\end{pmatrix}e^t + 2\begin{pmatrix}1\\1\end{pmatrix}te^t\right\} + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

$$= \begin{pmatrix}2\\3\end{pmatrix}e^t + 2\begin{pmatrix}1\\1\end{pmatrix}te^t + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

$$= \begin{pmatrix}3\\2\end{pmatrix}e^t + 2\begin{pmatrix}1\\1\end{pmatrix}te^t$$

Hence,

$$x' = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}x + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

7.3 26a $Ax = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{n1} \cdots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}$ (3)

$$(Ax, y) = \left[\sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j \right] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} x_j \overline{y}_k$$

$$A^{T} y = \begin{bmatrix} a_{11} \cdots a_{n1} \\ \vdots \\ a_{1n} \cdots a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{i1} y_i \\ \vdots \\ \sum_{i=1}^{n} a_{in} y_i \end{bmatrix}$$

$$(x, A^{T} y) = [x_1, \ldots, x_n] \begin{bmatrix} \sum_{i=1}^{n} a_{i1} y_i \\ \vdots \\ \sum_{i=1}^{n} a_{in} y_i \end{bmatrix}$$

$= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} \overline{y}_i x_k$, since A is real.

$\therefore (Ax, y) = (x, A^{T} y)$

26b From part a,

$$(Ax, y) = \sum_{k=1}^{n} \sum_{j=1}^{n} \overline{a_{kj}} x_j \overline{y_k}$$

$$= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} \overline{a_{kj}} \overline{y_k} = (x, A^{*} y)$$

26c If A is Hermetian, $A^{*} = A$. The result follows immediately from part a.